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Nonoverlapping partitions of a surface *

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Abstract The colouring of planar domains is considered through the tight packing of rectangular regions. It is demonstrated that a maximal number of colours in a neighbourhood is achieved through the introduction of ribboned regions. This number can be reduced to four in the brick model with a special choice of colours in the surrounding region. An exceptional planar domain found by interweaving a ribboned region with a compact hexagonal configuration of isometric circles of a Schottky group requires an additional colour. The equivalent tight packing of isometric circles of the Schottky group provides a method for deriving the number of colours required to cover a Riemann surface. The chromatic number is derived for both orientable surfaces of genus $g \geq 3$ and nonorientable surfaces of genus $g \geq 4$.

Key words planar domains, ribboned regions, isometric circles, minimal number, Schottky problem, Riemann surfaces.

2020 Mathematics Subject Classification 05C15, 14H42, 30C20, 30F45, 52C26.

1 Introduction

The geographical areas in a colouring of a map may share boundaries without any overlap. This partitioning is characteristic also of isometric circles in a Schottky covering of a Riemann surface. If there is a region which is separate from the shaded areas, it can be expanded through a transformation of the sphere such that the isometric circles only cover a fraction of the surface. Then the ordering of the circles would be covered by four colours to distinguish between neighbouring but nonoverlapping disks. It will be demonstrated in section 3 that there is an exceptional configuration of isometric circles interwoven with a ribboned region that potentially requires an additional colour. If the ribbon is divided into portions, there is a relabelling of the different areas with only four colours. However, it is not necessary to have a partition of the ribboned region. A ribbon with an undivided domain must be labelled by a single letter, and then another colour would be required.
The number of colourings on a globe or surface with handles also may be determined. There is no Schottky covering of a sphere, and, if no region remains after the shading of the isometric disks, this uniformization does not exist because the domain of ordinary points consists of the empty set. A formula for the chromatic number has been given for arbitrary genus $g \geq 1$. An analysis of the pairing of the isometric circles in Theorem 4.1, which have different colours upon adjacency, yields this

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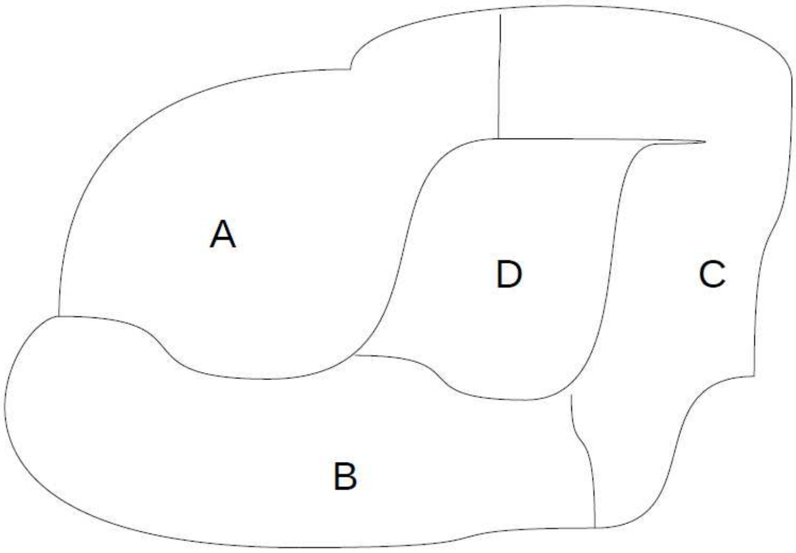

Fig. 1: Regions in a planar domain.

value of the maximum number of colours, provided that the surface does not satisfy the condition of orientability.

2 Colourings of planar regions

A typical group of regions in a map will appear in Fig. 1. Only four colours are necessary for these neighbouring areas. The brick model only requires three colours, and the insertion of a thin region connecting different blocks introduces one new colour. The inclusion of two thin ribboned regions in Fig. 2 also requires initially four colours, since the region above the higher ribbon and below the the top level is divided into three domains with different colours. However, at each stage of this partition, if the corresponding portion of the ribbon is labelled with the letter other than the two regions which are being divided, the number of regions can be reduced to three. The use of additional nonoverlapping ribboned regions in the sequence does not necessitate any new colours, since the thin regions can be labelled by similar markings.

The points of adjacency in a hexagonal configuration of isometric circles of equal size are not necessarily indicators of different colourings. However, if the boundaries are expanded to thin shaded regions, this configuration could have seven colours, although the minimum number immediately surrounding the central circle would be three, unless there are three adjoining adjacent circles, which would require four colours. When the circles have equal size, this configuration may be extended indefinitely. The sizes of the isometric circles can be decreased to cover essentially any planar region, with the infinitesimal remainders included in stretched domains.

Suppose that the size decreases geometrically with the level number. Since the available angle is reduced, the sizes also must become less for the circles to be nonoverlapping. However, if the number of levels is finite, the last level can be coloured alternately, and the same method may be used for the interior levels. The identical conclusion will not hold for the hexagonal configuration because each interior circle is surrounded by six circles of equal size. An alternating sequence of colours exists such that the neighbouring regions are labelled differently. The levels are distinguished and each circle is surrounded by different colourings. The proof of this last statement is given as follows. There are six circles surrounding each circle not located at the center of the configuration. Two of these circles belong to the same level, while four are positioned in the adjacent levels. Given two ribboned regions, the maximum number of colours would be required. The Schottky group represented by the configuration with the ribbon continuing indefinitely through an infinite number of circles could be considered to

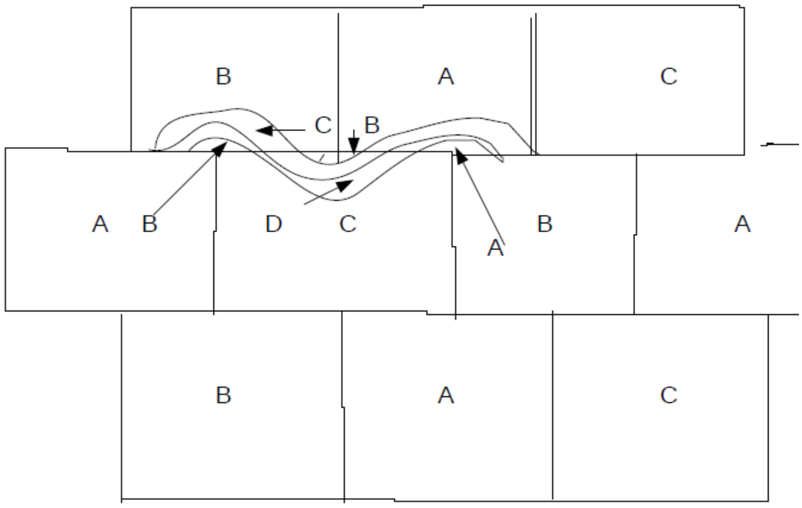

Fig. 2: Interweaving of two ribboned regions with a brick model.

define a uniformization of a surface with a boundary.

3 The number of colours of a map

The occurrence of four colours in a map [1, 2] follows from the Schottky uniformization of Riemann surfaces. There would be several stages of the discussion. First, it is necessary to demonstrate that any nonoverlapping partition of a planar surface can be conformally transformed to a Schottky covering of a Riemann surface. This result would require the classical retrosection theorem regarding the uniformization of every closed surface of finite genus by a Schottky group. Secondly, the chromatic number must be shown to be a conformal invariant. Thirdly, the maximal required number of colours arises in the closest packing of the isometric circles. A minimum of three colours may be used for six circles of equal radius surrounding a central circle. More generally, four colours are required for the neighbouring circles around a circle in a hexagonal configuration (Fig. 3) when the number of levels is greater than 3. By analogy with the brick model consisting of three colours, a new colour is introduced with the interweaving of a ribbon through the isometric circles.

Lemma 3.1. *Any nonoverlapping partition of a planar surface can be conformally transformed to a Schottky covering a Riemann surface.*

Proof. The nonoverlapping of the partition allows a pairing of like colours with the regions joined to create a handle. If there are unequal numbers of colours, the remaining regions may be paired according to location. This method allows the construction of a Riemann surface corresponding to the planar partition. Therefore, by the retrosection theorem, this Riemann surface may be uniformized by a Schottky group. The boundary of each region generally would not be a circle. Each of the subdomains in a partition could not be the isometric disks of a classical Schottky group. However, these curves characterize nonclassical groups that may be embedded in the set of Fuchsian Schottky groups. It has been proven that all Fuchsian Schottky groups are related to similarity transformations to classical Schottky groups. Therefore, there exists a conformal transformation to a planar domain with isometric circles of a classical Schottky group. □

Lemma 3.2. *The chromatic number is a conformal invariant.*

Proof. The chromatic number is determined by the maximum number of colourings of a graph on a Riemann surface. A conformal transformation will not change the angles between the edges of the graph. Since conformal transformations do not introduce self-intersections of the edges of a graph

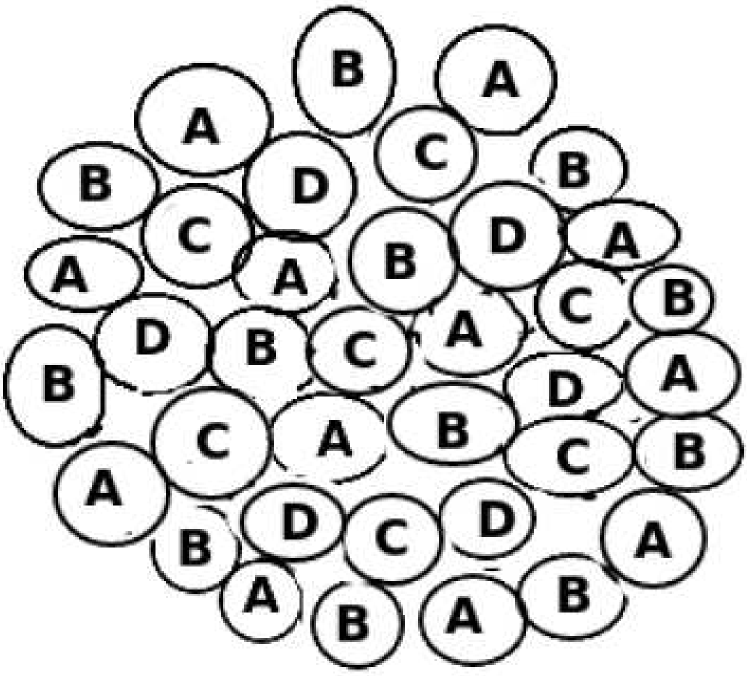

Fig. 3: Compact hexagonal configuration of isometric circles.

on the surface, and the colouring of the surface would not be altered, the chromatic number is an invariant. □

Theorem 3.3. *The interweaving of ribboned regions through the closest packing of isometric circles in a hexagonal configuration introduces a new colour which may be removed upon relabelling of portions of the ribbon.*

Proof. It would be expected that the maximum number of colours that is necessary for the colouring of the nonoverlapping disks would occur in a closest packing of the isometric circles. For circles of equal size, this packing results in a hexagonal configuration. Beyond the central circle, there are $6\ell - 6$ circles at level $\ell \geq 2$.
Although a typical circle in this configuration is surrounded by six circles, it is not identical to the brick model, since three can be circles in the next level, two at the same level and one in the preceding level comprising the six neighbouring circles, while the remaining configurations number two each respectively. The (2,2,2) configurations, which are identical to the brick models, only require three colours. The interweaving of ribboned regions introduces one colour. The (3,2,1) configurations, which occur in the directions of the vertices of the hexagon, include four colours. Matching with the brick model would cause an irregularity if only three colours occur in the pattern. Instead of three different consecutive colours, the sequence in Fig. 3 involves an alternating sequence which changes in the neighbouring levels. Therefore, there must be four colours in the most compact configuration of isometric circles of equal radius. Given the occurrence of four letters in the hexagonal configuration, the effect of the inclusion of ribboned remains to be determined.
Consider the slice through the hexagonal configuration. Then the sequence of circles CACBC ... at one equal level with respect to the splice and DBAD... at the next level (Fig. 4). This neighbourhood can be mapped to the brick model with four colours. If a single ribboned region is included amongst these brick size domains, it may be noted that it cuts out two other domains which could be labelled B above the boundary between A and D and A below the boundary of B and C (Fig. 4). Nevertheless,

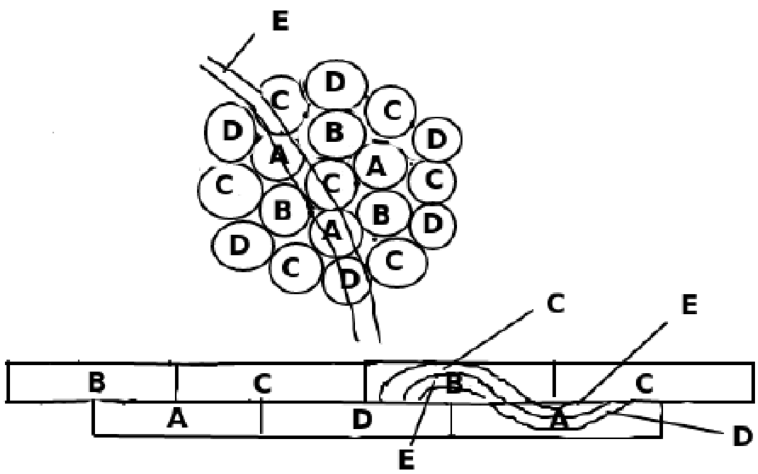

Fig. 4: Interweaving of a ribboned region with a hexagonal set of isometric circles and a brick model.

the ribboned region can be labelled E because it begins at the brick C and it is contiguous with D. A relabelling of the first portion of the ribbon to be C and the second portion to be D would require only four colours. It is not necessary, however, to partition the ribboned region, and another colour is required when there it is undivided.

A similar result occurs for the interweaving of two thin ribboned regions. When the ribbons cuts out the regions which are labelled C above the boundary between A and D, and A below the boundary between B and C, the two ribbons might be labelled B and D (Fig. 4) . It follows, therefore, that an extra colour is necessary when ribboned-size regions are interwoven through a compact hexagonal configuration of circles of equal radius.

The introduction of thin ribboned regions may be used to maximize the number of colours in a planar domain. The Schottky group with a fundamental domain represented by a hexagonal configuration of circles with a ribboned region may be considered to be a uniformizing group of a surface with a boundary. With an infinite number of isometric circles, this group would be uniformization of an infinite-genus surface with a boundary. □

4 Colourings of surfaces

An upper bound for the number of colours needed for graphs on orientable surfaces of genus g are given in [3]. The upper bound for the chromatic number

$$\gamma(g) = \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor \qquad g = 1,\ g \geq 4 \tag{4.1}$$

was proven for surfaces except the Klein bottle [4]. If g is set equal to zero, the lower bound would equal 4. In light of Theorem 3.3, the bound might be increased to cover hexagonal configurations of circles interwoven with a ribboned region which appears to require five colours.

The validity of the formula for $g \geq 1$ coincides with the existence of a Schottky uniformization of a closed Riemann surface of genus g for $g \geq 1$.

Theorem 4.1. *The maximum number of colours required to cover a surface of genus $g \geq 4$, without regard to orientability, is $\lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$.*

Proof. It may be noted that the increase with the genus is the same order as the number of levels in a hexagonal configuration of isometric circles of the Schottky group uniformizing a Riemann surface

of genus g. The level number for the tightest packing of circles of equal size is determined to be $\{\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\}$ [5]. The colourings between the levels may be selected to be different to ensure that there is no overlap with adjacent levels. It differs for large g from the chromatic number only by a factor of $3\sqrt{2}$. Three colours may be used at each level according to the colouring of the disks in the hexagonal configuration. The factor of $\sqrt{2}$ could arise from the existence of additional colourings required for the handles since the chromatic number of the torus is 7. The inclusion of the handle introduces of two colours at each side in the pairing of isometric, and replacing $24g$ by $48g$ gives the order of the chromatic number of the surface of genus $g \gg 1$ without the condition of orientability.
The constants must be adjusted to give the maximum number of colours at a given value of the genus. Instead of $\frac{1+\sqrt{9+48g}}{2}$, consider $\frac{a+\sqrt{b+48g}}{2}$. When $g=1$, either the next integer above or below this value must equal 7. If

$$\frac{a+\sqrt{b}}{2}+1=5, \frac{a+\sqrt{b+48}}{2}=7 \tag{4.2}$$

the solution is $a=7$ and $b=1$.
The exceptional nonorientable surfaces of genus 2 or 3 cannot have a chromatic number less than that of the closed orientable spheres with 2 or 3 nonoverlapping handles attached. It will be determined that the maximum number of colours can exceed $\lfloor\frac{7+\sqrt{1+48g}}{2}\rfloor$ for these surfaces.
When $g=4$, $\lfloor\frac{7+\sqrt{1+48g}}{2}\rfloor$ equals 10, which is sufficiently large since the chromatic number of orientable surfaces equals 10 for orientable surfaces of this genus. If $g \geq 5$, the handles created will require the pairing of the isometric circles with different colours at different levels only if these handles cross and form a nonorientable surface. The maximum number of colours is determined by the adjacency or crossing of the handles which cannot occur more than the tangency of the isometric circles in each level.
The closed surfaces of finite genus $g \geq 2$ may be conformally mapped to spheres with g handles attached. The maximal number of colours that would be required in the neighbourhood of each handle would not change when the handles do not overlap. Then the surface is orientable and another value may be derived for the chromatic number.

□

Theorem 4.2. *A chromatic number of 10 is sufficient for an orientable surface of genus* $g \geq 3$.

Proof. There are two colours at the base of each handle since the neighbourhoods on the sphere may be contiguous. The rest of the handle can be conformally mapped to a cylinder, which represents a planar domain with two edges identified. It is proven above in Theorem 3.3 that the supremum of the set of minimum numbers of colours necessary for the covering of planar domains is 5. The colours at the bases of the handle must be different from those on the cylinder. Bending each of the five regions with different colours on the planar domain to the edge which meets each base of the handle, the maximum number of colours on a surface with one handle is 7. When there is a second nonoverlapping handle, two new colours may be required at the bases yielding a maximum value of 9. If a third handle is attached to the surface, it may be conformally mapped to a planar domain, with disks removed, which can be covered by regions of five colours by Theorem 3.3. Since the colours on the cylinder must be different from the bases, and the chromatic number is an invariant under conformal transformations by Lemma 3.2, and it would equal 10. Five new colours therefore are introduced as a result of potential adjacency of the three handles. The addition of more handles to the surface leaves remainder to be a sphere with disks removed, that can be stereographically projected to the extended complex plane with disks removed, which may be covered by 5 colours. Considering the union of adjacent handles as a local region which with five colours, new handles no longer would require any new colours, since the theoretical explanation for the sufficiency of this set is analogous to that for planar domains. The chromatic number of the orientable surface of genus g then may be reduced to 10 for $g \geq 3$.

□

5 Conclusion

The four-colour map theorem is investigated through a listing of all possible topological combinations in a plane. A configuration that is most likely to yield the maximum number of colours makes use of

the interweaving of ribboned regions with a tight packing of rectangular domains. Although it seems that a ribboned diagram may depict a neighbourhood with five colours, this number can be reduced to four after a special permutation in the standard regions.
The ribboned regions also can be included in a configuration of isometric circles of a Schottky group. It is demonstrated that an additional fourth colour is required for a hexagonal configuration consisting of more than three levels. The introduction of a ribboned region appears to require an another colour when it is adjacent to the interiors of disks labelled by each of the four colours.
The Schottky group is the uniformizing group of an intermediate covering of a Riemann surface of genus $g \geq 1$. Therefore, it may be used in the analogue of the theorem for these surfaces. The chromatic number of orientable and nonorientable surfaces of genus $g \geq 1$ is derived in this work through a pairing of the isometric circles and the maximum number of colours on the handles.
These results are consistent with the five-colour map theorem [3] in graph theory and the existence of four colouring of maps [1] representing uniformizations of closed Riemann surfaces with a pairing of isometric circles. The introduction of ribboned regions is not compatible with this mapping and represents another class of manifolds that can have boundaries.

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