FUZZY IRRESOLVABLE SETS AND FUZZY OPEN HEREDITARILY IRRESOLVABLE SPACES

G. Thangaraj¹,*, S. Lokeshwari²

1. INTRODUCTION

The notion of fuzzy sets as an approach to a mathematical representation of vagueness in everyday language was introduced by L.A. Zadeh [16] in his classical paper in the year 1965. The potential of fuzzy notion was realized by the researchers and has successfully been applied in all branches of Mathematics. In 1968, C.L. Chang [2] introduced the concept of fuzzy topological spaces. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. In 1943, E. Hewitt [3] introduced the concepts of resolvable and irresolvable spaces in classical topology. The concept of open hereditarily irresolvable spaces was introduced by E.K. van Douwen in [15]. The concept of fuzzy resolvable sets in fuzzy topological spaces was introduced and studied in [9]. In continuation of this work, the notions of fuzzy hereditarily irresolvable spaces and fuzzy open hereditarily irresolvable spaces are introduced and studied by means of fuzzy irresolvable sets. In this paper, several characterizations of fuzzy irresolvable sets in fuzzy topological spaces are established. The existence of fuzzy regular closed sets and fuzzy regular open sets in fuzzy topological spaces is established by means of fuzzy irresolvable sets. By means of fuzzy resolvable and irresolvable sets, the notions of fuzzy hereditarily irresolvable spaces and fuzzy open hereditarily irresolvable spaces are introduced and studied.

2. PRELIMINARIES

In order to make the exposition self-contained, some basic notions and results used in the sequel, are given. In this work by \((X,T)\) or simply by \(X\), we will denote a fuzzy topological space due to Chang (1968). Let \(X\) be a
non-empty set and \( I \) the unit interval \([0,1]\). A fuzzy set \( \lambda \) in \( X \) is a mapping from \( X \) into \( I \). The fuzzy set \( 0_\lambda \) is defined as \( 0_\lambda(x) = 0 \), for all \( x \in X \) and the fuzzy set \( 1_\lambda \) is defined as \( 1_\lambda(x) = 1 \), for all \( x \in X \).

**Definition 2.1** [2]: Let \( (X, T) \) be a fuzzy topological space and \( \lambda \) be any fuzzy set in \( (X, T) \). The interior and the closure of \( \lambda \), are defined respectively as follows:

(i). \( \text{int}(\lambda) = \bigvee \{ \mu / \mu \leq \lambda, \mu \in T \} \)

(ii). \( \text{cl}(\lambda) = \bigwedge \{ \mu / \lambda \leq \mu, 1-\mu \in T \} \).

**Definition 2.1** [2]: For a family \( \{ \lambda_i / i \in I \} \) of fuzzy sets in \((X,T)\), the union \( \psi = \bigvee_i (\lambda_i) \) and intersection \( \delta = \bigwedge_i (\lambda_i) \), are defined respectively as

(iii). \( \psi(\lambda) = \sup \{ \lambda_i(\lambda) / x \in X \} \)

(iv). \( \delta(\lambda) = \inf \{ \lambda_i(\lambda) / x \in X \} \).

**Lemma 2.1** [1]: For a fuzzy set \( \lambda \) of a fuzzy topological space \( X \),

(i). \( 1-\text{int}(\lambda) = \text{cl}(1-\lambda) \) and (ii). \( 1-\text{cl}(\lambda) = \text{int}(1-\lambda) \).

**Definition 2.2**: A fuzzy set \( \lambda \) in a fuzzy topological space \((X,T)\), is called a

(i) **fuzzy dense set** if there exists no fuzzy closed set \( \mu \in (X,T) \) such that \( \lambda \leq \mu < 1 \). That is, \( \text{cl}(\lambda) = 1 \), in \((X,T)\) [5].

(ii) **fuzzy nowhere dense set** if there exists no non-zero fuzzy open set \( \mu \in (X,T) \) such that \( \mu < \text{cl}(\lambda) \). That is, \( \text{int}(\lambda) = 0 \), in \((X,T)\) [5].

(iii) **fuzzy simply open set** if \( \text{Bd}(\lambda) \) is a fuzzy nowhere dense set in \((X,T)\). That is, \( \lambda \) is a fuzzy simply open set in \((X,T)\) if \( \text{cl}(\lambda) \land \text{cl}(1-\lambda) \), is a fuzzy nowhere dense set in \((X,T)\) [7].

(iv) **fuzzy simply* open set** if \( \lambda \leq \mu \lor \delta \), where \( \mu \) is a fuzzy open set and \( \delta \) is a fuzzy no where dense set in \((X,T)\) [8].

(v) **fuzzy somewhere dense set** if \( \text{int} (\lambda) \neq 0 \) in \((X,T)\) [14].

(vi) **fuzzy resolvable set** in \((X,T)\) if for each fuzzy closed set \( \mu \in (X,T) \), \( \text{cl}(\mu \land \lambda) \land \text{cl}(1-\lambda) \) is a fuzzy nowhere dense set in \((X,T)\) [10].

(vii) **fuzzy regular- open set** in \((X,T)\) if \( \lambda \leq \mu \lor \delta \) and **fuzzy regular – closed set** in \((X,T)\) if \( \lambda = \text{cl} (\lambda) \) [1].

**Definition 2.3**: A fuzzy topological space \((X,T)\) is called a

(i) **fuzzy strongly irresolvable space** if for every fuzzy dense set \( \lambda \) in \((X,T)\), \( \text{int} (\lambda) = 1 \) in \((X,T)\) [13].

(ii) **fuzzy hyperconnected space** if every non - null fuzzy open subset of \((X,T)\), is fuzzy dense in \((X,T)\) [4].

(iii) **fuzzy globally disconnected space** if each fuzzy semi-open set is fuzzy open in \((X,T)\) [11].

**Theorem 2.1** [12]: If \( \lambda \) is a fuzzy somewhere dense set in a fuzzy topological space \((X,T)\), then there exist a fuzzy regular closed set \( \eta \) in \((X,T)\) such that \( \eta \leq \text{cl}(\lambda) \).

**Theorem 2.2** [9]: If \( \lambda \) is a fuzzy nowhere dense set in a fuzzy globally disconnected space \((X,T)\), then \( \lambda \) is a fuzzy resolvable set in \((X,T)\).

**Theorem 2.3** [9]: If \( \lambda \) is a fuzzy dense set in a fuzzy strongly irresolvable and fuzzy globally disconnected space \((X,T)\), then \( \lambda \) is a fuzzy resolvable set in \((X,T)\).

**Theorem 2.4** [9]: If \( \lambda \) is a fuzzy simply* open and fuzzy dense set in a fuzzy strongly irresolvable space \((X,T)\), then \( \lambda \) is a fuzzy resolvable set in \((X,T)\).

**Theorem 2.5** [10]: If \( \lambda \) is a fuzzy resolvable set in a fuzzy topological space \((X,T)\), then for each fuzzy closed set \( \mu \in (X,T) \), \( \lambda \land (1-\lambda) \land \mu \) is a fuzzy nowhere dense set in \((X,T)\).

**Theorem 2.6** [10]: If \( \lambda \) is a fuzzy open in a fuzzy hyper-connected space \((X,T)\), then \( \lambda \) is a fuzzy resolvable set in \((X,T)\).

**Theorem 2.7** [1]: In a fuzzy topological space

(a). The closure of a fuzzy open set is a fuzzy regular closed set.

(b). The interior of a fuzzy regular closed set is a fuzzy regular open set.
3. FUZZY IRRESOLVABLE SETS

**Definition 3.1:** Let \((X,T)\) be a fuzzy topological space. A fuzzy set \(\lambda\) is called a fuzzy irresolvable set if for a fuzzy closed set \(\mu\) in \((X,T)\), \(\text{Int}\{\text{cl}(\mu \wedge \lambda) \wedge \text{cl}(\mu \wedge (1-\lambda))\} \neq 0\), where \(1-\mu \in T\).

**Example 3.1:** Let \(X = \{a, b, c\}\). Consider the fuzzy sets \(\alpha, \beta, \gamma\) defined on \(X\) as follows:
\(\alpha : X \rightarrow [0, 1]\) is defined as \(\alpha(a) = 0.5, \alpha(b) = 0.4, \alpha(c) = 0.7\).
\(\beta : X \rightarrow [0, 1]\) is defined as \(\beta(a) = 0.6, \beta(b) = 0.5, \beta(c) = 0.6\).
\(\gamma : X \rightarrow [0, 1]\) is defined as \(\gamma(a) = 0.4, \gamma(b) = 0.6, \gamma(c) = 0.3\).
\(\lambda : X \rightarrow [0, 1]\) is defined as \(\lambda(a) = 0.4, \lambda(b) = 0.5, \lambda(c) = 0.5\).

Then, \(T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \vee \beta \vee \gamma, \alpha \vee \beta \vee \gamma \}_T\) is a fuzzy topology on \(X\). On computation, for a fuzzy closed set \(1 - \alpha\), \(\text{Int}\{\text{cl}(1-\alpha) \wedge \lambda \wedge \{1-(1-\alpha) \wedge (1-\lambda))\} = \text{Int}\{1-(\alpha \vee \beta)\} = \beta \wedge \gamma \neq 0\), in \((X,T)\). Hence, \(\lambda\) is a fuzzy irresolvable set in \((X,T)\).

**Proposition 3.1:** If \(\lambda\) is a fuzzy irresolvable set in a fuzzy topological space \((X,T)\), then \(\text{bd} (\lambda)\) is a fuzzy somewhere dense set in \((X,T)\).

**Proof:** Let \(\lambda\) be any fuzzy irresolvable set \(\text{in} (X,T)\). Then, for any fuzzy closed set \(\mu \in (X,T)\), \(\text{Int}\{\text{cl}(\lambda \wedge \mu) \wedge \text{cl}(1-(1-\lambda) \wedge \mu)\} \neq 0\), in \((X,T)\). Now, \(\text{Int}\{\text{cl}(\lambda \wedge \mu) \wedge \text{cl}(1-(1-\lambda) \wedge \mu)\} \neq 0\), in \((X,T)\). Thus, \(\text{Int}\{\text{cl}(\lambda \wedge \mu) \wedge \text{cl}(1-(1-\lambda) \wedge \mu)\} \neq 0\), in \((X,T)\). Hence, there exists a fuzzy regular closed set \(\eta\) in \((X,T)\) such that \(\eta \subseteq \text{bd}(\lambda)\). Since \(\text{bd}(\lambda)\) is a fuzzy nowhere dense set in \((X,T)\) and \(\text{Int}\{\text{bd}(\lambda)\} \neq 0\), in \((X,T)\).

**Remark 3.1:** In view of the above proposition one will have the following result:

“If \(\lambda\) is a fuzzy irresolvable set in a fuzzy topological space \((X,T)\), then \(\lambda\) is not a fuzzy simply open set in \((X,T)\)”.

**Proposition 3.2:** If \(\lambda\) is a fuzzy irresolvable set in a fuzzy topological space \((X,T)\), then there exists a fuzzy regular closed set \(\eta\) in \((X,T)\) such that \(\eta \subseteq \text{bd}(\lambda)\).

**Proof:** Let \(\lambda\) be any fuzzy irresolvable set \(\text{in} (X,T)\). Then, for any fuzzy closed set \(\mu \in (X,T)\), \(\text{Int}\{\text{cl}(\lambda \wedge \mu) \wedge \text{cl}(1-(1-\lambda) \wedge \mu)\} \neq 0\), in \((X,T)\). Hence, \(\text{Int}\{\text{cl}(\lambda \wedge \mu) \wedge \text{cl}(1-(1-\lambda) \wedge \mu)\} \neq 0\), in \((X,T)\).

**Proposition 3.3:** If \(\lambda\) is a fuzzy irresolvable set in a fuzzy topological space \((X,T)\), then \(\lambda\) and \((1-\lambda)\) are fuzzy somewhere dense sets in \((X,T)\).

**Proof:** Let \(\lambda\) be any fuzzy irresolvable set \(\text{in} (X,T)\). Then, for any fuzzy closed set \(\mu \in (X,T)\), \(\text{Int}\{\text{cl}(\lambda \wedge \mu) \wedge \text{cl}(1-(1-\lambda) \wedge \mu)\} \neq 0\), in \((X,T)\). Hence, \(\text{Int}\{\text{cl}(\lambda \wedge \mu) \wedge \text{cl}(1-(1-\lambda) \wedge \mu)\} \neq 0\), in \((X,T)\).

**Proposition 3.4:** If a fuzzy open set \(\lambda\) is a fuzzy irresolvable set in a fuzzy topological space \((X,T)\), then \(\lambda\) is not a fuzzy dense set in \((X,T)\).

**Proof:** Let \(\lambda\) be any fuzzy open set \(\text{in} (X,T)\). Then, by proposition 3.3, \((1-\lambda)\) is a fuzzy somewhere dense set in \((X,T)\) and \(\text{Int}(1-\lambda) \neq 0\), in \((X,T)\). Hence, \(\text{Int}(1-\lambda) \neq 0\), in \((X,T)\).
Proposition 3.5: If $\lambda$ is a fuzzy closed set in a fuzzy topological space $(X,T)$ in which fuzzy open sets are fuzzy irresolvable sets, then $\text{int}(\lambda) \neq \emptyset$, in $(X,T)$.

Proof: Let $\lambda$ be a fuzzy closed set in $(X,T)$. Then, $1-\lambda$ is a fuzzy open set in $(X,T)$. Then, by proposition 3.4, $\text{cl}(1-\lambda) \neq 1$, and thus $1-\text{int}(\lambda) \neq \emptyset$, in $(X,T)$. Hence $\text{int}(\lambda) \neq \emptyset$, in $(X,T)$.

Proposition 3.6: If $\lambda$ is a fuzzy set defined on $X$ in a fuzzy topological space $(X,T)$ in which fuzzy open sets are fuzzy irresolvable sets, then there exists a fuzzy regular open set $1$, in $(X,T)$.

Proof: Let $\lambda$ be a fuzzy set defined on $X$ in $(X,T)$. If $\text{int}(\lambda)$ is a non-zero fuzzy open set in $(X,T)$, then by hypothesis, $\text{int}(\lambda)$ is a fuzzy irresolvable set in $(X,T)$. By proposition 3.4, $\text{cl}(\text{int}(\lambda)) \neq 1$, in $(X,T)$. This implies that $1-\text{cl}(\text{int}(\lambda)) \neq 0$ and thus $\text{int}(1-\lambda) \neq \emptyset$, in $(X,T)$. Hence $1-\lambda$ is a fuzzy somewhere dense set in $(X,T)$.

Proposition 3.7: If $\lambda$ is a fuzzy set defined on $X$ in a fuzzy topological space $(X,T)$ in which fuzzy open sets are fuzzy irresolvable sets, then $1$ is a fuzzy somewhere dense set in $(X,T)$.

Proof: Let $\lambda$ be a fuzzy set defined on $X$ in the fuzzy topological space $(X,T)$. By the Theorem 2.1, there exists a fuzzy regular closed set $\eta$ in $(X,T)$ such that $\eta \leq \text{cl}(1-\lambda)$. Then, $\eta \leq 1-\text{int}(\lambda)$. This implies that $\text{int}(\lambda) \leq 1-\eta$, in $(X,T)$. Now $1-\eta$ is a fuzzy regular open set in $(X,T)$. Let $\delta = 1-\eta$. Thus, there exists a fuzzy regular open set $\delta$ in $(X,T)$ such that $\text{int}(\lambda) \leq \delta$.

Proposition 3.8: If $\lambda$ is a fuzzy closed set in a fuzzy topological space $(X,T)$ in which fuzzy open sets are fuzzy irresolvable sets, then $\lambda$ is a fuzzy somewhere dense set in $(X,T)$.

Proof: Let $\lambda$ be a fuzzy closed set in $(X,T)$ in which fuzzy open sets are fuzzy irresolvable sets. Then, by proposition 3.5, $\text{int}(\lambda) \neq \emptyset$, in $(X,T)$. Now $\text{int}(\lambda) = \text{int}(\lambda) \neq \emptyset$, in $(X,T)$. Hence, $\lambda$ is a fuzzy somewhere dense set in $(X,T)$.

Proposition 3.9: If $\lambda$ is a fuzzy somewhere dense set in a fuzzy topological space $(X,T)$ in which fuzzy open sets are fuzzy irresolvable sets, and if $\lambda \leq \mu$, then $\mu$ is a fuzzy somewhere dense set in $(X,T)$.

Proof: Let $\lambda$ be a fuzzy somewhere dense set in $(X,T)$ in which fuzzy open sets are fuzzy irresolvable sets. Then, by proposition 3.8, $\lambda$ is a fuzzy somewhere dense set in $(X,T)$ and thus $\text{intcl}(\lambda) \neq \emptyset$. Now $\lambda \leq \mu$ implies that $\text{intcl}(\lambda) \leq \text{intcl}(\mu)$. Then $\text{intcl}(\mu) \neq \emptyset$. Hence $\mu$ is a fuzzy somewhere dense set in $(X,T)$.

Proposition 3.10: If $\lambda$ is a fuzzy somewhere dense set in a fuzzy topological space $(X,T)$ in which fuzzy open sets are fuzzy irresolvable sets, then $\text{int}(\lambda \wedge (1-\lambda)) = 0$, in $(X,T)$.

Proof: Let $\lambda$ be a fuzzy somewhere dense set in $(X,T)$. Then, $\text{intcl}(\lambda) \neq \emptyset$. By hypothesis, the fuzzy open set $\text{intcl}(\lambda)$, is a fuzzy resolvable set in $(X,T)$ such that $\delta \leq \text{cl}(\lambda)$ and $1-\delta \geq 1-\text{cl}(\lambda)$. Now $\text{int}(1-\delta) \geq \text{int}[1-\text{cl}(\lambda)]$ implies that $\text{int}(1-\delta) \geq 1-\text{cl}[\text{cl}(\lambda)]$. Thus, $\text{int}(1-\delta) \geq 1-\text{cl}(\lambda)$ in $(X,T)$ (since $\text{cl}(\lambda) = 1-\text{cl}(\lambda)$). Let $\gamma = 1-\delta$. Then $\gamma$ is a fuzzy closed set with $\text{int}(\gamma) \neq \emptyset$, in $(X,T)$. Since the non-zero fuzzy open set $\text{intcl}(\lambda)$ is a fuzzy resolvable set in $(X,T)$, by theorem 2.5, for the fuzzy closed set $\gamma$, $\text{intcl}(\lambda) \wedge (1-\text{intcl}(\lambda)) \wedge \gamma = 0$, in $(X,T)$. But $\text{int}(\lambda) \wedge (1-\text{int}(\lambda)) \wedge \gamma \leq \text{intcl}(\lambda) \wedge (1-\text{int}(\lambda)) \wedge \gamma \leq \text{intcl}(\lambda) \wedge (1-\text{int}(\lambda)) \wedge \gamma = 0$, in $(X,T)$. This implies that $\text{intcl}(\lambda) \wedge \text{int}(\lambda) \wedge (1-\text{int}(\lambda)) = 0$ and then $\text{int}(\lambda) \wedge (1-\text{intcl}(\lambda)) \wedge \gamma = 0$. Now $\text{int}(\lambda) \leq \text{intcl}(\lambda), \text{int}(\gamma) \leq 1-\text{cl}(\lambda), 1-\text{cl}[\text{int}(\lambda)] \geq 1-\text{cl}(\lambda)$ implies that $\text{int}(\lambda) \wedge (1-\text{cl}(\lambda)) \leq \text{intcl}(\lambda) \wedge (1-\text{intcl}(\lambda)) \wedge \text{int}(\gamma)$ and thus $\text{int}(\lambda) \wedge (1-\text{cl}(\lambda)) = 0$. This implies that $\text{int}(\lambda) \wedge \text{int}(1-\lambda) = 0$, in $(X,T)$ and $\text{int}(\lambda) \wedge (1-\lambda) = 0$, in $(X,T)$.

Proposition 3.11: If $\lambda$ is a fuzzy somewhere dense set in a fuzzy topological space $(X,T)$ in which fuzzy open sets are fuzzy irresolvable sets, $\text{cl}[\lambda \vee (1-\lambda)] = 1$, in $(X,T)$.

Proof: Let $\lambda$ be a fuzzy somewhere dense set in $(X,T)$. Since $(X,T)$ is a fuzzy topological space in which fuzzy open sets are fuzzy irresolvable sets, by proposition 3.10, $\text{int}[\lambda \vee (1-\lambda)] = 0$, in $(X,T)$. Then $1-\text{int}[\lambda \vee (1-\lambda)] = 1$. This implies that $1-\text{int}[\lambda \vee (1-\lambda)] = 1$, and then $\text{cl}(1-\lambda) \neq 1$. Now $\text{cl}(1-\lambda) = \text{cl}((1-\lambda) \vee (1-\lambda)) = \text{cl}(1-\lambda)$, implies that $\text{cl}[\lambda \vee (1-\lambda)] = 1$, in $(X,T)$.
4. FUZZY HEREDITARILY IRRESOLVABLE SPACES

**Definition 4.1:** A fuzzy topological space $(X, T)$ is called a fuzzy hereditarily irresolvable space if there is no non-zero fuzzy resolvable set in $(X, T)$.

**Proposition 4.1:** If $\lambda$ is a fuzzy set in a fuzzy topological space $(X, T)$ in which fuzzy closed sets have zero interiors, then $\lambda$ is a fuzzy irresolvable set in $(X, T)$.

**Proof:** Let $\lambda$ be a non-zero fuzzy set defined on $X$ in $(X, T)$. Then, for a fuzzy closed set $\mu$ in $(X, T)$ by hypothesis, $\text{int}(\mu) = 0$. Now, $\text{int}(\text{cl}(\lambda \wedge \mu) \setminus \text{cl}(1 - \lambda) \setminus \mu) = \text{int}(\{\text{cl}(\lambda \wedge \mu) \setminus \text{cl}(1 - \lambda) \setminus \mu\}) = \text{int}(\{\text{cl}(\lambda \wedge \mu) \setminus \text{cl}(1 - \lambda) \setminus \mu\}) \subseteq \text{int}(\{\text{cl}(\lambda \wedge \mu) \setminus \text{cl}(1 - \lambda) \setminus \mu\}) \setminus \mu = 0$. Thus, for a fuzzy closed set $\mu$, $\text{int}(\text{cl}(\lambda \wedge \mu) \setminus \text{cl}(1 - \lambda) \setminus \mu) = 0$ and hence $\lambda$ is a fuzzy irresolvable set in $(X, T)$.

**Proposition 4.2:** If $\lambda \leq \mu$, for each fuzzy closed set $\mu$ with $\text{int}(\mu) = 0$ in a fuzzy topological space $(X, T)$, then $\lambda$ is a fuzzy irresolvable set but not a fuzzy open set in $(X, T)$.

**Proof:** Let $\lambda$ be a fuzzy set defined on $X$ and $\mu$ a fuzzy closed set such that $\text{int}(\mu) = 0$ in $(X, T)$. Then, by proposition 4.1, $\lambda$ is a fuzzy irresolvable set in $(X, T)$. Since $\lambda \leq \mu$, $\text{int}(\lambda) \leq \text{int}(\mu)$ and $\text{int}(\mu) = 0$, implies that $\text{int}(\lambda) = 0$ and hence $\lambda$ is not a fuzzy open set in $(X, T)$.

**Proposition 4.3:** If $\lambda$ is a fuzzy set in a fuzzy topological space $(X, T)$ in which fuzzy open sets are fuzzy dense sets, then $\lambda$ is a fuzzy irresolvable set in $(X, T)$.

**Proof:** Let $\lambda$ be a non-zero fuzzy set in $(X, T)$. If $\mu$ is a non-zero fuzzy closed set in $(X, T)$, then $1 - \mu$ is a fuzzy open set in $(X, T)$ and by hypothesis, $\text{cl}(1 - \mu) = I$, $\text{int}(\mu)$, $\text{int}(1 - \mu) = 1$, implies that $\text{int}(\mu) = 0$ in $(X, T)$. Then, by proposition 4.1 $\lambda$ is a fuzzy irresolvable set in $(X, T)$.

**Proposition 4.4:** If $\lambda$ is a fuzzy set in a fuzzy topological space $(X, T)$ in which $\text{int}(\text{bd}(\lambda)) = 0$, then $\lambda$ is a fuzzy irresolvable set in $(X, T)$.

**Proof:** Let $\lambda$ be a non-zero fuzzy set defined on $X$ in $(X, T)$. By hypothesis, $\text{int}(\text{bd}(\lambda)) = 0$, in $(X, T)$. Now, for a fuzzy closed set $\mu$ in $(X, T)$, as in the proof of proposition 4.1, $\text{int}(\text{cl}(\lambda \wedge \mu) \setminus \text{cl}(1 - \lambda) \setminus \mu) = 0$ by proposition 4.1 $\lambda$ is a fuzzy irresolvable set in $(X, T)$.

**Remark 4.1:** In view of the propositions 4.1, 4.3, and 4.4, one will have the following results:

(i) “The fuzzy topological spaces in which fuzzy closed sets have zero interiors, are not fuzzy hereditarily irresolvable spaces.”

(ii) “The fuzzy topological spaces in which fuzzy open sets are fuzzy dense sets, are not fuzzy hereditarily irresolvable spaces.”

(iii) “The fuzzy topological spaces in which boundary of fuzzy sets have zero interiors, are not fuzzy hereditarily irresolvable spaces.”

**Proposition 4.5:** If $(X, T)$ is a fuzzy hyperconnected space, then $(X, T)$ is not a fuzzy hereditarily irresolvable space.

**Proof:** Let $(X, T)$ be a fuzzy hyperconnected space. Then, each fuzzy open set is a fuzzy dense set in $(X, T)$. If $\lambda$ is a non-zero fuzzy set defined on $X$, then by proposition 4.3, $\lambda$ is a fuzzy irresolvable set in $(X, T)$ and hence $(X, T)$ is not a fuzzy hereditarily irresolvable space.

**Proposition 4.6:** If there exists a fuzzy nowhere dense set in a fuzzy globally disconnected space $(X, T)$, then $(X, T)$ is not a fuzzy hereditarily irresolvable space.

**Proof:** Let $(X, T)$ be a fuzzy globally disconnected space. Suppose that $\lambda$ is a fuzzy nowhere dense set in $(X, T)$. Then, by theorem 2.2, $\lambda$ is a fuzzy irresolvable set in $(X, T)$ and hence $(X, T)$ is not a fuzzy hereditarily irresolvable space.

**Proposition 4.7:** If there exists a fuzzy dense set in a fuzzy strongly irresolvable and fuzzy globally disconnected space $(X, T)$, then $(X, T)$ is not a fuzzy hereditarily irresolvable space.

**Proof:** Let $(X, T)$ be a fuzzy strongly irresolvable and fuzzy globally disconnected space. Suppose that $\lambda$ is a fuzzy dense set in $(X, T)$, then, by theorem 2.3, $\lambda$ is a fuzzy irresolvable set in $(X, T)$ and hence $(X, T)$ is not a fuzzy hereditarily irresolvable space.
Proposition 4.8: If there exist a fuzzy simply open and fuzzy dense set in a fuzzy strongly irresolvable space \((X,T)\), then \((X,T)\) is not a fuzzy hereditarily irresolvable space.

Proof: Let \((X,T)\) be a fuzzy strongly irresolvable space. Suppose that \(\lambda\) is a fuzzy simply open and fuzzy dense set in \((X,T)\). Then, by theorem 2.4, \(\lambda\) is a fuzzy resolvable set in \((X,T)\) and hence \((X,T)\) is not a fuzzy hereditarily irresolvable space.

Remark 4.2: In view of the propositions 4.6, 4.7 and 4.8, one will have the following results:

(i) “The existence of fuzzy nowhere dense sets in a fuzzy globally disconnected space makes it a fuzzy non-hereditarily irresolvable space”.

(ii) “The existence of fuzzy dense sets in a fuzzy strongly irresolvable and fuzzy globally disconnected space makes it a fuzzy non-hereditarily irresolvable space”.

(iii) “The existence of fuzzy simply open and fuzzy dense sets in a fuzzy strongly irresolvable space makes it a fuzzy non-hereditarily irresolvable space”.

Proposition 4.9: If \((X,T)\) is a fuzzy hereditarily irresolvable space and if \(\lambda\) is a fuzzy set in \((X,T)\), then \(\lambda\) is a fuzzy irresolvable set in \((X,T)\).

Proof: Let \((X,T)\) be a fuzzy hereditarily irresolvable space and \(\lambda\) be a fuzzy set defined on \(X\) in \((X,T)\). Then, there is no non-zero fuzzy resolvable set in \((X,T)\), implies that for a fuzzy closed set \(\mu\) in \((X,T)\), \(\text{int} \{ \text{cl} (1 - \lambda) \} \neq 0\) and hence the fuzzy set \(\lambda\) is a fuzzy irresolvable set in \((X,T)\).

Proposition 4.10: If \(\lambda\) is a fuzzy closed set in a fuzzy hereditarily irresolvable space \((X,T)\), then

(i). \(\text{int} (\lambda) \neq 0;\)

(ii). \(\text{cl int}(\lambda) \neq 1;\)

(iii). For a fuzzy closed set \(\mu\) in \((X,T)\), \(\text{int} (\mu) \neq 0.\)

Proof: Let \(\lambda\) be a fuzzy closed set in \((X,T)\). Since \((X,T)\) is a fuzzy hereditarily irresolvable space, by proposition 4.8, \(\lambda\) is a fuzzy irresolvable set in \((X,T)\). Then, for a fuzzy closed set \(\mu \text{ int}(X,T)\), \(\text{int} \{ \text{cl} (1 - \lambda) \} \neq 0\) in \((X,T)\). Since \(\text{int} \{ \text{cl}(\lambda \mu) \cap \text{cl} (1 - \lambda)\} \neq 0\), \(\text{int} \{ \text{cl}(\lambda \mu) \} \neq 0\).

Proposition 4.11: If \((X,T)\) is a fuzzy hereditarily irresolvable space, then there is no non-zero fuzzy nowhere dense set in \((X,T)\).

Proof: Suppose that \(\lambda\) is a fuzzy nowhere dense set in \((X,T)\). Then \(\text{int} (\lambda) = 0\), in \((X,T)\). Now, for a fuzzy closed set \(\mu\) in \((X,T)\), \(\text{int} \{ \text{cl} (\lambda \mu) \} \neq 0\). In \((X,T)\), \(\text{int} (\lambda) = 0\), \(\text{int} (\mu) = 0\), and \(\text{int} (\lambda \mu) = 0\).

Proposition 4.12: If \((X,T)\) is a fuzzy hereditarily irresolvable space, then there is no non-zero fuzzy set \(\lambda\) in \((X,T)\) such that \(\text{cl int}(\lambda) = 1\).

Proof: Suppose that \(\lambda\) is a fuzzy set in \((X,T)\) such that \(\text{cl int}(\lambda) = 1\). Then, \(1 - \text{cl int}(\lambda) = 0\) and this implies that \(\text{int} (\lambda) = 0\) in \((X,T)\). Now, for a fuzzy closed set \(\mu\) in \((X,T)\), \(\text{int} (\lambda \mu) = \text{int} (1 - \lambda \mu) = 0\). In \((X,T)\), \(\text{int} (\lambda) = 0\), \(\text{int} (\mu) = 0\), and \(\text{int} (\lambda \mu) = 0\).
Proof: Let \( \lambda \) be a fuzzy set defined on \( X \) in \( (X, T) \). Since \( (X, T) \) is a fuzzy hereditarily irresolvable space, \( \lambda \) is a fuzzy irresolvable set in \( (X, T) \). Then, by proposition 3.1, \( \text{int} \{ \text{bd}(\lambda) \} \neq 0 \), in \( (X,T) \). Thus \( \lambda \) is not a fuzzy simply open set in \( (X,T) \).  

**Remark 4.3:** In view of the propositions 4.11 and 4.13, one will have the following result: 

“In fuzzy hereditarily irresolvable spaces, there are no fuzzy simply open sets and fuzzy nowhere dense set “.

**Proposition 4.14:** If \( (X,T) \) is a fuzzy hereditarily irresolvable space, then the fuzzy simply* open sets are fuzzy open sets in \( (X,T) \).

**Proof:** Let \( \lambda \) be a fuzzy simply* open set in \( (X,T) \). Then, \( \lambda = \mu \lor \delta \), where \( \mu \) is a fuzzy open set and \( \delta \) is a fuzzy nowhere dense set in \( (X,T) \). Since \( (X, T) \) is a fuzzy hereditarily irresolvable space, by proposition 4.11, there is no non-zero fuzzy nowhere dense set in \( (X,T) \) and hence \( \lambda = \mu \lor \theta = \mu \) and thus the fuzzy simply open sets in a fuzzy hereditarily irresolvable space \( (X,T) \) are fuzzy open sets in \( (X,T) \).

**5. FUZZY OPEN HEREDITARILY IRRESOLVABLE SPACES**

**Definition 5.1:** A fuzzy topological space \( (X, T) \) is called a fuzzy open hereditarily irresolvable space if each non-zero fuzzy open set is a fuzzy irresolvable space in \( (X,T) \).

**Example 5.1:** Let \( X = \{ a, b, c \} \). Consider the fuzzy sets \( \alpha, \beta, \gamma \) defined on \( X \) as follows:  

\[
\alpha: X \rightarrow [0, 1] \text{ is defined as } \alpha(a) = 0.5; \quad \alpha(b) = 0.4; \quad \alpha(c) = 0.7.
\]

\[
\beta: X \rightarrow [0, 1] \text{ is defined as } \beta(a) = 0.6; \quad \beta(b) = 0.5; \quad \beta(c) = 0.6.
\]

\[
\gamma: X \rightarrow [0, 1] \text{ is defined as } \gamma(a) = 0.4; \quad \gamma(b) = 0.6; \quad \gamma(c) = 0.3.
\]

Then, \( T = \{ 0, \alpha, \beta, \gamma, \alpha \lor \beta, \alpha \lor \gamma, \beta \lor \gamma, \alpha \land \beta, \alpha \land \gamma, \beta \land \gamma, \alpha \lor (\beta \land \gamma), \beta \lor (\alpha \lor \gamma), \alpha \lor \beta \lor \gamma \} \) is a fuzzy topology on \( X \). On computation, one can easily see that for a fuzzy closed set \( \delta \) in \( (X,T) \) and, for each non-zero fuzzy open set \( \mu \) in \( (X,T) \), \( \text{int} \{ \text{cl}(\{1-\delta\} \land \mu) \land \text{cl}((1-\delta) \land (1-\mu)) \} \neq 0 \), in \( (X,T) \). Hence the fuzzy topological space \( (X,T) \) is a fuzzy open hereditarily irresolvable space.

**Proposition 5.1:** If \( \lambda \) is a fuzzy closed set in a fuzzy open hereditarily irresolvable space \( (X,T) \), then \( \text{int}(\lambda) \neq 0 \), in \( (X,T) \).

**Proof:** Let \( \lambda \) be a fuzzy closed set in \( (X,T) \). Since \( (X,T) \) is a fuzzy open hereditarily irresolvable space, fuzzy open sets in \( (X,T) \) are fuzzy irresolvable sets in \( (X,T) \) and then, by proposition 3.5, \( \text{int}(\lambda) \neq 0 \), in \( (X,T) \).

**Proposition 5.2:** If \( \lambda \) is a fuzzy closed set in a fuzzy open hereditarily irresolvable space \( (X,T) \), then \( \lambda \) is a fuzzy somewhere dense set in \( (X,T) \).

**Proof:** Let \( \lambda \) be a fuzzy closed set in \( (X,T) \). Since \( (X,T) \) is a fuzzy open hereditarily irresolvable space, by proposition 5.1, \( \text{int}(\lambda) \neq 0 \), in \( (X,T) \). Now \( \text{int}(\lambda) = \text{int}(\lambda) \neq 0 \) in \( (X,T) \). Hence, \( \lambda \) is a fuzzy somewhere dense set in \( (X,T) \).

**Proposition 5.3:** If \( \lambda \) is a fuzzy set defined on \( X \) in a fuzzy open hereditarily irresolvable space then there exists a fuzzy regular open set \( \delta \) in \( (X,T) \) such that \( \text{int}(\lambda) \leq \delta \).

**Proof:** Let \( \lambda \) be a fuzzy set defined on \( X \) in \( (X,T) \). Since \( (X,T) \) is a fuzzy open hereditarily irresolvable space, by proposition 3.7, there exists a fuzzy regular open set \( \delta \) in \( (X,T) \) such that \( \text{int}(\lambda) \leq \delta \).

**Proposition 5.4:** If \( \lambda \) is a fuzzy set defined on \( X \) in a fuzzy open hereditarily irresolvable space \( (X,T) \), then \( 1-\lambda \) is a fuzzy somewhere dense set in \( (X,T) \).

**Proof:** Let \( \lambda \) be a fuzzy set defined on \( X \) in \( (X,T) \). Then, \( \text{int}(\lambda) \) is a fuzzy open set in \( (X,T) \). Since \( (X,T) \) is a fuzzy open hereditarily irresolvable space, by proposition 3.4, \( \text{int}(\lambda) \) is not a fuzzy dense set in \( (X,T) \). That is, \( \text{cl} \text{int}(\lambda) \neq 1 \) and then \( 1-\text{cl} \text{int}(\lambda) \neq 0 \). This implies that \( \text{int} \{1-\lambda\} \neq 0 \). Hence \( 1-\lambda \) is a fuzzy somewhere dense set in \( (X,T) \).
Proposition 5.5: If a fuzzy topological space \((X, T)\) is a fuzzy open hereditarily irresolvable space, then \(\text{int cl} \ (\lambda) \neq 0\), for any non-zero fuzzy set \(\lambda\) in \((X, T)\).

**Proof:** Let \((X, T)\) be a fuzzy open hereditarily irresolvable space and \(\lambda\) be a fuzzy set defined on \(X\) such that \(\text{int cl} \ (\lambda) \neq 0\), in \((X, T)\). Suppose that \(\text{int cl} \ (\lambda) = 0\), in \((X, T)\). Then, \(\text{cl} \ (1 - \lambda) = 1 - \text{int}(\lambda) = 1\) and then \(1 - \lambda\) will be a fuzzy dense set in the fuzzy open hereditarily irresolvable space \((X, T)\), a contradiction to \(1 - \lambda\) being a fuzzy somewhere dense set in \((X, T)\), [by proposition 5.4] and thus if \(\text{int cl} \ (\lambda) \neq 0\), for a non-zero fuzzy set in a fuzzy open hereditarily irresolvable space \((X, T)\), then \(\text{int} \ (\lambda) \neq 0\), in \((X, T)\).

Proposition 5.6: If \((X, T)\) is a fuzzy hereditarily irresolvable space, then \((X, T)\) is a fuzzy open hereditarily irresolvable space.

**Proof:** Let \(\lambda\) be a fuzzy open set in \((X, T)\). Since \((X, T)\) is a fuzzy hereditarily irresolvable space, the fuzzy open set \(\lambda\) is a fuzzy irresolvable set in \((X, T)\). Hence the fuzzy open set \(\lambda\) is a fuzzy irresolvable set in \((X, T)\), implies that \((X, T)\) is a fuzzy open hereditarily irresolvable space.

Proposition 5.7: If \((X, T)\) is a fuzzy hyperconnected space, then \((X, T)\) is not a fuzzy open hereditarily irresolvable space.

**Proof:** Let \(\lambda\) be a fuzzy open set in the fuzzy hyperconnected space \((X, T)\). Then, by theorem 2.6, \(\lambda\) is a fuzzy irresolvable set in \((X, T)\). Thus the fuzzy open set \(\lambda\) is not a fuzzy irresolvable set in \((X, T)\), implies that \((X, T)\) is not a fuzzy open hereditarily irresolvable space.

Remark 5.1: In view of the above propositions 4.5 and 5.6, one will have the following result:

“Fuzzy hyperconnected spaces are neither fuzzy hereditarily irresolvable spaces and nor fuzzy open hereditarily irresolvable spaces.”

Proposition 5.8: If \(\lambda\) is a fuzzy open set in a fuzzy open hereditarily irresolvable space \((X, T)\), then \(\lambda\) is not a fuzzy dense set in \((X, T)\).

**Proof:** Let \(\lambda\) be a fuzzy open set in \((X, T)\). Since \((X, T)\) is a fuzzy open hereditarily irresolvable space, the fuzzy open set \(\lambda\) is a fuzzy irresolvable set in \((X, T)\) and then by proposition 3.4, \(\lambda\) is not a fuzzy dense set in \((X, T)\).

Proposition 5.9: If \((X, T)\) is a fuzzy open hereditarily irresolvable space, then \((X, T)\) is not a fuzzy hyperconnected space.

**Proof:** Let \(\lambda\) be a fuzzy open set in \((X, T)\). Since \((X, T)\) is a fuzzy open hereditarily irresolvable space, the fuzzy open set \(\lambda\) is a fuzzy irresolvable set in \((X, T)\), and then by proposition 3.4, \(\lambda\) is not a fuzzy dense set in \((X, T)\). Hence \((X, T)\) is not a fuzzy hyperconnected space.

REFERENCES

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