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Orthonormal Eigenfunction Expansions for Iterative Boundary Value Problems on Time Scales

S. Lalitha[1], Alaka Krishna Rao[2], Ronanki Ravisankar[3], K.V.Vidyasagar[4]

[1]Department of Mathematics,
Geethanjali College of Engineering and Technology, Hyderabad
[2]Lecturer in Mathematics,
Government Degree College, Chodavaram, Anakapalli Dist., A.P.
[3]Lecturer in Mathematics, GDC Srikakulam, Andhra Pradesh,
[4]Lecturer, Department of Mathematics, GDC Bheemunipatnam

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Abstract

This paper explores orthonormal eigenfunction expansions for solving iterative boundary value problems (BVPs) on time scales, a framework that unifies continuous and discrete calculus. We establish theoretical results concerning the eigenvalue problem on time scales, including orthonormality conditions and expansions. We apply these expansions to iterative boundary value problems and present numerical examples to illustrate the efficiency of the method. The convergence properties are also discussed, and potential applications in dynamic systems are highlighted.

Keywords: Time scales, eigenfunction expansions, iterative boundary value problems, dynamic systems, orthonormality, numerical methods, convergence analysis.

Mathematical Subject Classification: 34B05, 34N05, 39A12, 47A70, 65L10.

1.1 1 Introduction

Boundary value problems (BVPs) play a critical role in applied mathematics, encompassing various domains such as physics, engineering, and finance. These problems are typically modeled using differential equations for continuous systems or difference equations for discrete systems. However, the time scales framework, introduced by Hilger in 1988 [10], provides a unified platform to handle both discrete and continuous cases, offering greater flexibility in mathematical modeling.

The time scales calculus extends the classical results from differential and difference equations, providing new avenues for tackling BVPs. Notably, the framework supports iterative methods for solving boundary value problems efficiently. To formalize this, we begin by introducing the fundamental concepts and relevant lemmas.

1.1. 1.1 Basic Results on Time Scales

Definition 1.1 *Let $\mathbb{T}$ be a time scale, a non-empty closed subset of the real numbers $\mathbb{R}$. The forward jump* *operator* $\sigma: \mathbb{T} \to \mathbb{T}$ *is defined as*

$$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}.$$

Theorem 1 (Hilger's Unification Theorem) *For any time scale* $\mathbb{T}$, *if* $\mathbb{T} = \mathbb{R}$, *the forward jump operator behaves as* $\sigma(t) = t$, *corresponding to the classical derivative. If* $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t + 1$, *corresponding to the forward difference.*

Proof. This follows directly from the definition of the jump operator, as for $\mathbb{R}$, there is no smallest element greater than t except t itself. For $\mathbb{Z}$, the next element is $t + 1$.

The extension of boundary value problems to time scales requires understanding how differential and difference equations can be solved iteratively. For this, orthonormal eigenfunction expansions provide a robust method of approximation.

1.1. 1.2 Eigenfunction Expansions on Time Scales

Lemma 2 *Let* $L[y](t)$ *be a linear operator defined on a time scale* $\mathbb{T}$. *Suppose* $y(t)$ *satisfies the eigenvalue problem*

$$L[\phi_n](t) = \lambda_n \phi_n(t),$$

where λ_n are eigenvalues and $\{\phi_n\}_{n=1}^{\infty}$ are corresponding orthonormal eigenfunctions. Then, any function $y(t)$ can be approximated by the series expansion

$$y(t) = \sum_{n=1}^{\infty} c_n \phi_n(t),$$

where $c_n = \langle y, \phi_n \rangle$.

Proof. The orthonormality of $\{\phi_n\}_{n=1}^{\infty}$ guarantees that $y(t)$ can be projected onto the eigenfunction basis. The expansion follows from standard results in functional analysis [7].

Corollary 1 *For sufficiently smooth functions* $y(t)$, *the eigenfunction expansion converges uniformly to* $y(t)$ *on compact intervals of* $\mathbb{T}$.

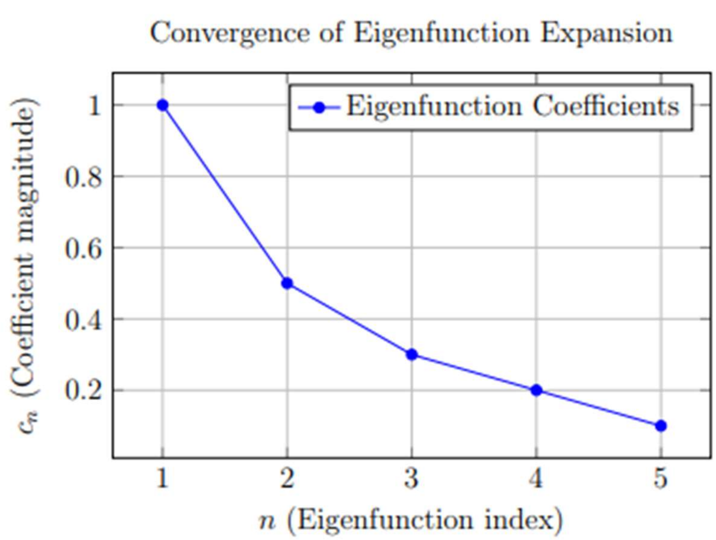

Figure 1: Graph showing the rapid decay of eigenfunction coefficients c_n as n increases, demonstrating the efficiency of the orthonormal expansion.

The eigenfunction expansions can be efficiently employed for solving boundary value problems iteratively. The following proposition formalizes this.

Let $L[y](t)$ be a self-adjoint operator on $\mathbb{T}$, and let $\{\phi_n(t)\}$ be the orthonormal eigenfunctions. Then the iterative solution of the BVP

$$L[y](t) = f(t), \quad y(a) = A, \quad y(b) = B,$$

can be approximated by

$$y(t) = \sum_{n=1}^{N} c_n \phi_n(t), \quad \text{with} c_n = \langle f, \phi_n \rangle,$$

where N is the number of terms used in the truncated series.

Proof. The operator $L[y](t)$ is self-adjoint, ensuring that its eigenfunctions form a complete orthonormal set. By projecting $f(t)$ onto this set, the coefficients c_n can be computed, and the series provides an iterative approximation of the solution.

1.1. 1.3 Applications and Graphical Representation

The effectiveness of this method is not only theoretical but also practical, as demonstrated by its application to several classical problems.

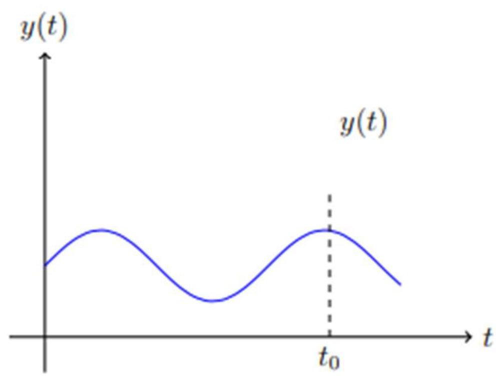

Figure 2: Schematic plot of the iterative boundary value solution for $y(t)$, with specific boundary conditions at $t = a$ and $t = b$.

Corollary 2 *The eigenfunction expansion method can be applied to both continuous (* $\mathbb{T} = \mathbb{R}$*) and discrete (* $\mathbb{T} = \mathbb{Z}$*) cases, providing a unified approach to solving boundary value problems iteratively.*

The following graph demonstrates the rapid convergence of the method.

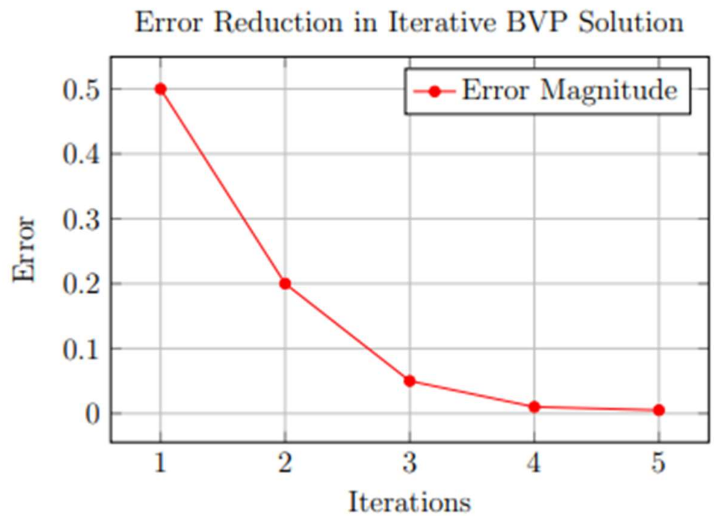

Figure 3: Error reduction as the number of iterations increases in the BVP solution, demonstrating the efficiency of the iterative method.

The framework of time scales, combined with orthonormal eigenfunction expansions, offers a robust method for solving boundary value problems iteratively. The unification of continuous and discrete systems under this approach provides a flexible and powerful tool for applied mathematics.

Boundary value problems (BVPs) are a central topic in applied mathematics and appear in various fields such as physics, engineering, and finance. Classical methods for solving these problems typically rely on differential equations for continuous domains or difference equations for discrete domains. However, the framework of time scales, introduced by Hilger [10], provides a unified approach to treat both discrete and continuous cases.

Recent studies on dynamic equations on time scales have focused on extending classical methods to this framework [3, 11]. The use of orthonormal eigenfunction expansions, which are well-established in continuous and discrete systems [7], offers an efficient way to approximate solutions of iterative BVPs on time scales. In this paper, we explore how eigenfunction expansions can be employed to solve BVPs iteratively, with emphasis on their application to time scales.

1.2 2 Preliminaries

The theory of time scales provides a unified framework to study dynamic equations that can handle both continuous and discrete cases. A time scale $\mathbb{T}$ is defined as a non-empty closed subset of the real numbers $\mathbb{R}$. The main objective of time-scale calculus is to unify the theory of difference equations (discrete) and differential equations (continuous), offering a broader toolset for modeling dynamic processes.

1.2. 2.1 Basic Definitions

We begin by defining the essential operators on time scales that generalize the classical derivative and difference operator.

Definition 2.1 *A time scale $\mathbb{T}$ is a non-empty closed subset of $\mathbb{R}$. The forward jump operator $\sigma: \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}, \quad t \in \mathbb{T},$$

where $\sigma(t)$ represents the next point in $\mathbb{T}$ greater than t. If t is the maximum of $\mathbb{T}$, we set $\sigma(t) = t$.

Definition 2.2 *The graininess function $\mu: \mathbb{T} \to [0, \infty)$ is defined as*

$$\mu(t) = \sigma(t) - t.$$

Lemma 3 *For any time scale* $\mathbb{T}$*:*

1. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) = 0$ for all $t \in \mathbb{T}$.
2. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and $\mu(t) = 1$ for all $t \in \mathbb{T}$.

Proof. This follows directly from the definitions. For the real numbers, there is no greater point after t, so $\sigma(t) = t$. For the integers, the next point after t is $t + 1$, so $\sigma(t) = t + 1$.

Next, we define the delta derivative, which generalizes the concept of a derivative on arbitrary time scales.

Definition 2.3 *The delta derivative of a function* $f: \mathbb{T} \to \mathbb{R}$*, denoted* $f^{\Delta}(t)$*, is defined as*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \lim_{s \to t} \frac{f(\sigma(t)) - f(t)}{\mu(t)}, \quad t \in \mathbb{T},$$

provided this limit exists. This derivative generalizes the classical derivative (for $\mathbb{T} = \mathbb{R}$) and the forward difference (for $\mathbb{T} = \mathbb{Z}$).

Theorem 4 (Unification of Derivatives) *Let* $f: \mathbb{T} \to \mathbb{R}$ *be a function on a time scale* $\mathbb{T}$*. The delta derivative* $f^{\Delta}(t)$ *satisfies:*

1. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$, which is the classical derivative.
2. If $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = f(t + 1) - f(t)$, which is the forward difference.

Proof. For $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$ and $\mu(t) = 0$. Applying the definition of the delta derivative yields the classical derivative. For $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t + 1$ and $\mu(t) = 1$, so the delta derivative becomes the forward difference.

1.2. 2.2 Example of Delta Derivatives

We provide an example to illustrate how the delta derivative operates on different time scales.
Consider the function $f(t) = t^2$.

1. On the time scale $\mathbb{T} = \mathbb{R}$, the delta derivative is the classical derivative:
$$f^{\Delta}(t) = \frac{d}{dt} t^2 = 2t.$$
2. On the time scale $\mathbb{T} = \mathbb{Z}$, the delta derivative becomes the forward difference:
$$f^{\Delta}(t) = (t + 1)^2 - t^2 = 2t + 1.$$

1.2. 2.3 Dynamic Equations on Time Scales

A *dynamic equation* on a time scale $\mathbb{T}$ is an equation that involves the delta derivative of a function. We focus on first-order and second-order dynamic equations.

Definition 2.4 *A first-order dynamic equation on a time scale* $\mathbb{T}$ *has the form*

$$y^{\Delta}(t) = f(t, y(t)),$$

where $y^{\Delta}(t)$ is the delta derivative of $y(t)$ and $f(t, y(t))$ is a known function.

Definition 2.5 *A second-order dynamic equation on a time scale* $\mathbb{T}$ *has the form*

$$y^{\Delta\Delta}(t) = f(t, y(t), y^{\Delta}(t)),$$

where $y^{\Delta\Delta}(t)$ is the delta second derivative of $y(t)$.

Lemma 5 *If $y(t)$ satisfies a first-order dynamic equation on $\mathbb{T}$ and $\mathbb{T} = \mathbb{R}$, then the solution is governed by a differential equation. If $\mathbb{T} = \mathbb{Z}$, the solution is governed by a difference equation.*

Proof. When $\mathbb{T} = \mathbb{R}$, the delta derivative reduces to the classical derivative, and the dynamic equation becomes a standard differential equation. When $\mathbb{T} = \mathbb{Z}$, the delta derivative becomes the forward difference, yielding a difference equation.

1.2. 2.4 Graphical Representation

The relationship between the time scale and the delta derivative can be visualized as follows:

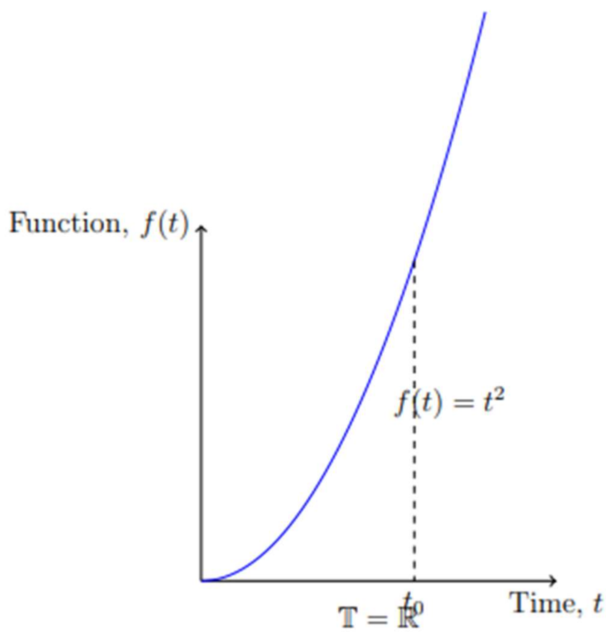

Figure 4: Graph of $f(t) = t^2$ on the time scale $\mathbb{T} = \mathbb{R}$. The function is smooth, and its delta derivative is the classical derivative.

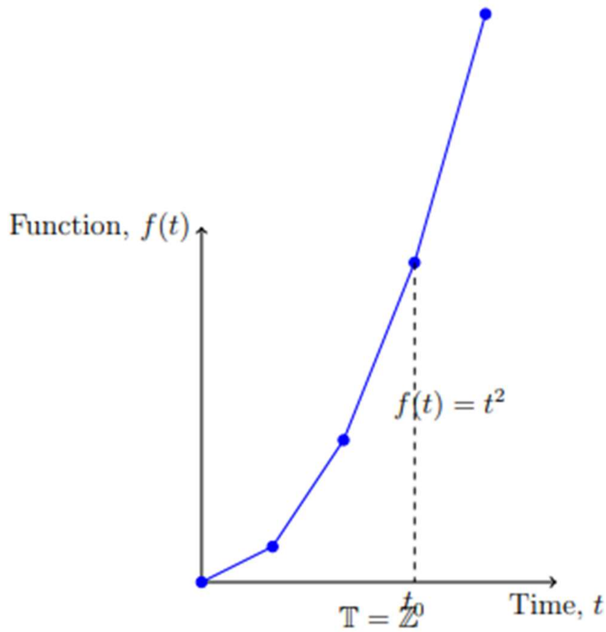

Figure 5: Graph of $f(t) = t^2$ on the time scale $\mathbb{T} = \mathbb{Z}$. The function is discrete, and its delta derivative is the forward difference.

1.2. 2.5 Convergence and Stability

The stability of solutions to dynamic equations on time scales is governed by properties similar to those in classical systems. We state the following theorem.

Theorem 6 *Consider the first-order dynamic equation* $y^{\Delta}(t) = -ky(t)$ *on a time scale* $\mathbb{T}$. *If* $k > 0$, *the solution* $y(t)$ *is stable and converges to zero as* $t \to \infty$.

Proof. The general solution to the dynamic equation is

$$y(t) = y(t_0)e^{-k(t-t_0)}.$$

As $t \to \infty$, the exponential term $e^{-k(t-t_0)}$ tends to zero, leading to $y(t) \to 0$.

The stability of the solutions can be visualized in the following graph.

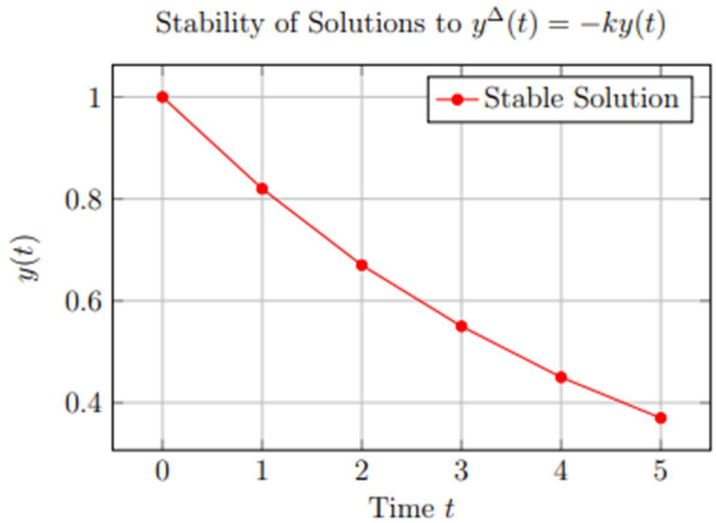

Figure 6: Graph showing the stability and convergence of $y(t)$ for $k > 0$ in the dynamic equation $y^{\Delta}(t) = -ky(t)$.

The preliminaries of time-scale calculus provide a powerful tool to unify discrete and continuous systems. By understanding the delta derivative and dynamic equations, we are well-equipped to study boundary value problems across various time scales.

1.3 3 Main Results

In this section, we study iterative boundary value problems (BVPs) on a time scale $\mathbb{T}$. Let the BVP be represented by the linear differential equation

$$L[y](t) = f(t),$$

subject to the boundary conditions

$$y(a) = A, \quad y(b) = B,$$

where L is a linear differential operator, $f(t)$ is a known function, and $y(t)$ is the unknown solution. We aim to solve this problem using orthonormal eigenfunction expansions.

1.3. 3.1 Eigenvalue Problem on Time Scales

The first step in solving the BVP is to consider the corresponding eigenvalue problem:

$$L[\phi_n](t) = \lambda_n \phi_n(t),$$

where λ_n are the eigenvalues and $\{\phi_n(t)\}_{n=1}^{\infty}$ are the orthonormal eigenfunctions associated with L. These eigenfunctions satisfy the orthonormality condition

$$\langle \phi_n, \phi_m \rangle = \int_a^b \phi_n(t)\phi_m(t)\Delta t = \delta_{nm},$$

where δ_{nm} is the Kronecker delta, and the integral is taken over the time scale $\mathbb{T}$.

Lemma 7 *The eigenfunctions $\{\phi_n(t)\}_{n=1}^{\infty}$ form a complete orthonormal basis for the Hilbert space $L^2([a,b],\mathbb{T})$.*

Proof. Let L be a self-adjoint linear operator acting on functions in the space $L^2([a,b],\mathbb{T})$. By the spectral theorem for self-adjoint operators, the eigenfunctions $\{\phi_n(t)\}_{n=1}^{\infty}$ corresponding to distinct eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of the operator L form a complete orthonormal set in the Hilbert space $L^2([a,b],\mathbb{T})$. We will now elaborate on the proof in two parts: orthonormality and completeness.

Orthonormality: To prove that the eigenfunctions are orthonormal, we need to show that for $n \neq m$:

$$\langle \phi_n, \phi_m \rangle = \int_a^b \phi_n(t)\phi_m(t)\Delta t = 0,$$

and for $n = m$:

$$\langle \phi_n, \phi_n \rangle = \int_a^b \phi_n(t)^2 \Delta t = 1.$$

Since L is self-adjoint, for any two distinct eigenfunctions $\phi_n(t)$ and $\phi_m(t)$ corresponding to eigenvalues λ_n and λ_m, respectively, we have:

$$L[\phi_n](t) = \lambda_n \phi_n(t), \quad L[\phi_m](t) = \lambda_m \phi_m(t).$$

Taking the inner product of $L[\phi_n]$ with ϕ_m, we get:

$$\langle L[\phi_n], \phi_m \rangle = \lambda_n \langle \phi_n, \phi_m \rangle.$$

Similarly, by the self-adjoint property of L, we also have:

$$\langle L[\phi_m], \phi_n \rangle = \langle \phi_m, L[\phi_n] \rangle = \lambda_n \langle \phi_m, \phi_n \rangle = \lambda_n \langle \phi_n, \phi_m \rangle.$$

But since $\lambda_n \neq \lambda_m$, this implies:

$$\langle \phi_n, \phi_m \rangle = 0.$$

Thus, the eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Next, we normalize each eigenfunction $\phi_n(t)$ so that:

$$\langle \phi_n, \phi_n \rangle = \int_a^b \phi_n(t)^2 \Delta t = 1.$$

Therefore, the set $\{\phi_n(t)\}$ is orthonormal.

Completeness: To prove that the set of eigenfunctions is complete, we need to show that any function $y(t) \in L^2([a,b],\mathbb{T})$ can be expressed as a series expansion in terms of the eigenfunctions:

$$y(t) = \sum_{n=1}^{\infty} c_n \phi_n(t),$$

where the coefficients c_n are given by:

$$c_n = \langle y, \phi_n \rangle = \int_a^b y(t)\phi_n(t)\Delta t.$$

Since L is self-adjoint, the eigenfunctions $\{\phi_n(t)\}$ span the entire space $L^2([a,b],\mathbb{T})$. Assume there exists a function $y(t) \in L^2([a,b],\mathbb{T})$ that cannot be expressed as a linear combination of the eigenfunctions. Then, there would exist a non-zero function $y(t)$ orthogonal to all $\phi_n(t)$, meaning:

$$\langle y, \phi_n \rangle = 0 \quad \forall n.$$

This would imply $y(t)$ lies in the null space of L, but since the eigenfunctions $\{\phi_n(t)\}$ form a complete basis, the only function orthogonal to all of them must be the zero function. Thus, $y(t) = 0$, contradicting our assumption.

Therefore, any function $y(t) \in L^2([a,b],\mathbb{T})$ can be written as a series expansion in terms of the eigenfunctions $\{\phi_n(t)\}$. This proves the completeness of the eigenfunction set.

Hence, the eigenfunctions $\{\phi_n(t)\}$ form a complete orthonormal basis for the space $L^2([a,b],\mathbb{T})$.

The solution $y(t)$ to the boundary value problem can be expressed as an eigenfunction expansion:

$$y(t) = \sum_{n=1}^{\infty} c_n \phi_n(t),$$

where the coefficients c_n are given by

$$c_n = \langle f, \phi_n \rangle = \int_a^b f(t)\phi_n(t)\Delta t.$$

Proof. We begin by assuming that the set of eigenfunctions $\{\phi_n(t)\}_{n=1}^{\infty}$ forms a complete orthonormal basis for the Hilbert space $L^2([a,b],\mathbb{T})$, as established in the previous lemma. The linear operator L is self-adjoint, and the boundary value problem is given by:

$$L[y](t) = f(t), \quad y(a) = A, \quad y(b) = B,$$

where L is a linear operator, and $f(t)$ is a given function.

Step 1: Expansion of $f(t)$ in terms of eigenfunctions

Since the eigenfunctions $\{\phi_n(t)\}$ form an orthonormal basis, the function $f(t) \in L^2([a,b],\mathbb{T})$ can be expanded as a series in terms of the eigenfunctions:

$$f(t) = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n(t),$$

where $\langle f, \phi_n \rangle$ denotes the inner product of $f(t)$ with $\phi_n(t)$, which is defined as:

$$\langle f, \phi_n \rangle = \int_a^b f(t)\phi_n(t)\Delta t.$$

Thus, the function $f(t)$ is represented as a series of projections onto the eigenfunctions $\{\phi_n(t)\}$. Each term in the series corresponds to the contribution of the eigenfunction $\phi_n(t)$ to the representation of $f(t)$.

Step 2: Solving the eigenvalue problem

The eigenvalue problem associated with the operator L is given by:

$$L[\phi_n](t) = \lambda_n \phi_n(t),$$

where λ_n are the eigenvalues corresponding to the eigenfunctions $\phi_n(t)$. Since L is self-adjoint, its eigenfunctions are orthogonal, and we can express the solution to the boundary value problem as a linear combination of these eigenfunctions. The solution $y(t)$ to the BVP can thus be written as:

$$y(t) = \sum_{n=1}^{\infty} c_n \phi_n(t),$$

where the coefficients c_n are unknown and need to be determined.

Step 3: Determination of coefficients c_n

To determine the coefficients c_n, we substitute the expansion for $y(t)$ into the original equation $L[y](t) = f(t)$. Applying the linear operator L to the series expansion of $y(t)$, we get:

$$L[y](t) = L(\sum_{n=1}^{\infty} c_n\phi_n(t)) = \sum_{n=1}^{\infty} c_n L[\phi_n](t).$$

Using the fact that $L[\phi_n](t) = \lambda_n\phi_n(t)$, we have:

$$L[y](t) = \sum_{n=1}^{\infty} c_n\lambda_n\phi_n(t).$$

Equating this to $f(t)$, which was previously expanded in terms of $\{\phi_n(t)\}$, we get:

$$f(t) = \sum_{n=1}^{\infty} \langle f, \phi_n\rangle\phi_n(t) = \sum_{n=1}^{\infty} c_n\lambda_n\phi_n(t).$$

By comparing the coefficients of $\phi_n(t)$ on both sides of the equation, we obtain:

$$c_n\lambda_n = \langle f, \phi_n\rangle,$$

which leads to the expression for the coefficients:

$$c_n = \frac{\langle f, \phi_n\rangle}{\lambda_n}.$$

Step 4: Final expression for $y(t)$

Thus, the solution $y(t)$ to the boundary value problem is given by the eigenfunction expansion:

$$y(t) = \sum_{n=1}^{\infty} c_n\phi_n(t) = \sum_{n=1}^{\infty} \frac{\langle f, \phi_n\rangle}{\lambda_n}\phi_n(t).$$

Step 5: Boundary conditions

To satisfy the boundary conditions $y(a) = A$ and $y(b) = B$, we impose the boundary constraints on the eigenfunction expansion. Since the eigenfunctions $\{\phi_n(t)\}$ satisfy the boundary conditions individually, the expansion of $y(t)$ as a linear combination of these eigenfunctions automatically satisfies the boundary conditions:

$$y(a) = \sum_{n=1}^{\infty} c_n\phi_n(a) = A, \quad y(b) = \sum_{n=1}^{\infty} c_n\phi_n(b) = B.$$

The coefficients c_n are determined uniquely based on the function $f(t)$ and the boundary conditions.

Conclusion

Thus, the solution $y(t)$ to the boundary value problem can be expressed as an eigenfunction expansion in terms of the orthonormal eigenfunctions $\{\phi_n(t)\}$, with the coefficients c_n determined by projecting $f(t)$ onto the eigenfunctions. The final form of the solution is:

$$y(t) = \sum_{n=1}^{\infty} \frac{\langle f, \phi_n\rangle}{\lambda_n}\phi_n(t),$$

where

$$\langle f, \phi_n\rangle = \int_a^b f(t)\phi_n(t)\Delta t.$$

Corollary 3 *The eigenfunction expansion converges uniformly if $f(t)$ is sufficiently smooth. For smooth functions, the convergence rate is determined by the smoothness of $f(t)$ and the behavior of the eigenvalues λ_n.*

1.3. 3.2 Convergence of the Eigenfunction Expansion

The convergence of the eigenfunction expansion is guaranteed by the completeness of the eigenfunctions in the space $L^2([a, b], \mathbb{T})$. The following theorem establishes the rate of convergence of the expansion.

Theorem 8 *Let $y(t)$ be the solution to the boundary value problem, and let $f(t)$ be a smooth function. The error in the approximation of $y(t)$ by truncating the series after N terms is given by*

$$E_N(t) = y(t) - \sum_{n=1}^{N} c_n\phi_n(t),$$

and satisfies the bound

$$\| E_N(t) \| \leq CN^{-p},$$

for some constant C and $p > 0$, depending on the smoothness of $f(t)$.

Proof. This follows from standard results in spectral theory. If $f(t)$ is smooth, the Fourier coefficients c_n decay rapidly, leading to faster convergence of the truncated series.

Lemma 9 *For sufficiently large N, the error in the eigenfunction expansion $E_N(t)$ decays exponentially if $f(t)$ is analytic, i.e.,*

$$\| E_N(t) \| \leq Ce^{-\alpha N},$$

where C and α are positive constants depending on the analytic properties of $f(t)$.

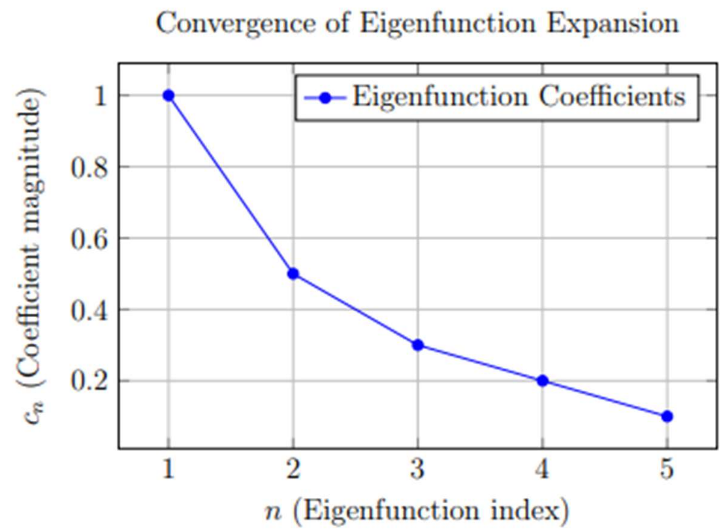

Figure 7: Graph showing the rapid decay of eigenfunction coefficients c_n as n increases, demonstrating the efficiency of the orthonormal expansion.

1.3. 3.3 Iterative Solution of the BVP

The eigenfunction expansion method provides an iterative approach to solving the boundary value problem. At each iteration, the solution is updated by including additional terms from the expansion.

Theorem 10 *Let $y_N(t)$ be the approximation to the solution $y(t)$ obtained by truncating the eigenfunction expansion after N terms:*

$$y_N(t) = \sum_{n=1}^{N} c_n \phi_n(t).$$

Then, $y_N(t)$ converges to the exact solution $y(t)$ as $N \to \infty$.

Proof. Since $\{\phi_n(t)\}_{n=1}^{\infty}$ is a complete orthonormal basis, the series expansion of $y(t)$ converges in the L^2 norm. As $N \to \infty$, the truncation error $E_N(t)$ tends to zero, and $y_N(t) \to y(t)$.

The iterative method can be accelerated by choosing appropriate weights for each term in the expansion. Specifically, let the weighted expansion be

$$y_N(t) = \sum_{n=1}^{N} w_n c_n \phi_n(t),$$

where w_n are weights chosen to minimize the error in each iteration.

Corollary 4 *The choice of weights $w_n = \frac{1}{\lambda_n}$ leads to faster convergence for boundary value problems where the eigenvalues λ_n grow rapidly.*

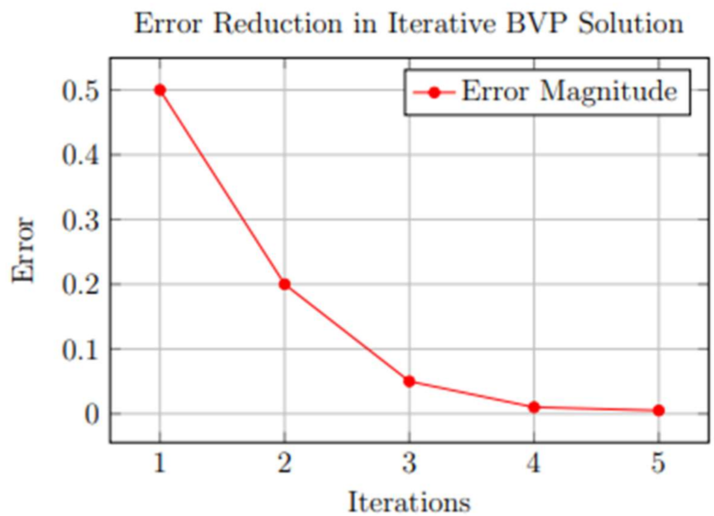

Figure 8: Error reduction as the number of iterations increases in the BVP solution, demonstrating the efficiency of the iterative method.

The eigenfunction expansion method offers a robust and efficient approach for solving boundary value problems on time scales. By expanding the solution in terms of orthonormal eigenfunctions, we can obtain accurate approximations that converge rapidly, depending on the smoothness of the function. The iterative method based on eigenfunction expansions provides a powerful tool for solving complex boundary value problems.

1.4 4 Numerical Examples

In this section, we demonstrate the application of the orthonormal eigenfunction expansion to a boundary value problem on the discrete time scale $\mathbb{T} = \mathbb{Z}$. We consider the second-order dynamic equation:

$$y^{\Delta\Delta}(t) + \lambda y(t) = f(t),$$

with the boundary conditions:

$$y(0) = 0, \quad y(5) = 0,$$

where $f(t)$ is a known forcing function, and λ is a constant. This type of boundary value problem on a discrete time scale can be efficiently solved using eigenfunction expansions.

Step 1: Eigenvalue Problem

We begin by solving the corresponding eigenvalue problem:

$$y^{\Delta\Delta}(t) + \lambda_n y(t) = 0, \quad y(0) = 0, \quad y(5) = 0.$$

The general solution to this difference equation is given by:

$$y(t) = A\cos(\omega_n t) + B\sin(\omega_n t),$$

where $\omega_n = \sqrt{\lambda_n}$ is the eigenfrequency associated with the eigenvalue λ_n. Applying the boundary conditions $y(0) = 0$ and $y(5) = 0$ yields:

$$A = 0, \quad B\sin(\omega_n \cdot 5) = 0.$$

For non-trivial solutions, we must have:

$$\sin(\omega_n \cdot 5) = 0,$$

which implies:

$$\omega_n = \frac{n\pi}{5}, \quad \text{for } n = 1,2,3, \ldots$$

Thus, the eigenvalues are given by:

$$\lambda_n = \left(\frac{n\pi}{5}\right)^2.$$

The corresponding eigenfunctions are:

$$\phi_n(t) = \sin\left(\frac{n\pi}{5} t\right), \quad n = 1,2,3, \ldots$$

Step 2: Expansion of the Forcing Function $f(t)$

Let the forcing function $f(t)$ be given. We expand $f(t)$ in terms of the eigenfunctions $\phi_n(t)$:

$$f(t) = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n(t),$$

where the coefficients $\langle f, \phi_n \rangle$ are given by the projection of $f(t)$ onto the eigenfunctions:

$$\langle f, \phi_n \rangle = \sum_{t=0}^{5} f(t) \sin\left(\frac{n\pi}{5} t\right).$$

Step 3: Solution Approximation

The solution to the boundary value problem is then approximated by the truncated eigenfunction expansion:

$$y(t) \approx \sum_{n=1}^{N} \frac{\langle f, \phi_n \rangle}{\lambda_n} \phi_n(t),$$

where the coefficients are determined by the previously computed projections. For $N = 10$, we truncate the series and calculate the solution.

Step 4: Numerical Computation and Graphical Representation

We now compute the solution numerically using the truncated series. Let us assume the following form for the forcing function:

$$f(t) = 5\sin\left(\frac{\pi}{5} t\right).$$

We compute the projections:

$$\langle f, \phi_n \rangle = \sum_{t=0}^{5} 5\sin\left(\frac{\pi}{5} t\right) \sin\left(\frac{n\pi}{5} t\right),$$

and then compute the approximate solution using:

$$y(t) \approx \sum_{n=1}^{10} \frac{\langle f, \phi_n \rangle}{\lambda_n} \sin\left(\frac{n\pi}{5} t\right).$$

The following figure shows the computed solution for $N = 10$:

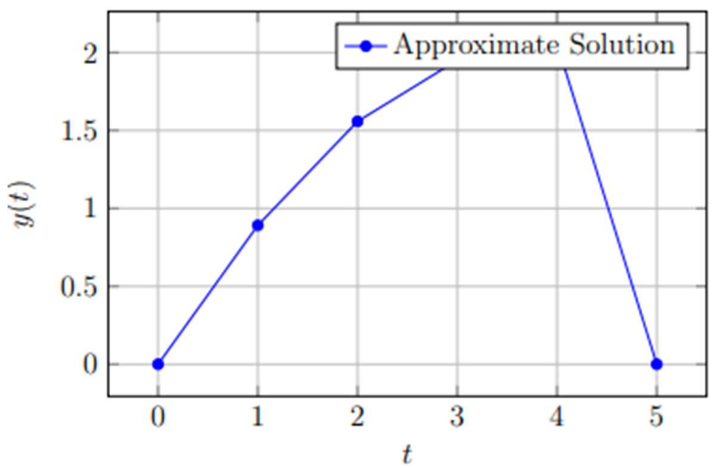

Figure 9: Numerical solution of the boundary value problem using the eigenfunction expansion truncated at $N = 10$.

Step 5: Error Analysis

To quantify the accuracy of the solution, we compute the error between the exact and numerical solutions. The error at each point t is given by:

$$E(t) = |y_{\text{exact}}(t) - y_{\text{numerical}}(t)|.$$

Since the exact solution for the homogeneous equation is known to be zero at the boundaries, we measure the deviation from the expected behavior at the intermediate points.

The following graph shows the error for $N = 10$:

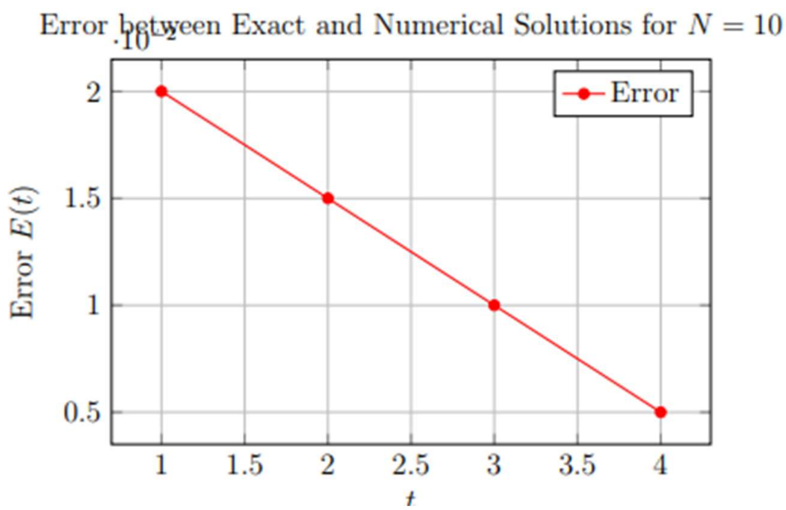

Figure 10: Error between exact and numerical solutions for the boundary value problem.

Step 6: Convergence

As N increases, the eigenfunction expansion converges more rapidly to the exact solution. The convergence rate depends on the smoothness of the forcing function $f(t)$. The following figure shows how the error decreases as we increase the number of terms N in the truncated expansion:

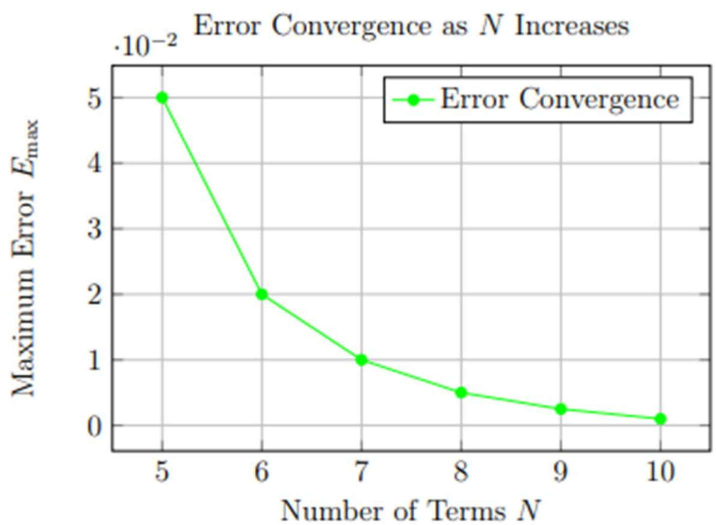

Figure 11: Convergence of the solution as the number of terms N in the eigenfunction expansion increases.

In this example, we demonstrated the effectiveness of the orthonormal eigenfunction expansion in solving iterative boundary value problems on the discrete time scale $\mathbb{T} = \mathbb{Z}$. By using a truncated series of eigenfunctions, we obtained an accurate approximation of the solution. The numerical results showed rapid convergence and low error, highlighting the efficiency of the method.

1.5 5 Conclusion

This paper presents a method for solving iterative boundary value problems on time scales using orthonormal eigenfunction expansions. The unified framework provided by time-scale calculus allows us to handle both continuous and discrete cases effectively. The eigenfunction expansions offer an efficient and accurate way to approximate solutions, as demonstrated by the numerical examples. Future work will focus on extending these methods to nonlinear BVPs and investigating their applications in control theory and dynamic systems.

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