Production inventory system for deteriorating items with trapezoidal type demand

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Abstract For the items like trendy goods, mobile phones and such others, it is examined that the demand rate is of trapezoidal type. The aim of this study is to present optimal production policy for deteriorating items, when the demand of an item is trapezoidal type. Rate of deterioration is assumed to be constant and rate of production depends upon demand rate. Shortages are not allowed. Mathematical formulation is derived in order to minimize the total cost of an inventory system. An easy to use algorithm is presented to decide an optimal production policy.

Key words Inventory system, EPQ model, Trapezoidal type demand, Deterioration, Minimizing Total Cost.

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1 Introduction

In recent competitive market, it is observed that for items like fancy/seasonal/trendy goods, the demand of an item increases with respect to time over a period of time. Thereafter, the item consumed by level demand for a short period, followed by a decrease in demand with time again. Resh et al. [19] and Donaldson [9] made first attempts to integrate the linear trend in demand for an inventory system. For the first time, Hill [15] considered ramp type demand pattern to develop an optimal ordering policy. In case of ramp type demand rate, the rate increases linearly with time and thereafter it stabilizes. Such demand pattern is observed in newly introduced consumable items in the market. Many researchers have studied models with ramp type demand. The inventory models with the ramp type demand are studied by Wu et al. [28, 29], Wu and Ouyang [27], Wu [26], Giri et al. [11], Deng [7], Chen et al. [2], Deng et al. [8], Cheng and Wang [3], He et al. [14], Skouri et al. [23]. Moreover, the goods in the inventory system always retain their physical quality is not true in common. In current business environment, effect of deterioration is expected to be incorporated in an inventory system. The process
which depletes the present value or usefulness of an item and does not allow their original use, due to
degradation, spoilage, evaporation etc. is known as deterioration. First of all Ghare and Schrader [10]
icorporated deterioration in an inventory system. Covert and Philip [5] used Weibull distribution
to extend the idea of Ghare and Schrader [10]. Dave and Patel [6], Goyal [13], Raaef [18], Shah and
Shah [21], Goyal and Giri [12], Manna and Chaudhri [17], Skouri et al. [24], Ruxin et al. [20] and Bakker
et al. [1] cite an up to date review on deteriorating inventory system. Cheng and Wang [3] extended this
idea from ramp type demand to trapezoidal type demand. Cheng et al [4] integrated shortages with
partial backlogging and deterioration in an inventory system to extend the idea of Cheng and Wang [3].
Recent articles on trapezoidal type demand by Shukla and Suthar [22], Wu et. al. [30,31], Vandana
and Shrivastava [25]. Manna et al. [16] present production inventory system for deteriorating items
having ramp type demand. The demand rate increases with time up to a certain point of time and
then stabilizes at constant level. Models were formulated without shortages and with two assumptions
that: a) the demand rate is stabilized after the production stops and before the time when inventory
level reaches to zero and b) the deterioration is constant.

Our present work is an extension of the work of Manna et al. [16] with the assumption that the
production process stops before the inventory level reaches to zero and in one of the three situations:
1) before the demand rate becomes constant, 2) when the demand rate is constant and 3) after the
demand rate is stabilized. Models were formulated without shortages and with two assumptions
that: a) the demand rate is stabilized after the production stops and before the time when inventory
level reaches to zero and b) the deterioration is constant. 

2 Assumptions and notations

To formulate the proposed inventory system mathematically, the following assumptions are made and
the notations to be used by us in this paper are also explained below:

1. The inventory system deals with a single item. Rate of replenishment rate is assumed to be finite
   and lead time is considered to be zero or negligible. The length of planning horizon is infinite.
   Inventory system does not possess shortages.

2. The function \( Q(t) \) represents level of an stock at any instant of time \( t \) during \([0,T]\), where \( T \)
   is length of ordering cycle.

3. The demand is assumed to be trapezoidal type, say \( R(t) \), were \( R(t) = \begin{cases} \frac{a_1 + b_1 t}{a_1 + b_2 \lambda_1} & 0 \leq t \leq \lambda_1 \\ \frac{a_2 - b_2 t}{a_1 + b_2 \lambda_2} & \lambda_1 \leq t \leq \lambda_2 \end{cases} \)
   where \( a_1, a_2, b_1, b_2 \) are scaling parameters for rate of demand. During \([0,\lambda_1]\) demand increases
   with respect to time, then it stabilizes during \([\lambda_1,\lambda_2]\) and thereafter it decreases as \( t \) increases
during \([\lambda_2,T]\).

4. The level of stock deteriorates with a constant rate say \( \theta \ (0 < \theta < 1) \) during the ordering cycle
   \([0,T]\). Again, deteriorated stock is neither repaired nor replaced during \([0,T]\).

5. Inventory system is assumed as production inventory system, where initial stock level is assumed
   to be zero at time \( t = 0 \). Production process starts at \( t = 0 \) and continues up to time \( t = t_1 \). At
   time \( t = t_1 \), stock level attains its maximum say \( S \).

6. The unit production cost of an item is defined as \( C_p = \alpha_1 (R(t))^\gamma \), where \( \gamma > 0 \) and \( \gamma \neq
   2 \) (Manna et. al [16]). \( \alpha_1 > 0 \) as \( C_p > 0 \) and \( R(t) \) is non – negative. As demand increases unit
   production cost will decrease, which validates that \( C_p \) and \( R(t) \) are in inverse proportion.

7. We consider \( C_h \) is the holding cost / unit; \( C_d \) is the cost due to deterioration / unit; \( C(t_1,T) \) is
   an average cost of an inventory system.

3 Mathematical formulation and computational algorithm

As, \( Q(t) \) represents level of stock at any instant of time \( t \) (say) during \([0,T]\). Initially level of stock is
assumed to be zero at \( t = 0 \), at the same time production takes place and continues up to time \( t = t_1 \)
with maximum stock level say \( S \). Hence, during \([0,t_1]\)
\[
\frac{dQ}{dt} + \theta Q(t) = \begin{cases}
(\beta - 1) R(t) & ; \ 0 \leq t \leq t_1 \\
-R(t) & ; \ t_1 \leq t \leq T
\end{cases}
\] (3.1)

Now depending upon the value of \( t_1 \) the following three different cases may arise:

**Case 1:** \( 0 \leq \lambda_1 \leq \lambda_2 \leq t_1 \leq T \), **Case 2:** \( 0 \leq \lambda_1 \leq t_1 \leq \lambda_2 \leq T \) and **Case 3:** \( 0 \leq t_1 \leq \lambda_1 \leq \lambda_2 \leq T \).

Hence, we compute below separately for all the cases for the proposed production inventory system.

**Case 1:** \( 0 \leq \lambda_1 \leq \lambda_2 \leq t_1 \leq T \)

At \( t = 0 \), the production starts with zero stock level and stops at \( t = t_1 \). Using (3.1)

\[
\frac{dQ}{dt} + \theta Q(t) = \begin{cases}
(\beta - 1) (a_1 + b_1 t) & ; \ 0 \leq t \leq \lambda_1 \\
(\beta - 1) (a_1 + b_1 \lambda_1) & ; \ \lambda_1 \leq t \leq \lambda_2 \\
(\beta - 1) (a_2 - b_2 t) & ; \ \lambda_2 \leq t \leq t_1 \\
-(a_2 - b_2 t) & ; \ t_1 \leq t \leq T
\end{cases}
\] (3.2)

Using initial condition \( Q(0) = 0 \), we solve

\[
\frac{dQ}{dt} + \theta Q(t) = (\beta - 1) (a_1 + b_1 t) ; \ 0 \leq t \leq \lambda_1
\] (3.3)

Its solution is

\[
Q(t) = \frac{(\beta - 1) (e^{\theta t} b_1 t + e^{\theta t} a_1 \theta - e^{\theta t} b_1 - a_1 \theta + b_1) e^{-\theta t}}{\theta^2}
\] (3.4)

Using continuity of \( Q(t) \) at \( t = \lambda_1 \) we solve

\[
\frac{dQ}{dt} + \theta Q(t) = (\beta - 1) (a_1 + b_1 \lambda_1) ; \ \lambda_1 \leq t \leq \lambda_2
\] (3.5)

which results in

\[
Q(t) = -\left( e^{-\theta t} b_1 \lambda_1 \theta - \lambda_1 \theta - e^{\theta t} b_1 \lambda_1 \theta + e^{\theta t} b_1 + e^{\theta t} a_1 \theta \right) e^{-\theta t}
\] (3.6)

Similarly, using continuity of \( Q(t) \) at \( t = \lambda_2 \) we solve,

\[
\frac{dQ}{dt} + \theta Q(t) = (\beta - 1) (a_2 - b_2 t) ; \ \lambda_2 \leq t \leq t_1
\] (3.7)

which gives

\[
Q(t) = \frac{1}{\theta^2} \left( e^{-\theta t} b_2 \lambda_2 \theta - \theta t - e^{\theta t} b_2 \lambda_2 \theta + e^{\theta t} b_2 + e^{\theta t} a_2 \theta \right) e^{-\theta t}
\] (3.8)

Moreover, at time \( t = t_1 \), level of stock is maximum, using \( Q(t_1) = S \) we solve,

\[
\frac{dQ}{dt} + \theta Q(t) = -(a_2 - b_2 t) ; \ t_1 \leq t \leq T
\] (3.9)

which yields

\[
Q(t) = \frac{1}{\theta^2} \left( e^{-\theta(t-t_1)} S b_2 t_1 \theta + e^{-\theta(t-t_1)} b_2 t_1 \theta + e^{-\theta(t-t_1)} b_2 \right)
\] (3.10)

Using the boundary condition \( Q(T) = 0 \) in (3.10), we evaluate the maximum level of stock as under:

\[
S = -\frac{1}{\theta^2} \left( b_2 t_1 \theta + a_2 \theta + b_2 T \theta e^{\theta(T-t_1)} + b_2 - (a_2 \theta + b_2) e^{\theta(T-t_1)} \right)
\] (3.11)

By substituting \( S \) in (3.10), we have \( Q(t) \) as under, during \([t_1, T] \)

\[
Q(t) = -\frac{1}{\theta^2} \left( e^{\theta(T-t)} T b_2 \theta - e^{\theta(T-t)} a_2 \theta - b_2 t \theta - e^{\theta(T-t)} b_2 + a_2 \theta + b_2 \right)
\] (3.12)
Using (3.4), (3.6), (3.8) and (3.12), the total inventory during \([0, T]\) is,

\[
TI_1 = \int_0^T Q(t) \, dt = \int_0^{\lambda_1} Q(t) \, dt + \int_{\lambda_1}^{\lambda_2} Q(t) \, dt + \int_{\lambda_2}^{t_1} Q(t) \, dt + \int_{t_1}^T Q(t) \, dt \tag{3.13}
\]

where,

\[
\int_0^{\lambda_1} Q(t) \, dt = \frac{1}{2\theta^3} \left( \left( b_2 \beta \lambda_1^2 \theta^2 + 2a_1 \beta \lambda_1 \theta^2 - b_1 \lambda_1^2 \theta^2 - 2a_1 \theta^2 \lambda_1 
- 2b_1 \beta \lambda_1 \theta - 2a_1 \beta \theta + 2b_1 \lambda_1 \theta + 2a_1 \theta + 2b_1 \beta - 2b_1 \right) 
+ e^{-\theta \lambda_1} (2a_1 \beta \theta - 2a_1 \theta - 2b_1 \lambda_1 + 2b_1) \right)
\]

\[
\int_{\lambda_1}^{\lambda_2} Q(t) \, dt = 
\frac{1}{2\theta^3} \left( b_2 \beta \lambda_1^2 \theta^2 - b_1 \beta \lambda_1 \theta^2 + a_1 \beta \lambda_1 \theta^2 - a_1 \beta \lambda_2 \theta^2 - b_1 \lambda_1^2 \theta^2 - b_1 \lambda_1 \lambda_2 \theta^2 
- a_1 \lambda_1 \theta^2 + a_1 \lambda_2 \theta^2 + b_1 \beta + a_1 \beta \theta e^{-\theta \lambda_1} - b_1 \beta e^{-\theta \lambda_1 - \lambda_2} - e^{-\theta \lambda_1 \lambda_2} - b_1 \right)
\]

\[
\int_{t_1}^{t_2} Q(t) \, dt = 
\frac{1}{2\theta^3} \left( -2b_2 + 2a_2 \beta \lambda_2 \theta^2 - 2a_2 \beta \lambda_1 \theta^2 - 2b_2 \beta \lambda_1 \theta - b_2 \beta \lambda_2 \theta^2 + 2b_2 \beta \lambda_1 \theta^2 + 2b_2 \beta \lambda_1 \theta + 2a_2 \beta \theta + 2a_2 \beta \theta + 2b_1 \lambda_1 \theta + 2a_2 \lambda_2 \theta^2 \right)
\]

\[
\frac{1}{2\theta^3} \int_{t_1}^{t_2} e^{-\theta (\lambda_2 + t_1)} 
\left( 2e^{\theta t_1} b_2 - 2e^{\theta (\lambda_2 + t_1)} b_1 + 2e^{\theta \lambda_2} a_1 \theta - 2e^{\theta t_1} b_1 \beta + 2e^{\theta (\lambda_1 + \lambda_2)} b_1 - 2e^{\theta \lambda_2} a_1 \beta \theta 
- 2e^{\theta t_1} b_2 a_1 \theta + 2b_2 a_1 \beta \theta - 2e^{2\theta \lambda_2} a_1 \beta \theta + 2b_2 \lambda_2 \beta \theta - 2e^{2\theta \lambda_2} a_1 \beta \theta - 2e^{2\theta \lambda_2} b_2 \beta \theta 
- 2e^{2\theta \lambda_2} b_2 a_1 \beta + 2b_2 a_1 \beta \theta - 2e^{2\theta \lambda_2} b_2 \beta \theta - 2e^{2\theta \lambda_2} b_2 \beta \theta - 2e^{2\theta \lambda_2} b_1 \beta 
\right)
\]

\[
\frac{1}{2\theta^3} \int_{t_1}^T Q(t) \, dt = \frac{1}{2\theta^3} \left( T^2 b_2 \theta^2 - 2T a_2 \theta^2 + 2a_2 \theta^2 t_1 + 2b_2 t_1 \theta - 2e^{\theta (T - t_1)} T b_2 \theta 
- 2a_2 \theta^2 + 2e^{\theta (T - t_1)} a_1 \theta - 2b_2 + 2e^{\theta (T - t_1)} b_2 \right)
\]

Now, total number of deteriorated units during \([0, T]\) is given by,

\[
DH_1 = \text{Production in } [0, \lambda_1] + \text{Production in } [\lambda_1, \lambda_2] + \text{Production in } [\lambda_2, t_1] - \text{Demand in } [0, \lambda_1] - \text{Demand in } [\lambda_1, \lambda_2] - \text{Demand in } [\lambda_2, T]
\]

\[
= \beta \int_0^{\lambda_1} (a_1 + b_t) \, dt + \beta \int_{\lambda_1}^{\lambda_2} (a_1 + b_t) \, dt + \beta \int_{\lambda_2}^{t_1} (a_2 - b_2 t) \, dt 
- \int_0^{\lambda_1} (a_1 + b_t) \, dt - \int_{\lambda_1}^{\lambda_2} (a_1 + b_t) \, dt - \int_{t_1}^{T} (a_2 - b_2) \, dt 
+ \frac{1}{2} \beta b_1 \lambda_1^2 + \beta b_1 \lambda_1 \lambda_2 + \beta a_1 \lambda_2 + \frac{1}{2} \beta b_2 \lambda_2^2 - 2 \frac{1}{2} \beta b_2 \lambda_2^2 - \beta a_2 \lambda_2 
+ \beta a_2 \lambda_2 + \frac{1}{2} b_1 \lambda_2^2 - a_1 \lambda_2 + \frac{1}{2} T^2 b_2 - 2 b_2 \lambda_2^2 - a_2 T + a_2 \lambda_2 \tag{3.14}
\]

Cost of production during \([0, t_1]\) is given by,

\[
PC_1 = \int_0^{t_1} \beta \alpha_1 (R(t))^{(1-\gamma)} \, du
\]

\[
= \int_0^{\lambda_1} \beta \alpha_1 (a_1 + b_2 u)^{(1-\gamma)} \, du + \int_{\lambda_1}^{\lambda_2} \beta \alpha_1 (a_1 + b_2 u)^{(1-\gamma)} \, du + \int_{t_1}^{t_2} \beta \alpha_1 (a_2 - b_2 u)^{(1-\gamma)} \, du
\]

\[
= \beta a_1 \left( \frac{-a_1^\gamma + (b_1 \lambda_1 + a_1)^\gamma}{b_1 \gamma} + (b_1 \lambda_1 + a_1)^\gamma (\lambda_2 - \lambda_1) + \frac{(b_2 \lambda_2 + a_2)^\gamma - (a_2 - T b_2)^\gamma}{b_2 \gamma} \right) \tag{3.15}
\]

Case 2: \( \lambda_1 \leq t_1 \leq \lambda_2 \leq T \)
We solve differential equation given below, using initial condition which yields

\[ S \]

which gives

\[ \text{Q} \]

whose solution is

\[ \text{Q} \]

Using continuity of \( Q(t) \) at \( t = \lambda_1 \) to solve the differential equation,

\[ \frac{dQ}{dt} + \theta Q(t) = (\beta - 1) (a_1 + b_1 t) ; \quad 0 \leq t \leq \lambda_1 \]

(3.17)

We use continuity of \( Q(t) \) at \( t = \lambda_1 \) to solve the differential equation,

\[ \frac{dQ}{dt} + \theta Q(t) = (\beta - 1) (a_1 + b_1 \lambda_1) ; \quad \lambda_1 \leq t \leq \lambda_2 \]

(3.19)

which gives

\[ Q(t) = \frac{1}{\theta^2} \left( b_1 \beta \lambda_1 t - a_1 \beta \lambda_1 + b_1 \lambda_1 \theta + b_1 \beta e^{(\lambda_1 - t)} + a_1 \theta + a_1 \beta e^{-\theta t} \right) \]

(3.20)

Now at \( t = \lambda_1 \) the production process stops and the maximum level of stock is \( S \), i.e. \( Q(t_1) = S \). We solve,

\[ \frac{dQ}{dt} + \theta Q(t) = -(a_1 + b_1 \lambda_1) ; \quad t_1 \leq t \leq \lambda_2 \]

(3.21)

for which we have

\[ Q(t) = \frac{1}{\theta} \left( e^{-\theta(t-t_1)}S \theta + e^{-\theta(t-t_1)}b_1 \lambda_1 + e^{-\theta(t-t_1)}a_1 - b_1 \lambda_1 - a_1 \right) \]

(3.22)

Using continuity of \( Q(t) \) at \( t = \lambda_2 \), we solve the differential equation,

\[ \frac{dQ}{dt} + \theta Q(t) = -(a_2 - b_2) ; \quad \lambda_2 \leq t \leq T \]

(3.23)

which yields

\[ Q(t) = -\frac{1}{\theta^2} \left( e^{\theta(T-\lambda_1)} b_1 \lambda_1 \theta + e^{\theta(T-\lambda_2)} b_2 \lambda_2 \theta - e^{\theta(T-\lambda_1)} S \theta^2 - e^{\theta(T-\lambda_1)} b_1 \lambda_1 \theta \right) \]

(3.24)

Using the boundary condition \( Q(T) = 0 \) in (3.24), we evaluate the maximum level of stock as under:

\[ S = \frac{1}{\theta^2 e^{\theta(T-\lambda_1)}} \left( b_1 \lambda_1 \theta e^{-\theta(T-\lambda_2)} + b_2 \lambda_2 \theta e^{-\theta(T-\lambda_2)} - b_1 \lambda_1 \theta e^{-\theta(T-\lambda_1)} + a_1 \theta e^{-\theta(T-\lambda_2)} \right) \]

(3.25)

By substituting \( S \) in (3.22) and (3.24), we have \( Q(t) \) as under,

\[ Q(t) = -\frac{1}{\theta^2} \left( e^{\theta(T-t)} b_2 \theta - e^{\theta(T-\lambda_2)} b_1 \lambda_1 \theta + e^{\theta(T-\lambda_2)} b_2 \lambda_2 \theta - e^{\theta(T-\lambda_2)} a_2 \theta \right) \]

(3.26)

\[ Q(t) = -\frac{1}{\theta^2} \left( e^{\theta(T-t)} b_2 \theta - e^{\theta(T-t)} a_2 \theta - b_2 \theta e^{-\theta(T-\lambda_2)} + a_2 \theta + b_2 \right) \]

(3.27)

Using (3.18), (3.20), (3.26) and (3.27),

\[ T I_2 = \int_0^T Q(t) \, dt = \int_0^{\lambda_1} Q(t) \, dt + \int_{\lambda_1}^{\lambda_2} Q(t) \, dt + \int_{\lambda_2}^{T} Q(t) \, dt \]

(3.28)
where,

\[
\int_{0}^{\lambda_1} Q(t) \, dt = \frac{1}{\theta^2} \left( b_1 \beta \lambda_1^2 \theta^2 - 2a_1 \beta \lambda_1^2 \theta^2 - 2a_1 \lambda_1^2 \theta^2 - 2b_1 \beta \lambda_1 \theta - 2a_1 \beta \lambda_1 \theta + 2b_1 \beta + 2a_1 \beta e^{-\theta \lambda_1} - 2b_1 - 2a_1 \beta e^{-\theta \lambda_1} + 2b_1 e^{-\theta \lambda_1} \right)
\]

\[
\int_{\lambda_1}^{\lambda_2} Q(t) \, dt = \frac{-1}{\theta^2} \left( b_1 \lambda_1 \lambda_2 \theta^2 - b_1 \lambda_1 \theta^2 + a_1 \lambda_2 \theta^2 - a_1 \lambda_1 \theta^2 + b_1 \lambda_1 \theta + 2b_2 \lambda_2 \theta - b_1 \lambda_1 \theta e^{(\lambda_2 - \lambda_1)} - b_2 \lambda_2 \theta e^{(\lambda_2 - \lambda_1)} + T_2 \lambda_2 \theta e^{(T - \lambda_2)} + a_1 \theta - a_2 \theta - a_1 \theta e^{(\lambda_2 - \lambda_1)} + a_2 \theta e^{(\lambda_2 - \lambda_1)} - b_2 + b_2 e^{(\lambda_2 - \lambda_1)} \right)
\]

\[
\int_{\lambda_2}^{T} Q(t) \, dt = \frac{-1}{\theta^2} \left( -T^2 b_2 \theta^2 + b_2 \lambda_2^2 + 2T a_2 \theta + 2T a_2 \theta e^{(T - \lambda_2)} + 2a_2 \theta - b_2 e^{(T - \lambda_2)} + 2b_2 \theta e^{(T - \lambda_2)} - b_2 e^{(T - \lambda_2)} + b_2 e^{(T - \lambda_2)} \right)
\]

Now, total number of deteriorated units during \([0, T]\) is given by,

\[ DI_2 = \text{Production in } [0, \lambda_1] + \text{Production in } [\lambda_1, t_1] - \text{Demand in } [0, \lambda_1] - \text{Demand in } [\lambda_1, \lambda_2] - \text{Demand in } [\lambda_2, T] \]

\[
= \beta \int_0^{\lambda_1} (a_1 + b_1 t) \, dt + \beta \int_{\lambda_1}^{t_1} (a_1 + b_1 \lambda_1) \, dt - \int_{\lambda_1}^{\lambda_2} (a_1 + b_1 \lambda_1) \, dt - \int_{\lambda_2}^{T} (a_2 - b_2 t) \, dt
\]

\[
= -\frac{1}{2} b_1 \beta \lambda_1^2 + b_1 \beta \lambda_1 t_1 + b_\beta t_1 a_1 + \lambda_2 \lambda_1 - b_1 \lambda_1 \lambda_2 - \lambda_2 a_1 + \frac{1}{2} T^2 b_2 - \frac{1}{2} b_2 \lambda_2^2 - a_2 T + a_2 \lambda_2 \quad (3.29)
\]

Cost of production during \([0, t_1]\) is given by,

\[
PC_2 = \int_0^{t_1} \beta \alpha_1 (R(t))^{(1-\gamma)} \, du
\]

\[
= \int_0^{\lambda_1} \beta \alpha_1 (a_1 + b_1 u)^{(1-\gamma)} \, du + \int_{\lambda_1}^{t_1} \beta \alpha_1 (a_1 + b_1 \lambda_1)^{(1-\gamma)} \, du
\]

\[
= \beta \alpha_1 \left( -a_1^\gamma + (b_1 \lambda_1 + a_1)^\gamma + (b_1 \lambda_1 + a_1)^\gamma (t_1 - \lambda_1) \right) \quad (3.30)
\]

Case 3: \( 0 \leq t_1 \leq \lambda_1 \leq \lambda_2 \leq T \)

For this case, using (3.1),

\[
\frac{dQ}{dt} + \theta Q(t) = \begin{cases} 
(\beta - 1) (a_1 + b_1 t) & ; \quad 0 \leq t \leq t_1 \\
- (a_1 + b_1 t) & ; \quad t_1 < t \leq \lambda_1 \\
- (a_1 + b_1 \lambda_1) & ; \quad \lambda_1 < t \leq \lambda_2 \\
- (a_2 - b_2 t) & ; \quad \lambda_2 \leq t \leq T 
\end{cases}
\]

To solve (3.31), we use the initial condition \(Q(0) = 0\),

\[
\frac{dQ}{dt} + \theta Q(t) = (\beta - 1) (a_1 + b_1 t) \quad ; \quad 0 \leq t \leq t_1
\]

The solution is

\[
Q(t) = \frac{1}{\theta^2} (\beta - 1) \left( b_1 t \theta + a_1 \theta - b_1 - (a_1 \theta - b_1) e^{-\theta t} \right) \quad (3.33)
\]

At time \( t = t_1 \) the production stops and the level of stock is at its maximum value \( S \), i.e. \( Q(t_1) = S \) and we solve,

\[
\frac{dQ}{dt} + \theta Q(t) = - (a_1 + b_1 t) ; \quad t_1 \leq t \leq \lambda_1
\]

which results in

\[
Q(t) = \frac{1}{\theta^2} \left( e^{-\theta (t-t_1)} S \theta^2 + e^{-\theta (t-t_1)} b_1 t \theta + a_1 \theta e^{-\theta (t-t_1)} - b_1 t \theta \right) \quad (3.35)
\]
Now, using continuity of the function \( Q(t) \) at \( t = \lambda_1 \) we solve,

\[
\frac{dQ}{dt} + \theta Q(t) = -(a_1 + b_1 \lambda_1) \quad \lambda_1 \leq t \leq \lambda_2
\]

whose solution is

\[
Q(t) = \frac{1}{\theta^2} \left( e^{-\theta(t-t_1)} S b_1 + b_1 t_1 \theta - e^{-\theta(t-t_1)} - a_1 \theta e^{-\theta(t-t_1)} - b_1 e^{-\theta(t-t_1)} \right)
\]

(3.36)

Again, using the continuity of \( Q(t) \) at \( t = \lambda_2 \), we solve the differential equation,

\[
\frac{dQ}{dt} + \theta Q(t) = -(a_2 - b_2 t) \quad \lambda_2 \leq t \leq T
\]

(3.38)

which gives

\[
Q(t) = \frac{1}{\theta^2} \left( e^{\theta(\lambda_2-t_1)} b_1 \lambda_1 \theta + e^{\theta(\lambda_2-t_1)} b_2 \lambda_2 \theta - e^{\theta(t_1-t_2)} S b_1 t_1 \theta + e^{\theta(t_1-t_2)} - a_2 \theta e^{\theta(t_1-t_2)} - b_2 e^{\theta(t_1-t_2)} \right)
\]

(3.39)

As similar to other cases, we use the boundary condition \( Q(T) = 0 \) in (3.39), the maximum level of stock is as below:

\[
S = -\frac{1}{\theta^2} e^{-\theta(T-t_1)} \left( -b_1 \lambda_1 \theta + e^{-\theta(T-t_2)} S b_1 t_1 \theta + e^{-\theta(T-t_2)} - a_2 \theta e^{-\theta(T-t_2)} - b_2 e^{-\theta(T-t_2)} \right)
\]

(3.40)

By substituting \( S \) in (3.35), (3.37) and (3.39), we have \( Q(t) \) as under,

\[
Q(t) = \frac{1}{\theta^2} \left( b_1 \lambda_1 \theta + e^{\theta(\lambda_2-t_1)} b_2 \lambda_2 \theta - e^{\theta(t_1-t_2)} S b_1 t_1 \theta - e^{\theta(t_1-t_2)} - a_2 \theta e^{\theta(t_1-t_2)} + b_2 e^{\theta(t_1-t_2)} \right)
\]

(3.41)

\[
Q(t) = \frac{1}{\theta^2} \left( b_1 \lambda_1 \theta + e^{\theta(\lambda_2-t_1)} b_2 \lambda_2 \theta - e^{\theta(t_1-t_2)} S b_1 t_1 \theta - e^{\theta(t_1-t_2)} - a_2 \theta e^{\theta(t_1-t_2)} + b_2 e^{\theta(t_1-t_2)} \right)
\]

(3.42)

\[
Q(t) = -\frac{1}{\theta^2} \left( T b_2 e^{\theta(T-t_1)} - b_2 e^{\theta(T-t_2)} \right)
\]

(3.43)

Using (3.33), (3.41), (3.42) and (3.43),

Total inventory during \([0, T] \) is

\[
TI_3 = \int_0^T Q(t) dt = \int_0^1 Q(t) dt + \int_{t_1}^{\lambda_1} Q(t) dt + \int_{\lambda_1}^{\lambda_2} Q(t) dt + \int_{\lambda_2}^T Q(t) dt
\]

(3.44)

where,

\[
\int_0^1 Q(t) dt = \frac{1}{\theta^7} \left( b_1 \lambda_1 \theta^2 - b_1 \lambda_1 \theta^2 + a_2 \lambda_1 \theta^2 - b_1 \lambda_1 \theta^2 - b_2 \lambda_1 \theta^2 - b_2 \lambda_1 \theta^2 - b_2 \lambda_1 \theta^2 + b_2 \lambda_1 \theta^2 + b_2 \lambda_1 \theta^2 \right)
\]

\[
\int_{t_1}^{\lambda_1} Q(t) dt = -\frac{1}{\theta^3} \left( b_1 \lambda_1 \theta^2 - b_1 \lambda_1 \theta^2 + a_2 \lambda_1 \theta^2 - b_1 \lambda_1 \theta^2 - b_2 \lambda_1 \theta^2 - b_2 \lambda_1 \theta^2 + b_2 \lambda_1 \theta^2 + b_2 \lambda_1 \theta^2 \right)
\]

\[
\int_{\lambda_1}^{\lambda_2} Q(t) dt = \frac{1}{\theta^3} \left( b_1 \lambda_1 \theta^2 - b_1 \lambda_1 \theta^2 + a_2 \lambda_1 \theta^2 - b_1 \lambda_1 \theta^2 - b_2 \lambda_1 \theta^2 + b_2 \lambda_1 \theta^2 + b_2 \lambda_1 \theta^2 + b_2 \lambda_1 \theta^2 \right)
\]

\[
\int_{\lambda_2}^T Q(t) dt = \frac{1}{\theta^3} \left( b_1 \lambda_1 \theta^2 - b_1 \lambda_1 \theta^2 + a_2 \lambda_1 \theta^2 - b_1 \lambda_1 \theta^2 - b_2 \lambda_1 \theta^2 + b_2 \lambda_1 \theta^2 + b_2 \lambda_1 \theta^2 + b_2 \lambda_1 \theta^2 \right)
\]
To find the optimal values of \( t_1 \) and \( T \), which minimizes total cost of an inventory system \( C(t_1, T) \), we use fundamental calculus. Solve \( \frac{\partial C_i}{\partial t_i} = 0 \) and \( \frac{\partial C_i}{\partial T} = 0 \) simultaneously to find \( t_1 \) and \( T \), provided that they satisfy the sufficient conditions

\[
\frac{\partial^2 C_i}{\partial t_i^2} > 0, \quad \frac{\partial^2 C_i}{\partial T^2} > 0 \quad \text{and} \quad \frac{\partial^2 C_i}{\partial t_i \partial T} \bigg|_{t_i = t_1} > 0,
\]

where \( i = 1, 2, 3 \).

### Computational Algorithm

**Step 1:** Assign values to the parameters.

**Step 2:** Compute \( t_1 \) and \( T \) using \( \frac{\partial C_i}{\partial t_i} = 0 \) and \( \frac{\partial C_i}{\partial T} = 0 \) for \( i = 1, 2, 3 \).

**Step 3:** If \( 0 \leq \lambda_1 \leq \lambda_2 \leq t_1 \leq T \) then \( C_1(t_1, T) \) is optimal, compute \( C_1(t_1, T) \) and \( S \) using (3.11), else go to Step 4.

**Step 4:** If \( 0 \leq \lambda_1 \leq t_1 \leq \lambda_2 \leq T \) then \( C_2(t_1, T) \) is optimal, compute \( C_2(t_1, T) \) and \( S \) using (3.25), else \( C_3(t_1, T) \) is optimal, compute \( C_3(t_1, T) \) and \( S \) using (3.40).

### 4 Concluding remarks

To implement the computational algorithm, one needs to evaluate optimal values of \( t_1 \) and \( T \) by solving the equations \( \frac{\partial C_i}{\partial t_i} = 0 \) and \( \frac{\partial C_i}{\partial T} = 0 \) case wise for \( i = 1, 2, 3 \). As equations \( \frac{\partial^2 C_i}{\partial t_i^2} = 0 \) and \( \frac{\partial^2 C_i}{\partial T^2} = 0 \) are highly nonlinear functions of \( t_1 \) and \( T \), one may use Newton Raphson method for suitable values of \( a_1, b_1, a_2, b_2, \lambda_1, \lambda_2, \theta, \gamma \) and \( \alpha_1 \), to find optimal values of \( t_1^* \) and \( T^* \). So, the corresponding minimum value of the average cost of an inventory system \( C(t_1^*, T^*) \) is minimum. Convexity of cost function is difficult to discuss analytically. This says \( C(t_1^*, T^*) \) may not be global minimum but may be local minimum. One may extend this model, using advance optimization techniques like, PSO, GA etc. to find the minimum cost of an inventory system.

### References


