Odd graceful labeling in cycle with extended bistar

J. Jeba Jesintha, R. Jaya Glory and B. Sandiya

1,3. P.G. Department of Mathematics, Women’s Christian College, Affiliated to University of Madras, Chennai-600008, Tamil Nadu, India.

2. Research Scholar (Part-Time), P.G. Department of Mathematics, Women’s Christian College, Affiliated to University of Madras, Chennai-600008, Tamil Nadu, India.

1. E-mail: jjesintha75@yahoo.com, 2. E-mail: godblessglory@gmail.com

3. E-mail: sandiya2607@gmail.com

Abstract

Graph labeling creates a new direction towards research areas in graph theory and has various applications in coding approach, communication networks and many more. In 1991, Gnanajothi (Topics in Graph Theory, Ph.D. Thesis, Madurai Kamraj University, Tamilnadu, India) introduced a labeling method called odd graceful labeling. A graph $G$ with $q$ edges is odd graceful if there is an injection, $f : V(G) \rightarrow \{0, 1, 2, \ldots, (2q-1)\}$ such that when each $xy$ edge is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, (2q-1)\}$. In this paper, we prove the odd graceful labeling in a cycle with extended bistar.

Key words
Odd Graceful labeling, Extended bistar, Cycle.

2010 Mathematics Subject Classification
05C78.

1 Introduction

The course on graph labeling embarks with the initiation to $\beta$-valuation by Rosa [7] in 1967. Golomb [4] called this $\beta$-valuation as graceful labeling in 1972. A graph $G$ is said to admit a graceful labeling if there exists an injection from its $q$ edges to its vertex set $V$ given by $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ with the property that the resulting edge labels are also unique, where an edge incident with vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. In 1991, Gnanajothi [3] introduced a labeling method called the odd graceful labeling. A graph $G$ with $q$ edges is odd graceful if there is an injection, $f : V(G) \rightarrow \{0, 1, 2, \ldots, 2(q-1)\}$ such that when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, (2q-1)\}$. Gnanajothi [3] confirmed that the class of odd graceful graphs lies between the class of graphs with $\alpha$-labelings and the class of bipartite graphs. In [3], it is shown that the following graphs are odd graceful: the path $P_n$, the cycle $C_n$ if and only if $n$ is even, the comb $P_n \circ K_1$ (graphs obtained by linking a single pendant edge to each vertex of $P_n$), books,
crows $C_n \odot K_1$ (graphs obtained by connecting a single pendant edge to each vertex of $C_n$) if and only if $n$ is even, the disjoint union of copies of $C_4$, the one-point union of copies of $C_4$, caterpillars, rooted trees of height 2, the graphs obtained from $P_n (n \geq 3)$ by adding exactly two leaves at each vertex of degree 2 of $P_n$. Ibrahim Moussa [5] showed that the graph $C_m \sqcup P_n$ is odd graceful if $m$ is even. Eldergill [1] proved that the one-point union of any number of copies of $C_6$ is odd graceful. Sekar [8] showed that the splitting graph of $P_n$, the splitting graph of $C_n$ when $n$ is even, lobsters, banana trees and regular bamboo trees are odd graceful. For an overall embracing survey on graph theory we refer to the dynamic survey by Gallian [2].

Graph labeling creates huge applications in graph theory and has rigorous requisitions in coding theory, transmission networks, optimal circuits layouts and graph disintegration problems. The main objective of this paper is to prove that the extended bistar attachment with cycle is odd graceful.

**Definition 1.1.** [6] A Extended bistar $(K_{1,k} : n)$, as shown in Fig.1 below is the graph obtained by attaching a path of length $n$ with the center vertices of two copies of the star $K_{1,k}$ whose vertices are denoted by $r_1, r_2, \ldots, r_n, u_1, u_2, \ldots, u_k, s_1, s_2, \ldots, s_k$ and the edges by $r_1u_1, \ldots, r_nu_k, s_1 (1 \leq i \leq k)$ and $r_1r_j (1 \leq j \leq k)$.

![Fig. 1: The Extended bistar.](image)

**2 Main result**

We now prove the main result of this paper in the form of the Theorem 2.1 as follows:

**Theorem 2.1.** The graph $G$ obtained by attaching each vertex of a cycle $C_m$ with the isomorphic extended bistar of $(K_{1,k} : n)$ is odd graceful, where $m$ is even, $n$ is odd and when $m \equiv 0 \pmod{4}$.

**Proof.** Let $G$ be the graph obtained by attaching $m$ isomorphic copies of Extended bistar of $(K_{1,k} : n)$ graph at every vertex of cycle $C_m$ where $m \equiv 0 \pmod{4}$.

We narrate the graph $G$ as follows: the vertices in the cycle in $G$ are expressed as $C_1, C_2, C_3, \ldots, C_m$ in the clockwise direction. Extended bistar $E$ is described as follows: consider $m$ mirror images of isomorphic extended bistar of $(K_{1,k} : n)$. Denote the $m$ copies of $E$ as $E_1, E_2, \ldots, E_m$. The first internal vertices of the first copy of the extended bistar $E_1$ at the vertex $C_1$ of cycle are denoted by $u_1^1, u_2^1, u_3^1, \ldots, u_{n-1/2}^1$ and the second internal vertices of the first copy of the Extended bistar $E_1$ at the vertex $C_1$ of cycle are denoted as $v_1^1, v_2^1, v_3^1, \ldots, v_{n-1/2}^1$. The pendant vertices attached with $u_1^1$ are denoted as $x_1^1, x_2^1, x_3^1, \ldots, x_r^1$ and the pendant vertices attached with $v_1^1$ are denoted as $y_1^1, y_2^1, y_3^1, \ldots, y_r^1$.

The first internal vertices of the second copy of the Extended bistar $E^2$ at vertex $C_2$ of cycle are denoted as $u_1^2, u_2^2, u_3^2, \ldots, u_{n-1/2}^2$ and the second internal vertices of the second copy of the Extended bistar $E^2$ at vertex $C_2$ of cycle are denoted as $v_1^2, v_2^2, v_3^2, \ldots, v_{n-1/2}^2$. The pendant vertices attached with $u_1^2$ are denoted as $x_1^2, x_2^2, x_3^2, \ldots, x_r^2$ and the pendant vertices attached with $v_1^2$ are denoted as $y_1^2, y_2^2, y_3^2, \ldots, y_r^2$.

Correspondingly, the first internal vertices of the $m^{th}$ copy of the extended bistar $E^m$ at vertex $C_m$ of cycle are denoted as $u_1^m, u_2^m, u_3^m, \ldots, u_{n-1/2}^m$ and the second internal vertices of the $m^{th}$ copy of the extended bistar $E^m$ at vertex $C_m$ of cycle are denoted as $v_1^m, v_2^m, v_3^m, \ldots, v_{n-1/2}^m$. The pendant
vertices attached with \( u^1 \) are denoted as \( x^1_1, x^2_1, x^3_1, \ldots, x^m_1 \) and the pendant vertices attached with \( v^1 \) are denoted as \( y^1_1, y^2_1, y^3_1, \ldots, y^m_1 \).

Now we attach the \( m \) copies of the Extended bistar \( E^m \), namely as \( E^1, E^2, E^3, \ldots, E^m \) at the \( m \) vertices of the cycle \( C_m \) in such a way that the first internal vertices \( u^1_1, u^1_2, \ldots, u^1_{m-1/2} \) and the second internal vertices \( v^1_1, v^1_2, \ldots, v^1_{m-1/2} \) of the first copy of the Extended bistar \( E^1 \) are attached with the vertex \( C_1 \) which is attached with the cycle, the first internal vertices \( u^2_1, u^2_2, u^2_3, \ldots, u^2_{m-1/2} \) and the second internal vertices \( v^2_1, v^2_2, v^2_3, \ldots, v^2_{m-1/2} \) of the second copy of the Extended bistar \( E^2 \) are attached with the vertex \( C_2 \) which is attached with the cycle. In general, the first internal vertices \( u^m_1, u^m_2, u^m_3, \ldots, u^m_{m-1/2} \) and the second internal vertices \( v^m_1, v^m_2, v^m_3, \ldots, v^m_{m-1/2} \) of the \( m \)th copy of the Extended bistar \( E^m \) are attached with the vertices of the cycle \( C_m \) for which \( m \equiv (\text{mod}4) \). Thus we obtain the new graph \( G \) exhibited in Fig. 2.

Fig. 2: The graph \( G \) obtained by joining each vertex of \( C_m \) with the isomorphic copies of extended bistar of \( (K_{1,k} : n) \) graph.

The vertices of \( G \) are classified based on the parameters \( m \) as follows:

The number of pendant edges in the extended bistar are denoted by “\( r \)” while, the number of internal vertices in one copy of coconut tree are denoted as “\( n \)”.

**Case 1: When \( n \) is odd**

The vertex labels for the first internal vertices of the extended bistar \( (K_{1,k} : n) \) where \( n = 5, 9, 13, 17, \ldots \) are given below:

\[
f(u_{2i-1}^{2k}) = 2i - 2 + 2(k - 1)(n + 2r), \quad 1 \leq i \leq \frac{n-1}{4} \quad \text{and} \quad 1 \leq k \leq \frac{m}{2}.
\]

\[
f(u_{2i}^{2k-1}) = \begin{cases} 2q - 2r + 2i - 2(k - 1)(n + 2r) - 3, & 1 \leq i \leq \left(\frac{n-1}{4}\right), \quad 1 \leq k \leq \left(\frac{m}{2}\right) \\ 2q - 2r + 2i - 2(k - 1)(n + 2r) - 1, & 1 \leq i \leq \left(\frac{n-1}{4}\right), \quad \left(\frac{m}{2}\right) + 1 \leq k \leq \left(\frac{m}{2}\right) \end{cases}
\]

\[
f(u_{2i}^{2k}) = \begin{cases} 2q - 1 - 2(r + n) - (2i - 2) + 2(k - 1)(n + r), & 1 \leq i \leq \left(\frac{n-1}{4}\right), \quad 1 \leq k \leq \left(\frac{m}{2}\right) \\ 2q + 1 - 2(r + n) - (2i - 2) + 2(k - 1)(n + r), & 1 \leq i \leq \left(\frac{n-1}{4}\right), \quad \left(\frac{m}{2}\right) + 1 \leq k \leq \left(\frac{m}{2}\right) \end{cases}
\]

\[
f(u_{2i+1}^{2k}) = (2n + 2) + (2i - 2) + 2(k - 1)(n + 2r), \quad 1 \leq i \leq \left(\frac{n-1}{4}\right), \quad 1 \leq k \leq \left(\frac{m}{2}\right).
\]

The vertex labels for the second internal vertices of extended bistar \( (K_{1,k} : n) \) are given below;
\[ f \left( v_{2i-1}^{k-1} \right) = n - 1 + (2i - 2) + 2(k - 1)(n + 2r), \quad 1 \leq i \leq \frac{n - 1}{4}, \quad 1 \leq k \leq \frac{m}{2} \]

\[ f \left( v_{2i}^{k-1} \right) = \begin{cases} 2q - 1 - 2(n + 2r - 3) + (2i - 2) + 2(k - 1)(n + 2r), & 1 \leq i \leq \frac{(n-1)}{4}, \quad 1 \leq k \leq \frac{(m)}{2} \\ 2q + 1 - 2(n + 2r - 3) + (2i - 2) + 2(k - 1)(n + 2r), & 1 \leq i \leq \frac{(n-1)}{4}, \quad \frac{(m)}{2} + 1 \leq k \leq \frac{(m)}{2} \end{cases} \]

\[ f \left( v_{2i-1}^{2k} \right) = \begin{cases} 2q - 1 - 2(2r - n - 1) - (2i - 2) - 2(k - 1)(n + 2r), & 1 \leq i \leq \frac{(n-1)}{4}, \quad 1 \leq k \leq \frac{(m)}{2} \\ 2q + 1 - 2(2r - n - 1) - (2i - 2) - 2(k - 1)(n + 2r), & 1 \leq i \leq \frac{(n-1)}{4}, \quad \frac{(m)}{2} + 1 \leq k \leq \frac{(m)}{2} \end{cases} \]

\[ f \left( v_{2i}^{2k} \right) = 2(n + r) - 2 + (2i - 2) + 2(k - 1)(n + 2r), \quad 1 \leq i \leq \frac{(n-1)}{4}, \quad 1 \leq k \leq \frac{m}{2}. \]

Now, vertex labels for the cycle \( C_m \) are given below,

\[ f \left( C_{2j-1} \right) = \left( \frac{n - 1}{2} \right) + 2(k - 1)(n + 2r), \quad 1 \leq i \leq \left( \frac{n - 1}{4} \right), \quad 1 \leq k \leq \left( \frac{m}{2} \right) \]

\[ f \left( C_{2j} \right) = \begin{cases} 2q - 1 - 3 \left( \frac{n + 1}{2} \right) + 4r - 2(k - 1)(n + 2r), & 1 \leq i \leq \left( \frac{n - 1}{4} \right), \quad 1 \leq k \leq \left( \frac{m}{2} \right) \\ 2q - 1 - 3 \left( \frac{n + 1}{2} \right) + 4r - 2(k - 1)(n + 2r), & 1 \leq i \leq \left( \frac{n - 1}{4} \right), \quad \frac{(m)}{2} + 1 \leq k \leq \frac{(m)}{2} \end{cases} \]

The vertex labels for the pendant vertex at the first internal vertices of the extended bistar \( (K_{1,k} : n) \) now follow:

\[ f \left( x_i^{2k-1} \right) = \begin{cases} 2q - 1 - (2i - 2) - 2(k - 1)(n + 2r), & 1 \leq i \leq r, \quad 1 \leq k \leq \left( \frac{(n)}{4} \right) \\ 2q + 1 - (2i - 2) - 2(k - 1)(n + 2r), & 1 \leq i \leq r, \quad \frac{(n)}{4} + 1 \leq k \leq \frac{(m)}{2} \end{cases} \]

\[ f \left( x_i^{2k} \right) = 2n + (2i - 2) + 2(k - 1)(n + r), \quad 1 \leq i \leq r, \quad 1 \leq k \leq \left( \frac{m}{2} \right). \]

Further the vertex labels for the pendant vertex at the second internal vertices of the extended bistar \( (K_{1,k} : n) \) are:

\[ f \left( y_i^{2k-1} \right) = \begin{cases} 2q - 1 - 2(r - 1) - (n + 1) - (2i - 2) - 2(k - 1)(n + 2r), & 1 \leq i \leq r, \quad 1 \leq k \leq \left( \frac{(n)}{4} \right) \\ 2q + 1 - 2(r - 1) - (n + 1) - (2i - 2) - 2(k - 1)(n + 2r), & 1 \leq i \leq r, \quad \frac{(n)}{4} + 1 \leq k \leq \frac{(m)}{2} \end{cases} \]

\[ f \left( y_i^{2k} \right) = 2(n + r) + (2i - 2) + 2(k - 1)(n + r), \quad 1 \leq i \leq r, \quad 1 \leq k \leq \left( \frac{m}{2} \right). \]

Case 2: When \( n \) is odd

The vertex labels for the first internal vertices of the extended bistar \( (K_{1,k} : n) \) where \( n = 7, 11, 15, 19, \ldots \) are as given below:

\[ f \left( u_{2i}^{2k-1} \right) = 2i - 2 + 2(k - 1)(n + 2r), \quad 1 \leq i \leq \frac{n + 1}{4}, \quad 1 \leq k \leq \frac{m}{2} \]

\[ f \left( u_{2i-1}^{2k-1} \right) = \begin{cases} 2q - 2r + 2i - 2(k - 1)(n + 2r) - 3, & 1 \leq i \leq \left( \frac{n + 3}{4} \right), \quad 1 \leq k \leq \left( \frac{(n)}{4} \right) \\ 2q - 2r + 2i - 2(k - 1)(n + 2r) - 1, & 1 \leq i \leq \left( \frac{n + 3}{4} \right), \quad \frac{(n)}{4} + 1 \leq k \leq \frac{(m)}{2} \end{cases} \]

\[ f \left( u_{2i}^{2k} \right) = \begin{cases} 2q - 2i - 2(r + n) - (2i - 2) + 2(k - 1)(n + r), & 1 \leq i \leq \left( \frac{n + 2}{4} \right), \quad 1 \leq k \leq \left( \frac{(n)}{4} \right) \\ 2q + 2i - 2(r + n) - (2i - 2) + 2(k - 1)(n + r), & 1 \leq i \leq \left( \frac{n + 2}{4} \right), \quad \frac{(n)}{4} + 1 \leq k \leq \frac{(m)}{2} \end{cases} \]

\[ f \left( u_{2i-1}^{2k} \right) = (2n + 2) + (2i - 2) + 2(k - 1)(n + 2r), \quad 1 \leq i \leq \left( \frac{n + 1}{4} \right), \quad 1 \leq k \leq \left( \frac{m}{2} \right). \]

The vertex labels for the second internal vertices of the extended bistar \( (K_{1,k} : n) \) are now given below:

\[ f \left( v_{2i}^{2k-1} \right) = n - 1 + (2i - 2) + 2(k - 1)(n + 2r), \quad 1 \leq i \leq \frac{n + 1}{4}, \quad 1 \leq k \leq \frac{m}{2} \]

\[ f \left( v_{2i-1}^{2k-1} \right) = \begin{cases} 2q - 1 - 2(n + 2r - 3) + (2i - 2) + 2(k - 1)(n + 2r), & 1 \leq i \leq \left( \frac{n + 3}{4} \right), \quad 1 \leq k \leq \left( \frac{(n)}{4} \right) \\ 2q + 1 - 2(n + 2r - 3) + (2i - 2) + 2(k - 1)(n + 2r), & 1 \leq i \leq \left( \frac{n + 3}{4} \right), \quad \frac{(n)}{4} + 1 \leq k \leq \frac{(m)}{2} \end{cases} \]

\[ f \left( v_{2i}^{2k} \right) = 2(n + r) - 2 + (2i - 2) + 2(k - 1)(n + 2r), \quad 1 \leq i \leq \left( \frac{n + 1}{4} \right), \quad 1 \leq k \leq \left( \frac{m}{2} \right). \]
Now, the vertex labels for the cycle $C_m$ are given below:

$$f(C_{2j}) = \left(\frac{n-1}{2}\right) + 2(k-1)(n+2r), \quad 1 \leq k \leq \frac{m}{2}$$

$$f(C_{2j-1}) = \begin{cases} 
2q - 1 - 3\left(\frac{m}{2}\right) + 4r - 2(k-1)(n+2r), & 1 \leq k \leq \left(\frac{m}{2}\right) \\
2q + 1 - 2(i-2) - 2(k-1)(n+2r), & 1 \leq i \leq r, \left(\frac{m}{2}\right) + 1 \leq k \leq \left(\frac{m}{2}\right)
\end{cases}$$

The vertex labels for the first internal vertices of the extended bistar $\langle K_1,k : n \rangle$ are given below:

$$f(x_{i}^{2k-1}) = \begin{cases} 
2q - 1 - (2i-2) - 2(k-1)(n+2r), & 1 \leq i \leq r, 1 \leq k \leq \left(\frac{m}{2}\right) \\
2q + 1 - (2i-2) - 2(k-1)(n+2r), & 1 \leq i \leq r, \left(\frac{m}{2}\right) + 1 \leq k \leq \left(\frac{m}{2}\right)
\end{cases}$$

$$f(x_{i}^{2k}) = 2n + (2i-2) + 2(k-1)(n + r), \quad 1 \leq i \leq r, \quad 1 \leq k \leq \frac{m}{2}$$

Lastly, we give the vertex labels for the second internal vertices of extended bistar $\langle K_1,k : n \rangle$ as under

$$f(y_{i}^{2k-1}) = \begin{cases} 
2q - 1 - 2(r-1) - (n+1) - (2i-2) - 2(k-1)(n+2r), & 1 \leq i \leq r, 1 \leq k \leq \left(\frac{m}{2}\right) \\
2q + 1 - 2(r-1) - (n+1) - (2i-2) - 2(k-1)(n+2r), & 1 \leq i \leq r, \left(\frac{m}{2}\right) + 1 \leq k \leq \left(\frac{m}{2}\right)
\end{cases}$$

$$f(y_{i}^{2k}) = 2(n+r) + (2i-2) + 2(k-1)(n + r), \quad 1 \leq i \leq r, \quad 1 \leq k \leq \frac{m}{2}$$

From these equations, we see that the vertex labels are specified and also the edge labels can be computed and the edge labels are found to be odd and distinct. Therefore, the graph obtained by attaching each vertex $C_m$ with the extended bistar graph is odd graceful, when $m$ is even and $n$ is odd.

\[\square\]

**Illustration 2.2.** We illustrate the above theorem by means of the example depicted in Fig. 3 when $m = 4, k = 4, n = 9, q = 68, 2q - 1 = 135$.

Fig. 3: Eight isomorphic copies of the extended bistar $\langle K_{1,4} : 9 \rangle$ graph attached with the cycle $C_8$.

### 3 Conclusion

It is shown that the $m$-isomorphic copies of the extended bistar $\langle K_{1,k} : n \rangle$ where $n$ is odd, attached at each vertex of a cycle $C_m$ when $m \equiv (\text{mod}4)$ admits odd graceful labeling.
References


