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Harmonic Mean Cordial Labeling of Some Familes of Triangular Snake and Quadrilateral Snake Graphs

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ABSTRACT

A function $f: V(G) \to \{1,2\}$ on a graph $G = (V(G), e(G))$, where G is a simple, connected, finite and an undirected graph, is called Harmonic Mean Cordial if the induced function $f^*: E(G) \to \{1,2\}$ defined by $f^*(uv) = \lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1,2\}$, where $v_f(y)$ denote the number of vertices labelled as y and $e_f(y)$ denote the number of edges labelled as y respectively. A graph G that permits Harmonic Mean Cordial labeling is called a Harmonic Mean Cordial graph. The concepts that have have been obtained for the snake graph families have been used in applications of civil engineering, compture science, network optimization, coding theory, biology, etc. The current study has validated some theorems regarding Harmonic Mean Cordial labeling of several types of snake graphs and families of snake graphs such as double triangular snake graph, alternate triangular and quadrilateral snake graphs, double alternate triangular and quadrilateral snake graphs, spiked snake graph and alternate zig-zag triangular snake graph. The authors have also shown in past works the effectiveness of Harmonic Mean Cordial labeling in civil engineering applications in building bridges.

KEYWORDS
Double triangular snake graph, alternate triangular and quadrilateral snake graphs, double alternate triangular and quadrilateral snake graphs, spiked snake graph and alternate zigzag triangular snake graph..

AMS Subject Classification(2020): 05C78.

1. Introduction

The history of graph theory may be specifically traced to 1735, when the Swiss mathematician Leonhard Euler solved the Königsberg bridge problem. The Königsberg bridge problem was an old puzzle concerning the possibility of finding a path over every one of seven bridges that span a forked river flowing past an island, but without crossing any bridge twice. Euler argued that no such path exists. His proof involved only references to the physical arrangement of the bridges, but essentially he proved the first theorem in graph theory.

Graph theory was originated in 18th century was the origin of gt through kbp. The Konigsberg Bridge Problem is among the most difficult problems in mathematics, and it is the source of graph theory. In the eighteenth century, the capital of East Prussia in West Soviet Russia was Königsberg. There were seven bridges in this city that connected the two islands to the Pregel River's banks. Without swimming over the river or engaging in any other adventures, the task was to start at any of the four land locations, walk across each bridge precisely once, and then return to the starting position. Many thought the endeavor was unachievable since all of their efforts had failed. Swiss mathematician Leonard Euler (1707–1783) solved this well-known topic by converting it to a mathematical problem in 1736. Euler created a graph by replacing each land area with a vertex and each bridge with an edge connecting the relevant vertices. Nothing further was done in the field over the following century. The idea of trees was created in 1847 by G.R. Kirchhoff specifically for use in electric networks. In general, graph theory belongs to the field of combinatoric.s, but it also has strong ties to computer science and applied mathematics. Kirchhoff created a connected graph without cycles. Thomas Gutherie made the famous four-color issue's discovery in 1852. Then, in 1856, William Hamilton, P. Kirkman, and Thomas conducted study on polyhydra cycles and created the concept known as the Hamiltonian graph by observing travels that made precisely one stop at each point. H. Dudeney discussed the problem of riddles in 1913. In actuality, it took a century for Kenneth Appel and Wolfgang Haken to answer the four-color conundrum.

The main parameters and frequent research topics in graph theory include Graph colouring, Domination, Graph labeling, Energy of graphs, Decomposition of graphs and many more.

Graph theory is one of the fields of mathematics that is constantly growing. The field of mathematical science is vast and diverse. One of the main subjects in mathematics is graph theory, which is used to structural models. Graphs are helpful mathematical models for the thorough examination of a variety of concrete real-world circumstances.When vertices and edges meet certain conditions and are both labeled by integers, the graph is said to be labelled. If there are no limitations, there are an infinite number of labels in every graph. In the 1960s, graph labeling was first proposed by A. Rosa in [8]. Since then, several different graph labeling techniques have been researched.

A labeled graph is the one in which, under certain restrictions, either its edges or its vertices are labeled by numbers. There are an endless number of ways to label a graph if there are no restrictions. A. Rosa initially proposed graph labeling in the 1960s. Since then, a number of graph labeling strategies have been investigated. This also results in labelings for graphs that are harmonic, anti-magic, magic, k-equitable, etc. Annual updates to Joseph A. Gallian's dynamic survey on graph labeling are released [9]. For academics studying graph labeling, this survey offers a wealth of material.

2. Literature Review

Harmonic Mean Cordial labeling was initiated by J. Gowri and J. Jayapriya in 2021. In [2], J. Gowri and J. Jayapriya defined Harmonic Mean Cordial labeling and proved certain results on standard graphs as star graph, bistar graph and path. [1] Parejiya J., Jani D. B. and Hathi Y. M. proved theorems on HMC labeling of $CH_n \odot K_1$ and P_m x P_n. They also proved non-HMC labeling of certain graphs as complete bipartite graph $K_{m,n}$, $K_n \vee C_m$ and $C_m \vee C_n$ under relevant conditions. The authors discussed on results on I-cordial labeling of triangular, double triangular, triple triangular and alternate triangular snake graphs. S. K. Vaidya and N. B. Vyas proved results on product cordial labeling of alternate snake graphs such as alternate triangular, alternate quadrilateral, double alternate triangular and double alternate quadrilateral graphs under relevant restrictions. K. M. Babysmitha and K. Thirusangu worked on Distance Two labeling of certain snake graphs [6]. S. Meena and M. Sivasakhti worked on new results on Harmonic Mean labeling of graphs such as zig-zag triangular, alternate zig-zag triangular and spiked snake graphs.

Moreover, Yesha M. Hathi *et al.* have also worked on Harmonic Mean Cordial labeling of Dutch Windmill graph D_m^n, Triangular snake graph, Cycle, Fan graph, path union of a path and a cycle, graph obtained by taking m copies of a wheel, vertex switching of the centre most vertex in a path in their paper Harmonic Mean Cordial Labeling of Some Cycle and Path Related Graphs. The same authors in their above mentioned paper have also discussed about their research on application of HMC labeling on tied-arch bridges with vertical hangers.

Motivated by the above mentioned articles and results, we have investigated on HMC labeling of double triangular snake graph, alternate triangular snake graph, alternate quadrilateral snake graph, double alternate triangular snake graph, double alternate quadrilateral snake graph, spiked snake graph and alternate zig-zag triangular snake graph in this paper along with the respective illustrations for clear understanding.

3. Preliminaries

We begin by presenting the following definitions of Harmonic Mean Cordial labeling and some snake graphs for our research.

Let G = (V, E) be a simple, connected, finite and an undirected graph. A function $f: V(G) \rightarrow \{1,2\}$ is called Harmonic Mean Cordial if the induced function $f^*: E(G) \rightarrow \{1,2\}$ defined by $f^*(uv) = \lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)|$
≤ 1 and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1,2\}$, where $v_f(z)$ denote the number of vertices labelled as z and $e_f(z)$ denote the number of edges labelled as z respectively. A Graph G is called Harmonic Mean Cordial graph if it admits Harmonic Mean Cordial labeling in [2].

[33] A Triangular Snake graph is obtained from a path $u_1, u_2, u_3, \ldots, u_n$ by joining u_i and u_{i+1} a new

vertex v_i; $1 \leq i \leq n-1$, that is,by replacing each edge of a path by a triangle. [11] An alternate triangular snake $A(T_n)$ is obtained from a path $x_1, x_2, \ldots, x_n$ by joining x_i and x_{i+1} (alternately) to a new vertex y_i; $1 \leq i \leq n+1$. That is, C_3 replaces every alternate edge of a path. [11] An alternate quadrilateral snake $A(QS_n)$ is obtained from a path $x_1, x_2, x_3, .., x_n$ by joining x_i, x_{i+1} (alternately) to new vertices y_i, z_i respectively and then joining y_i and z_i, where $1 \leq i \leq n-1$. That is, C_4 takes the place of every alternate edge of a path. [11] A double alternate triangular snake $DA(T_n)$ consists of two alternate triangular snakes that have a common path. That is,$DA(T_n)$ is obtained from a path $x_1, x_2, x_3, .., x_n$ by joining x_i, x_{i+1} (alternately) to two new vertices y_i and z_i, where $1 \leq i \leq n-1$. [11] A double alternate quadrilateral snake DA(QSn) consists of two alternate quadrilateral snakes that have a common path. That is, it is obtained from a path $x_1, x_2, x_3, .., x_n$ by joining x_i and x_{i+1} (alternately) to new vertices y_i, z_i, y_i' and z_i' respectively and adding the edges $y_i y_i'$ and $z_i z_i'$. [6] A spiked snake graph $SS(4,n)$ is a graph obtained from a cyclic snake $S(4,n)$ with additional edges . It is a graph with vertex set $V(SS(4,n)) = \{x_1, x_2, \ldots, x_{n+1}\} \cup \{y_i, z_i, p_i, q_i / 1 \leq i \leq n\}$ and an edge set $E(SS(4,n)) = \{x_i y_i, y_i x_{i+1}, x_{i+1} p_i, p_i x_i, p_i q_i, y_i z_i / 1 \leq i \leq n\}$. [3] Let G be the graph obtained from the path $P_n = x_1, x_2, x_3, \ldots, x_n$ and let $s_1, s_2, s_3, \ldots, s_k, r_1, r_2, r_3, \ldots, r_k, t_1, t_2, t_3, \ldots, t_k$, where k = $\lfloor \frac{n}{3} \rfloor$ and by adding new edge$\{s_i x_{3i-1}, s_i x_{3i-2}, s_i x_{3i} / 1 \leq i \leq k\} \cup \{r_i x_{3i-1}, r_i x_{3i-2} / 1 \leq i \leq k\} \cup \{t_i x_{3i}, t_1 x_{3i+1} / 1 \leq i \leq k\}$ if $n \equiv 1,2 (mod 3)$. The family of graph is called alternate zig-zag triangle $AZ(T_n)$.

4. Results and Discussion

Theorem 4.1 *The Double Triangular Snake Graph* DT_n; $n \geq 2$ *and* $n \equiv 1(mod 2)$ *admits Harmonic Mean Cordial labeling.*

Proof. Let $G = DT_n$; $n \geq 2$ and $n \equiv 1(mod 2)$ be a double triangular snake graph with a vertex set $V = \{x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_{n-1}, y_1', y_2', y_3', \ldots, y_n'\}$, where $x_1, x_2, x_3, \ldots, x_n$ are the vertices on the path of DT_n joining each of x_i and x_{i+1} to a new vertex u_i in the upper half of DT_n and x_i and x_{i+1} joining to a new vertex y_i' in the lower half of DT_n for $1 \leq i \leq n-1$. Note that $|V(G)| = 3n-2$ and $|E(G)| = 5(n-1)$.

Define a labeling function $f: V(G) \to \{1,2\}$ as follows:

$f(y_i) = \{2 \ if \ \ 1 \leq i \leq \frac{n-1}{2} \ 2 \ if \ \ \frac{n-1}{2} < i \leq n-1$

$f(y_i') = \{2 \ if \ \ 1 \leq i \leq \frac{n-1}{2} \ 2 \ if \ \ \frac{n-1}{2} < i \leq n-1$

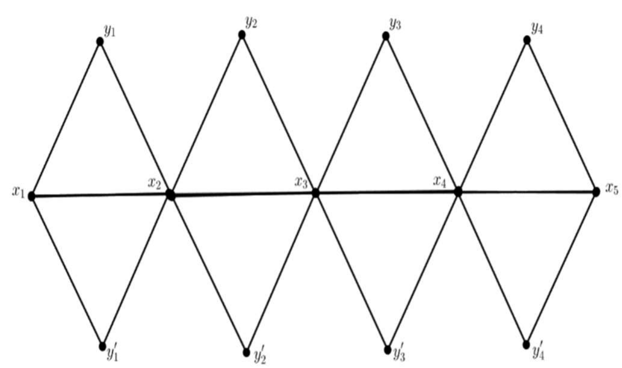
$f(x_i) = \{2 \ if \ \ 1 \leq i \leq \frac{n+1}{2} \ 2 \ if \ \ \frac{n+1}{2} < i \leq n$

Then, we get, $v_f(1) = \frac{3n-1}{2}$ and $v_f(2) = \frac{3n-3}{2} \Longrightarrow |v_f(1) - v_f(2)| = 1$.

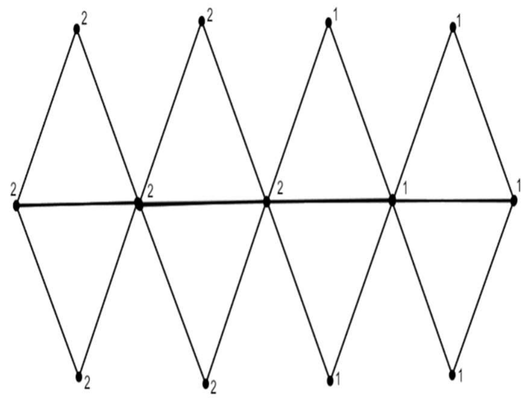
Furthermore, we have, $e_f(1) = \frac{5(n-1)}{2} = e_f(2) \Longrightarrow |e_f(1) - e_f(2)| = 0$.

As a result, Harmonic Mean Cordial labeling is admitted by the Double Triangular Snake Graph DT_n; $n \geq 2$ and $n \equiv 1(mod 2)$.

Illustration 4.1.1 HMC labeling of DT_5 is displayed in the subsequent figures.



DT_5



HMC labeling of DT_5

Theorem 4.2 *The Alternate Triangular Snake graph AT_m; $m \geq 4$ admits Harmonic Mean Cordial labeling with the exception of the case when $m \equiv 1(mod2)$ and $t \equiv 1(mod2)$.*

Proof. Let $G = AT_m$; $m \geq 4$ be an alternate triangular snake graph with a vertex set $V = \{x_1, x_2, x_3, \dots, x_m, y_1, y_2, y_3, \dots, y_t\}$, where t is the number of triangles produced in AT_m. Also, $x_1, x_2, x_3, \dots, x_m$ are the vertices on the path of AT_m joining each of x_i and x_{i+1} to a new vertex y_i for $1 \leq i \leq t$ alternatively.

We note here that $|V(G)| = \frac{3m}{2}$ for $m \equiv 0(mod2)$ and $|V(G)| = \frac{3m-1}{2}$ for $m \equiv 1(mod2)$. Moreover, $|E(G)| = 2m - 1$ for $m \equiv 0(mod2)$ and $|E(G)| = 2m - 2$ for $m \equiv 1(mod2)$.

Define a labeling function $f: V(G) \rightarrow \{1,2\}$ as follows:

Case 1 : $m \equiv 0(mod2)$ and $t \equiv 1(mod2)$

$f(x_i) = \{2\ if\ \ 1 \leq i \leq \frac{m}{2}\ 1\ if\ \ \frac{m}{2} < i \leq m$

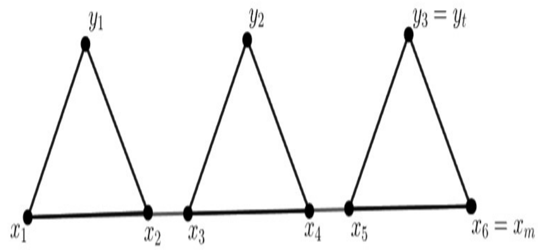
$f(y_i) = \{2\ if\ \ 1 \leq i \leq \frac{t+1}{2}\ 1\ if\ \ \frac{t+1}{2} < i \leq t$

From this, we obtain, $v_f(1) = \frac{\frac{3m}{2}+1}{2}$ and $v_f(2) = \frac{\frac{3m}{2}-1}{2} \Longrightarrow |v_f(1) - v_f(2)| = 1$.

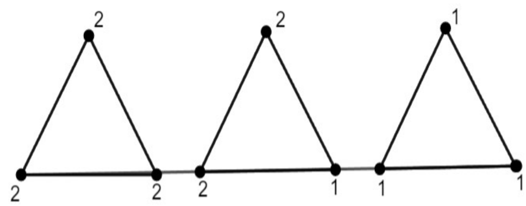
Furthermore, $e_f(1) = \frac{2m-2}{2} = m - 1$ and $e_f(2) = \frac{2m}{2} = m \Longrightarrow |e_f(1) - e_f(2)| = 1$.

Hence, the Alternate Triangular Snake graph AT_m; $m \geq 4, m \equiv 0(mod2)$ and $t \equiv 1(mod2)$ admits Harmonic Mean Cordial labeling.

Illustration 4.2.1 The following figures depict the HMC labeling of AT_6 as illustrated in 4.2.1.



AT_6



HMC labeling of AT_6

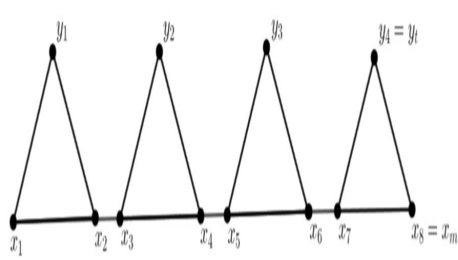
Case 2 : $m \equiv 0(mod2)$ and $t \equiv 0(mod2)$

$f(x_i) = \{2\ if\ \ 1 \le i \le \frac{m}{2}\ 1\ if\ \ \frac{m}{2} < i \le m$

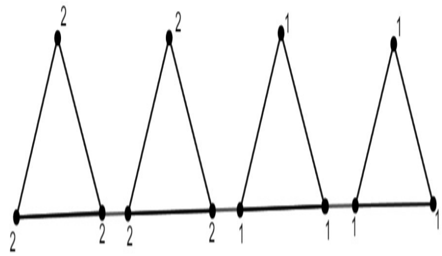
$f(y_i) = \{2\ if\ \ 1 \le i \le \frac{t}{2}\ 1\ if\ \ \frac{t}{2} < i \le t$

Consequently, we have, $v_f(1) = \frac{m+t}{2} = v_f(2) \Rightarrow |v_f(1) - v_f(2)| = 0$.
Furthermore, $e_f(1) = \frac{2m}{2} = m$ and $e_f(2) = \frac{2m-2}{2} = m - 1 \Rightarrow |e_f(1) - e_f(2)| = 1$.
Hence, the Alternate Triangular Snake graph $AT_m;\ m \ge 4$ and $m \equiv 0(mod2)$ and $t \equiv 0(mod2)$ admits Harmonic Mean Cordial labeling.

Illustration 4.2.2 The next figures display HMC labeling of AT_8.



AT_8



HMC labeling of AT_8

Case 3 : $m \equiv 1(mod2)$ and $t \equiv 0(mod2)$

$f(x_i) = \{2\ if\ \ 1 \leq i \leq \frac{m+1}{2}\ 1\ if\ \ \frac{m+1}{2} < i \leq m$

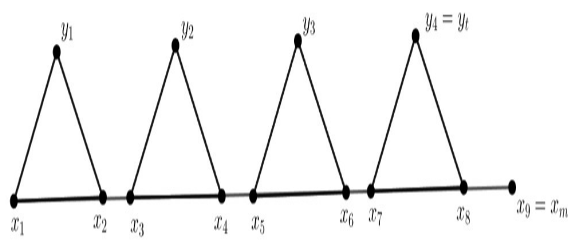
$f(y_i) = \{2\ if\ \ 1 \leq i \leq \frac{t}{2}\ 2\ if\ \ \frac{t}{2} < i \leq t$

Next, we obtain, $v_f(1) = \frac{m+t+1}{2}$ and $v_f(2) = \frac{m+t-1}{2} \Longrightarrow |v_f(1) - v_f(2)| = 1$.

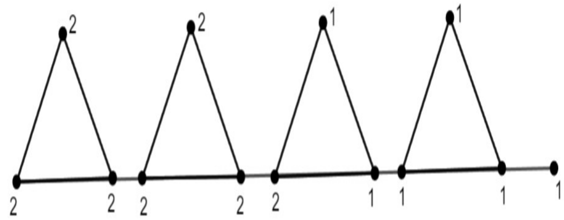
Furthermore, $e_f(1) = m - 1 = e_f(2) \Longrightarrow |e_f(1) - e_f(2)| = 0$.

Hence, the Alternate Triangular Snake graph $AT_m;\ m \geq 4$ and $m \equiv 1(mod2)$ and $t \equiv 0(mod2)$ admits Harmonic Mean Cordial labeling.

Illustration 4.2.3 HMC labeling of AT_9 is shown in the following figures.



AT_9



HMC labeling of AT_9

Theorem 4.3 *The Alternate Quadrilateral Snake graph $A(QS_m)$; $m \geq 8$ and $t \geq 2$; t is the number of quadrilaterals generated, admits Harmonic Mean Cordial labeling with the exception of the case when $m \equiv 0(mod2)$ and $t \equiv 1(mod2)$.*

Proof. Let $G = A(QS_n)$; $m \geq 8$ and $t \geq 2$; t is the number of quadrilaterals formed, be an alternate quadrilateral snake graph with a vertex set, $V = \{x_1, x_2, x_3, \dots, x_m, y_1, y_2, y_3, \dots, y_t, z_1, z_2, z_3, \dots, z_t\}$, where $x_1, x_2, x_3, \dots, x_m$ are the vertices on the path of $A(QS_n)$ and the vertices x_i and x_{i+1}, where $1 \leq i \leq n-1$ if n is even and $1 \leq i \leq n-2$ if n is odd, on the path are joined to the new vertices y_j and z_j and then, joining the vertices y_j and z_j; $1 \leq j \leq t$ alternately.

We note here that $|V(G)| = 2m$ if $m \equiv 0(mod2)$ and $|V(G)| = 2m - 1$ if $m \equiv 1(mod2)$. We also note that $|E(G)| = \frac{5m-2}{2}$ if $m \equiv 0(mod2)$ and $|E(G)| = \frac{5m-5}{2}$ if $m \equiv 1(mod2)$.

Define a labeling function $f: V(G) \rightarrow \{1,2\}$ as follows:

Case 1 : $m \equiv 0(mod\ 2)$ and $t \equiv 0(mod\ 2)$

$f(x_i) = \{2\ if\ \ 1 \leq i \leq \frac{m}{2}\ 1\ if\ \ \frac{m}{2} < i \leq m$

$f(y_i) = \{2\ if\ \ 1 \leq i \leq \frac{t}{2}\ 1\ if\ \ \frac{t}{2} < i \leq t$

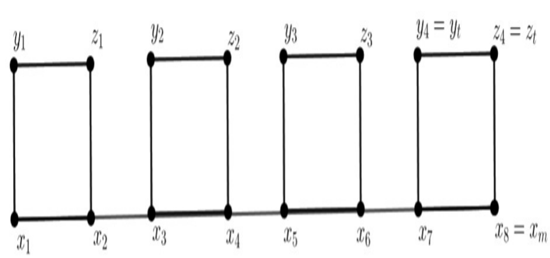
$f(z_i) = \{2\ if\ \ 1 \leq i \leq \frac{t}{2}\ 1\ if\ \ \frac{t}{2} < i \leq t$

Following this, $v_f(1) = \frac{m+t}{2} = v_f(2) \Longrightarrow |v_f(1) - v_f(2)| = 0$.

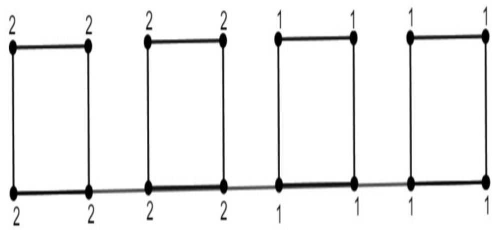
Furthermore, $e_f(1) = \frac{\frac{5m-2}{2}+1}{2}$ and $e_f(2) = \frac{\frac{5m-2}{2}-1}{2} \Longrightarrow |e_f(1) - e_f(2)| = 1$.

Thus, Harmonic Mean Cordial labeling is admitted for the Alternate Quadrilateral Snake graph $A(QS_m)$ Harmonic Mean Cordial labeling for $m \equiv 0(mod2)$ and $t \equiv 0(mod2)$.

Illustration 4.3.1 HMC labeling of $A(QS_8)$ is depicted in the following figures.



$A(QS_8)$



HMC labeling of $A(QS_8)$

Case 2 : $m \equiv 1(mod\ 2)$ and $t \equiv 0(mod\ 2)$

$f(x_i) = \{2\ if\ \ 1 \le i \le \frac{m+1}{2}\ 1\ if\ \ \frac{m+1}{2} < i \le m$

$f(y_i) = \{2\ if\ \ 1 \le i \le \frac{t}{2}\ 1\ if\ \ \frac{t}{2} < i \le t$

$f(z_i) = \{2\ if\ \ 1 \le i \le \frac{t}{2}\ 1\ if\ \ \frac{t}{2} < i \le t$

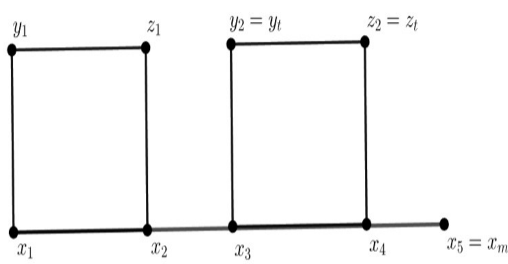
Then, $v_f(1) = \frac{m+2t+1}{2}$ and

$$v_f(2) = \frac{m + 2t - 1}{2} \Longrightarrow |v_f(1) - v_f(2)| = 1.$$

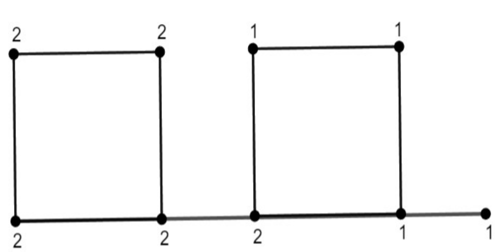
Furthermore, $e_f(1) = \frac{5(m-1)}{2} = e_f(2) \Longrightarrow |e_f(1) - e_f(2)| = 0.$

Thus, the Alternate Quadrilateral Snake graph $A(QS_m)$ admits Harmonic Mean Cordial labeling for $m \equiv 1(mod2)$ and $t \equiv 0(mod2)$.

Illustration 4.3.2 THe following figures illustrate HMC labeling of $A(QS_5)$.



$A(QS_5)$



HMC labeling of $A(QS_5)$

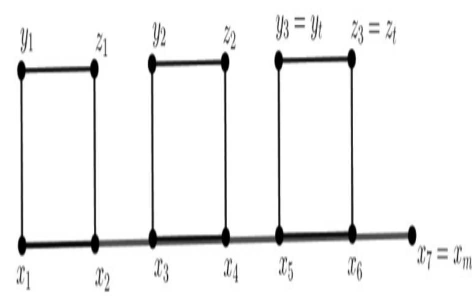
Case 3 : $m \equiv 1(mod\ 2)$ and $t \equiv 1(mod\ 2)$
$f(x_i) = \{2\ if\ \ 1 \leq i \leq \frac{m+1}{2}\ 1\ if\ \ \frac{m+1}{2} < i \leq m$

$f(y_i) = \{2\ if\ \ 1 \leq i \leq \frac{t+1}{2}\ 1\ if\ \ \frac{t+1}{2} < i \leq t$

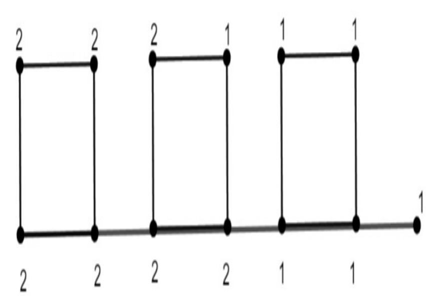
$f(z_i) = \{2\ if\ \ 1 \leq i \leq \frac{t-1}{2}\ 1\ if\ \ \frac{t-1}{2} < i \leq t$

Then, $v_f(1) = m - 1$ and $v_f(2) = m \Longrightarrow |v_f(1) - v_f(2)| = 1$.
Furthermore, $e_f(1) = m + 1$ and $e_f(2) = m \Longrightarrow |e_f(1) - e_f(2)| = 1$.
Hence, the Alternate Quadrilateral Snake graph $A(QS_m)$ admits Harmonic Mean Cordial labeling for $m \equiv 1(mod2)$ and $t \equiv 1(mod2)$.

Illustration 4.3.3 HMC labeling of $A(QS_7)$ is displayed in the following figures.



$A(QS_7)$



HMC labeling of $A(QS_7)$

Now, we prove for the case when the Alternate Quadrilateral Snake graph $A(QS_m)$; $m \geq 8$ and $t \geq 2$; t is the number of quadrilaterals created does not admit Harmonic Mean Cordial labeling, i.e., for the case when $m \equiv 0(mod2)$ and $t \equiv 1(mod2)$.

Once more, we observe in this instance, $|V(G)| = 2m - 1$ and $|E(G)| = \frac{5m-5}{2}$. Thus, we have, $v_f(1) = \frac{m+2t}{2} = v_f(2) \Longrightarrow |v_f(1) - v_f(2)| = 0$ which holds for the graph $A(QS_n)$ to be HMC in order to meet the criterion $|v_f(1) - v_f(2)| \leq 1$. Nevertheless, we still obtain at least $e_f(1) = \frac{m+2t}{2} + 2$ and $e_f(2) = \frac{m+2t}{2} \Longrightarrow |e_f(1) - e_f(2)| = 2 > 1$ if we use the best scenario to minimize $|e_f(1) - e_f(2)|$. Because of this, the Alternate

Quadrilateral Snake graph $A(QS_m)$; $m \geq 8$ and $t \geq 2$ is not Harmonic Mean Cordial for $m \equiv 0(mod2)$ and $t \equiv 1(mod2)$.

Hence, the Alternate Quadrilateral Snake graph $A(QS_m)$; $m \geq 8$ and $t \geq 2$; t is the number of quadrilaterals formed admits Harmonic Mean Cordial labeling except for the case when $m \equiv 0(mod2)$ and $t \equiv 1(mod2)$.

Theorem 4.4 *The Double Alternate Triangular Snake graph $DA(T_m)$; $m \geq 4$ and $t \geq 2$; t is the number of triangles formed in $DA(T_n)$ admits Harmonic Mean Cordial labeling except for except for the case when $m \equiv 0(mod2)$ and $t' \equiv 1(mod2)$, where $t' = \frac{t}{2}$.*

Proof. Let $G = DA(T_m)$; $m \geq 4$ and $t \geq 2$ be a double alternate triangular snake graph with a vertex set $V = \{x_1, x_2, x_3, \ldots, x_m, y_1, y_2, y_3, \ldots, y_t, z_1, z_2, z_3, \ldots, z_t\}$, where where $x_1, x_2, x_3, \ldots, x_n$ are the vertices on the path of DT_n joining each of x_i and x_{i+1} to a new vertex y_j in the upper half of $DA(T_n)$ and x_i and x_{i+1} joining to a new vertex z_j in the lower half of $DA(T_n)$ for $1 \leq i \leq n-1$ and $1 \leq j \leq t'$.

We note here that We note here that $|V(G)| = 2m$ if $m \equiv 0(mod2)$ and $|V(G)| = 2m-1$ if $m \equiv 1(mod2)$.Additionally, $|E(G)| = 3m-1$ if $m \equiv 0(mod2)$ and $|E(G)| = 3m-3$ if $m \equiv 1(mod2)$.

Define a labeling function $f: V(G) \rightarrow \{1,2\}$ as follows:

Case 1 : $m \equiv 0(mod\ 2)$ and $t' \equiv 0(mod\ 2)$

$f(x_i) = \{2\ if\ \ 1 \leq i \leq \frac{m}{2}\ 1\ if\ \ \frac{m}{2} < i \leq m$

$f(y_i) = \{2\ if\ \ 1 \leq i \leq \frac{t'}{2}\ 1\ if\ \ \frac{t'}{2} < i \leq t'$

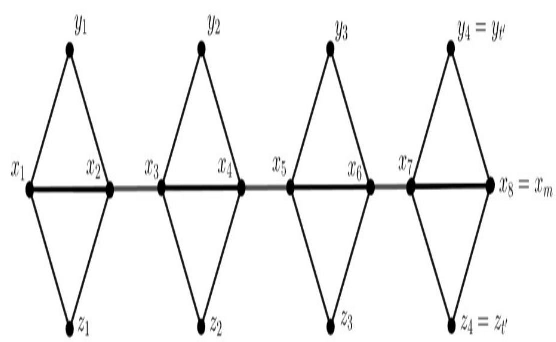
$f(z_i) = \{2\ if\ \ 1 \leq i \leq \frac{t'}{2}\ 1\ if\ \ \frac{t'}{2} < i \leq t'$

Then, $v_f(1) = \frac{m+t}{2} = v_f(2) \Longrightarrow |v_f(1) - v_f(2)| = 0$.

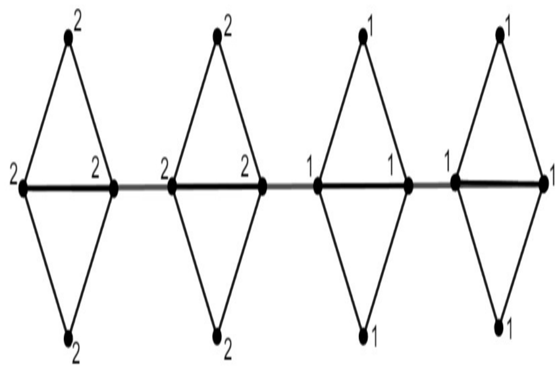
Furthermore, $e_f(1) = \frac{3m}{2}$ and $e_f(2) = \frac{3m}{2} - 1 \Longrightarrow |e_f(1) - e_f(2)| = 1$.

Hence, the Double Alternate Triangular Snake graph $DA(T_m)$; $m \equiv 0(mod2)$ and $t' \equiv 0(mod2)$ admits Harmonic Mean Cordial labeling.

Illustration 4.4.1 HMC labeling of $DA(T_8)$ is shown in the following figures.



$DA(T_8)$



HMC labeling of $DA(T_8)$

Case 2 : $m \equiv 1(mod\ 2)$ and $t' \equiv 0(mod\ 2)$

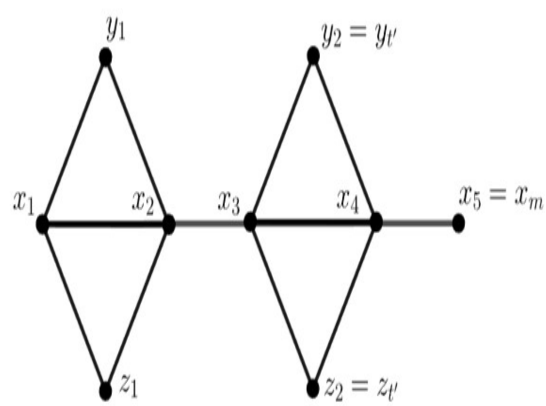
$f(x_i) = \{2\ if\ \ 1 \le i \le \frac{m+1}{2}\ 1\ if\ \ \frac{m+1}{2} < i \le m$

$f(y_i) = \{2\ if\ \ 1 \le i \le \frac{t'}{2}\ 1\ if\ \ \frac{t'}{2} < i \le t'$

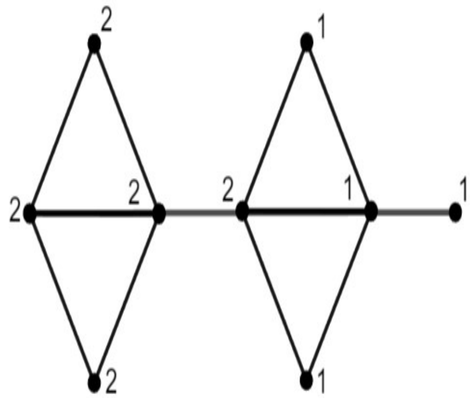
$f(z_i) = \{2\ if\ \ 1 \le i \le \frac{t'}{2}\ 1\ if\ \ \frac{t'}{2} < i \le t'$

Then, $v_f(1) = m - 1$ and $v_f(2) = m \Longrightarrow |v_f(1) - v_f(2)| = 1$.
Also, $e_f(1) = m + t' - 1 = e_f(2) \Longrightarrow |e_f(1) - e_f(2)| = 0$.
Hence, the Double Alternate Triangular Snake graph $DA(T_m)$; $m \equiv 1(mod2)$ and $t' \equiv 0(mod2)$ admits Harmonic Mean Cordial labeling.

Illustration 4.4.2 HMC labeling of $DA(T_5)$ is deoicted in the subsequent figures.



$DA(T_5)$



HMC labeling of $DA(T_5)$

Case 3 : $m \equiv 1(mod\ 2)$ and $t' \equiv 1(mod\ 2)$

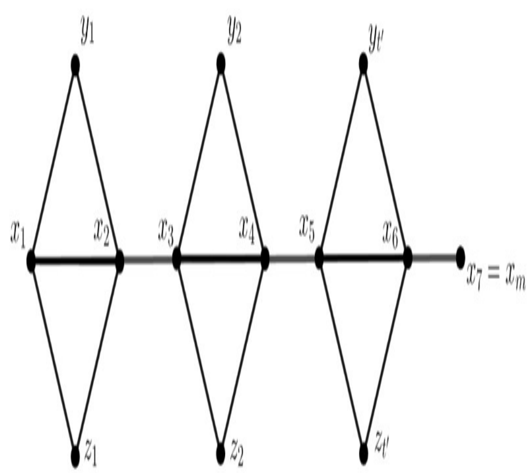
$f(x_i) = \{2\ if\ \ 1 \le i \le \frac{m+1}{2}\ 1\ if\ \ \frac{m+1}{2} < i \le m$

$f(y_i) = \{2\ if\ \ 1 \le i \le \frac{t'+1}{2}\ 1\ if\ \ \frac{t'+1}{2} < i \le t'$

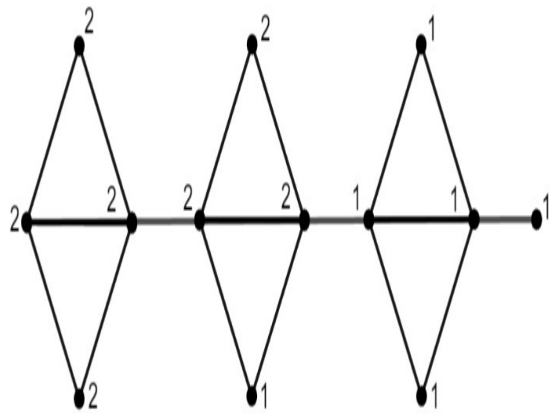
$f(z_i) = \{2\ if\ \ 1 \le i \le \frac{t'-1}{2}\ 1\ if\ \ \frac{t'-1}{2} < i \le t'$

Then, $v_f(1) = m - 1$ and $v_f(2) = m \Longrightarrow |v_f(1) - v_f(2)| = 1$.
Also, $e_f(1) = \frac{3(m-1)}{2} = e_f(2) \Longrightarrow |e_f(1) - e_f(2)| = 0$.
Hence, the Double Alternate Triangular Snake graph $DA(T_m)$; $m \equiv 1(mod2)$ and $t' \equiv 1(mod2)$ admits Harmonic Mean Cordial labeling.

Illustration 4.4.3 The following figures depict HMC labeling of $DA(T_7)$.



$DA(T_7)$



HMC labeling of $DA(T_7)$

Now, we prove for the case when the Double Alternate Triangular Snake graph $DA(T_m);\ m \geq 4$ and $t \geq 2$; t is the number of triangles formed in $DA(T_n)$ does not admit Harmonic Mean Cordial labeling, i.e., for the case when $m \equiv 0(mod2)$ and $t \equiv 1(mod2)$.

We again note here that for this case, $|V(G)| = 2m$ and $|E(G)| = 3m - 1$. So, to satisfy the condition $|v_f(1) - v_f(2)| \leq 1$, we get, $v_f(1) = \frac{m+2t'}{2} = v_f(2) \Longrightarrow$
$|v_f(1) - v_f(2)| = 0$ which holds for the graph $A(QS_n)$ to be HMC. But, if we apply the best possible situation to minimize $|e_f(1) - e_f(2)|$, we still get at least $e_f(1) = \frac{3m}{2} + 1$ and $e_f(2) = \frac{3m}{2} - 2 \Longrightarrow |e_f(1) - e_f(2)| = 3 > 1$. Hence, the Double Alternate Triangular Snake graph $DA(T_m);\ m \geq 4$ and $t \geq 2$ is not Harmonic Mean Cordial for $m \equiv 0(mod2)$ and $t' \equiv 1(mod2)$.

Hence, the Alternate Quadrilateral Snake graph $A(QS_m);\ m \geq 8$ and $t \geq 2$; t is the number of quadrilaterals formed admits Harmonic Mean Cordial labeling except for the case when $m \equiv 0(mod2)$ and $t \equiv 1(mod2)$.

Theorem 4.5 *Double Alternate Quadrilateral Snake graph* $DA(QS_n);\ n \geq 4$ *and* $t' \geq 2;\ t' = \frac{t}{2}$, *where* t *is the total number of quadrilateral formed in* $DA(QS_n)$ *admits Harmonic Mean Cordial labeling except for the cases :*

1) $m \equiv 0(mod\ 2)$ and $t' \equiv 1(mod\ 2)$ and 2) $m \equiv 1(mod\ 2)$ and $t' \equiv 1(mod\ 2)$.

Proof. Let $G = DA(QS_n);\ n \geq 4$ and $t' \geq 2;\ t' = \frac{t}{2}$, where t is the total number of quadrilateral formed in $DA(QS_n)$ be a double quadrilateral snake graph with a vertex set $V(G) = \{x_i, y_j, z_j, y'_j, z'_j / 1 \leq i \leq m, 1 \leq j \leq t'\}$.

We note that $|V(G)| = 3n$ if $m \equiv 0(mod2)$ and $|V(G)| = 3n - 2$ if $m \equiv 1(mod2)$. Also, $|E(G)| = 4n - 1$ if $m \equiv 0(mod2)$ and $|E(G)| = 4(n - 1)$ if $m \equiv 1(mod2)$.

Define a labeling function $f; V(G) \rightarrow \{1,2\}$ as follows:

Here, we already have, $t' = \frac{t}{2}$.

Case 1 : $m \equiv 0(mod\ 2)$ and $t' \equiv 0(mod\ 2)$

$f(x_i) = \{2\ if\ \ 1 \leq i \leq \frac{m}{2}\ 1\ if\ \ \frac{m}{2} < i \leq m$

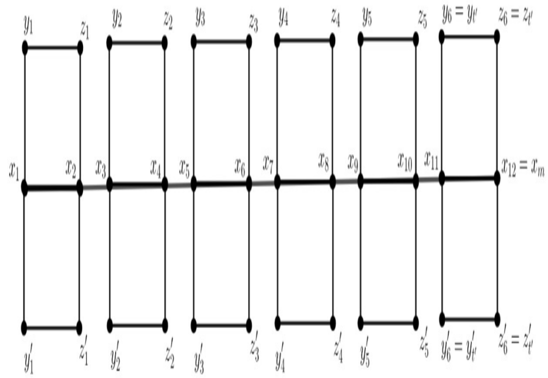
$f(y_i) = \{2\ if\ \ 1 \leq i \leq \frac{t'}{2}\ 1\ if\ \ \frac{t'}{2} < i \leq t'$

$f(z_i) = \{2\ if\ \ 1 \leq i \leq \frac{t'}{2}\ 1\ if\ \ \frac{t'}{2} < i \leq t'$

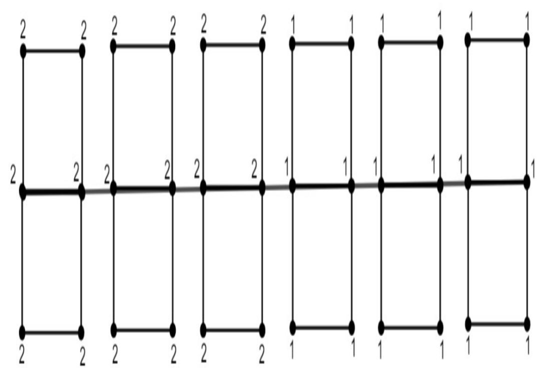
Then, we get, $v_f(1) = m + t' = v_f(2) \Longrightarrow |v_f(1) - v_f(2)| = 0$.
Also, $e_f(1) = 2m$ and $e_f(2) = 2m - 1 \Longrightarrow |e_f(1) - e_f(2)| = 1$.

Hence, the double alternate quadrilateral snake graph $DA(QS_n)$; $n \geq 4$ and $t' \geq 2$ admits Harmonic Mean Cordial labeling for $m \equiv 0(mod\ 2)$ and $t' \equiv 0(mod\ 2)$.

Illustration 4.5.1 HMC labeling of $DA(QS_{12})$ is shown in the following figures.



$DA(QS_{12})$



HMC labeling of $DA(QS_{12})$

Case 2 : $m \equiv 1(mod\ 2)$ and $t' \equiv 0(mod\ 2)$

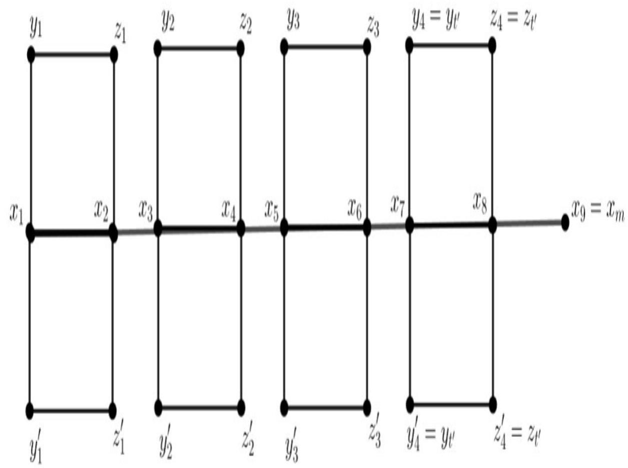
$f(x_i) = \{2\ if\ \ 1 \leq i \leq \frac{m+1}{2}\ 1\ if\ \ \frac{m+1}{2} < i \leq m$

$f(y_i) = \{2\ if\ \ 1 \leq i \leq \frac{t'}{2}\ 1\ if\ \ \frac{t'}{2} < i \leq t'$

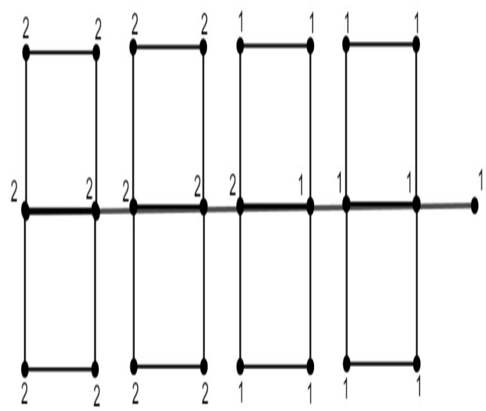
$f(z_i) = \{2\ if\ \ 1 \leq i \leq \frac{t'}{2}\ 1\ if\ \ \frac{t'}{2} < i \leq t'$

Then, we get, $v_f(1) = \frac{3n-1}{2}$ and $v_f(2) = \frac{3n-3}{2} \Longrightarrow |v_f(1) - v_f(2)| = 1$.
Also, $e_f(1) = 2(m-1) = e_f(2) \Longrightarrow |e_f(1) - e_f(2)| = 0$.
Hence, the double alternate quadrilateral snake graph $DA(QS_n)$; $n \geq 4$ and $t' \geq 2$ admits Harmonic Mean Cordial labeling for $m \equiv 1(mod\ 2)$ and $t' \equiv 0(mod\ 2)$.

Illustration 4.5.2 HMC labeling of $DA(QS_9)$ is displayed in the following figures.



$DA(QS_9)$



HMC labeling of $DA(QS_9)$

Now, we prove for the cases when the double alternate quadrilateral snake graph $DA(QS_n)$; $n \geq 4$ and $t' \geq 2$ and $t' = \frac{t}{2}$; t is the number of quadrilaterals formed in $DA(QS_n)$ does not admit Harmonic Mean Cordial labeling.

Case 1 : $m \equiv 0(mod\ 2)$ and $t' \equiv 1(mod\ 2)$

To satisfy the vertex condition of HMC labeling, we assign $\frac{3m}{2}$ number of vertices label 1. So, we will be labeling then the rest $\frac{3m}{2}$ vertices as label 2. This implies, $|v_f(1) - v_f(2)| = 0$. But, the vertices with label 2 will yield $e_f(1) = 2(n-1)$ and the vertices with label 1 will yield $e_f(2) = 2n + 1 \Longrightarrow |e_f(1) - e_f(2)| = 3 > 1$. Hence, the double alternate quadrilateral snake graph $DA(QS_n)$; $n \geq 4$ and $t' \geq 2$ does not admit Harmonic Mean Cordial labeling for $m \equiv 0(mod\ 2)$ and $t' \equiv 1(mod\ 2)$.

Case 2 : $m \equiv 1(mod\ 2)$ and $t' \equiv 1(mod\ 2)$

To satisfy the vertex condition of HMC labeling, we assign $\frac{3m-3}{2}$ number of vertices label 1. So, we will be labeling then the rest $\frac{3m-1}{2}$ vertices as label 2. This implies, $|v_f(1) - v_f(2)| = 1$. The vertices with label 2 will yield $e_f(1) = 2n - 1$ and the vertices with label 1 will yield $e_f(2) = 2n - 3 \Longrightarrow |e_f(1) - e_f(2)| = 2 > 1$. Hence, the double alternate quadrilateral snake graph $DA(QS_n);\ n \geq 4$ and $t' \geq 2$ does not admit Harmonic Mean Cordial labeling for $m \equiv 1(mod\ 2)$ and $t' \equiv 1(mod\ 2)$.

Hence, the Double Alternate Quadrilateral Snake graph $DA(QS_n);\ n \geq 4$ and $t' \geq 2;\ t' = \frac{t}{2}$, where t is the total number of quadrilateral formed in $DA(QS_n)$ admits Harmonic Mean Cordial labeling except for the cases :

1) $m \equiv 0(mod\ 2)$ and $t' \equiv 1(mod\ 2)$ and 2) $m \equiv 1(mod\ 2)$ and $t' \equiv 1(mod\ 2)$ respectively.

Theorem 4.6 *The Spiked Snake graph $SS(4,n)$ admits Harmonic Mean Cordial labeling for $n \geq 2$ except for $n \equiv 1(mod2)$.*

Proof. Let $G = SS(4,n);\ n \geq 2$ be a spiked snake graph with a vertex set, $V = \{x_i, y_j, z_j, p_j, q_j / 1 \leq i \leq n+1, 1 \leq j \leq n\}$. We note that $|V(G)| = 5n + 1$ and $|E(G)| = 6n$.

Define a labeling function $f: V(G) \to \{1,2\}$ as follows:

Consider $n \equiv 0(mod2)$.

$f(x_i) = \{2\ if\ \ 1 \leq i \leq \frac{n+2}{2}\ 1\ if\ \ \frac{n+2}{2} < i \leq n+1$

$$f(y_i) = f(z_i) = f(p_i) = f(q_i)$$

$= \{2\ if\ \ 1 \leq i \leq \frac{n}{2}\ 1\ if\ \ \frac{n}{2} < i \leq n$

Then, we get, $v_f(1) = \frac{5n}{2}$ and

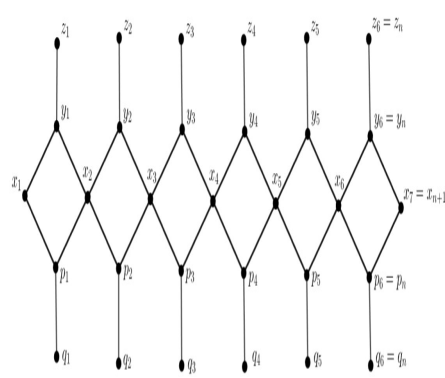
$v_f(2) = \frac{5n+2}{2} \Longrightarrow |v_f(1) - v_f(2)| = 1$.

Also, $e_f(1) = 3n = e_f(2) \Longrightarrow$

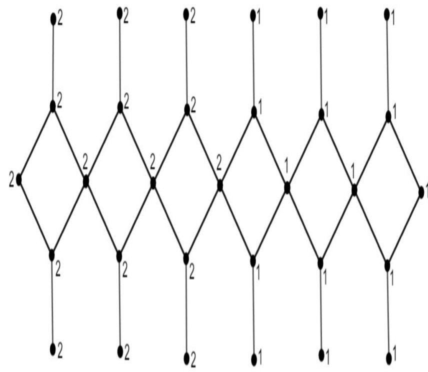
$$|e_f(1) - e_f(2)| = 0.$$

Hence, the Spiked Snake graph $SS(4,n);\ n \geq 2$ admits Harmonic Mean Cordial labeling for $n \equiv 0(mod2)$.

Illustration 4.6.1 HMC labeling of $SS(4,7)$ is displayed in the following figures.



$SS(4,7)$



HMC labeling of $SS(4,7)$

Now, we prove for the case when the spiked snake graph $SS(4,n)$ does not admit HMC labeling, i.e., for $n \equiv 1(mod2)$.

Here, we go through the best possible situation where we can minimize the condition for the number of vertices and edges labelled as 1 and 2 respectively.

$f(x_i) = \{2\ if\ \ 1 \le i \le \frac{n+1}{2}\ 1\ if\ \ \frac{n+1}{2} < i \le n+1$

$f(y_i) = f(z_i) = \{2\ if\ \ 1 \le i \le \frac{n+1}{2}\ 1\ if\ \ \frac{n+1}{2} < i \le n$

$f(p_i) = f(q_i) = \{2\ if\ \ 1 \le i \le \frac{n-1}{2}\ 1\ if\ \ \frac{n-1}{2} < i \le n$

Then, we get, $v_f(1) = \frac{5n+1}{2} = v_f(2) \Rightarrow |v_f(1) - v_f(2)| = 0$.

But, $e_f(1) = 3n+1$ and $e_f(2) = 3n-1 \Rightarrow |e_f(1) - e_f(2)| = 2 > 1$.

Hence, the Spiked Snake graph $SS(4,n)$; $n \ge 2$ does not admit Harmonic Mean Cordial labeling for $n \equiv 1(mod2)$.

Thus, the Spiked Snake graph $SS(4,n)$ admits Harmonic Mean Cordial labeling for $n \ge 2$ except for $n \equiv 1(mod2)$.

Theorem 4.7 *The Alternate Zig-Zag Triangular Snake graph* $AZ(T_n)$; $n \ge 7$ *admits Harmonic Mean Cordial labeling for the following cases except for the case when* $n \equiv 0(mod3)$ *:*

1) $n \equiv 1(mod2), n \equiv 2(mod3)$ and $t \equiv 1(mod2)$
2) $n \equiv 0(mod2), n \equiv 2(mod3)$ and $t \equiv 0(mod2)$
3) $n \equiv 1(mod2), n \equiv 1(mod3)$ and $t \equiv 0(mod2)$

Proof. Let $G = AZ(T_n)$; $n \ge 7$ be an alternate zig-zag triangular snake graph with a vertex set $V = \{x_i, y_j, v_j, u_j / 1 \le i \le n, 1 \le j \le t\}$. We note that $|V(G)| = n + 3t$ and

$|E(G)| = \{10t+1$ and $t \equiv 1(mod2) 10t + 1 if\ \ n \equiv 0(mod2), n \equiv 2(mod3)$ and $t \equiv 0(mod2) 10t if\ \ n \equiv 1(mod2), n \equiv 1(mod3)$ and $t \equiv 0(mod2)$.

Define a labeling function $f: V(G) \to \{1,2\}$ as follows:

Case 1 : $n \equiv 1(mod2), n \equiv 2(mod3)$ and $t \equiv 1(mod2)$

$f(x_1) = f(x_n) = 1.$

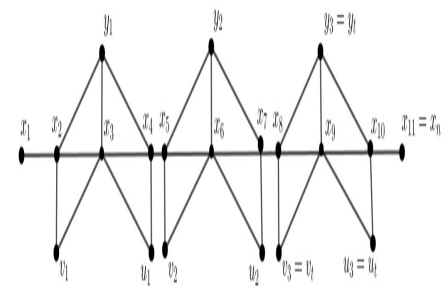
$f(x_i) = \{2\ if\ \ 2 \le i \le \frac{2n-t-5}{2}\ 1\ if\ \ \frac{2n-t-5}{2} < i \le n-1$

$f(y_i) = \{2\ if\ \ 1 \le i \le \frac{t+1}{2}\ 1\ if\ \ \frac{t+1}{2} < i \le t$

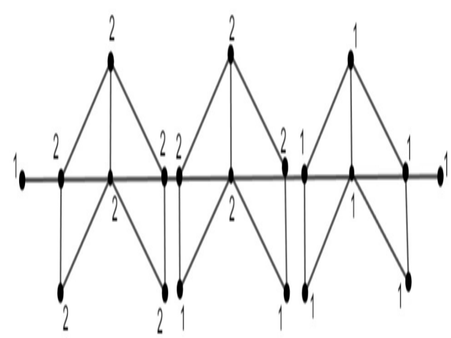
$f(v_i) = f(u_i) = \{2\ if\ \ 1 \le i \le \frac{t-1}{2}\ 1\ if\ \ \frac{t-1}{2} < i \le t$

Then, we get, $v_f(1) = \frac{n+3t}{2} = v_f(2) \Rightarrow |v_f(1) - v_f(2)| = 0$.
Also, $e_f(1) = \frac{10t+2}{2} = 5t + 1$ and $e_f(2) = \frac{10t}{2} = 5t \Rightarrow |e_f(1) - e_f(2)| = 1$.
Hence, the alternate zig-zag triangular snake graph $AZ(T_n)$; $n \geq 7$ admits Harmonic Mean Cordial labeling for $n \equiv 1(mod2), n \equiv 2(mod3)$ and $t \equiv 1(mod2)$.

Illustration 4.7.1 HMC labeling of AZT_{11} is shown in the following figures.



$AZ(T_{11})$



HMC labeling of $AZ(T_{11})$

Case 2 : $n \equiv 0(mod2), n \equiv 2(mod3)$ and $t \equiv 0(mod2)$

$$f(x_1) = 1$$
$$f(x_n) = 2.$$

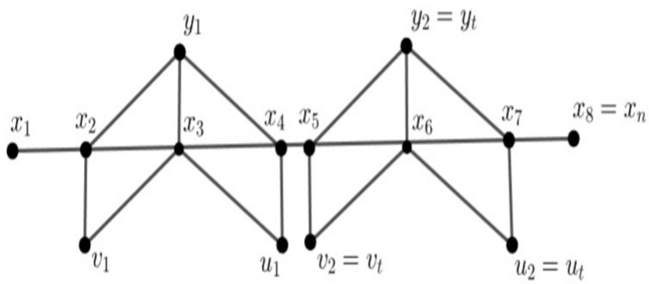
$$f(x_i) = \begin{cases} 2 & if \ 2 \leq i \leq \frac{n}{2} \\ 1 & if \ \frac{n}{2} < i \leq n-1 \end{cases}$$

$$f(y_i) = \begin{cases} 2 & if \ 1 \leq i \leq \frac{t}{2} \\ 1 & if \ \frac{t}{2} < i \leq t \end{cases}$$

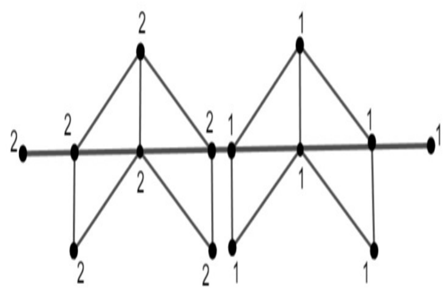
$$f(v_i) = f(u_i) = \begin{cases} 2 & if \ 1 \leq i \leq \frac{t}{2} \\ 1 & if \ \frac{t}{2} < i \leq t \end{cases}$$

Then, we get, $v_f(1) = n - 1 = v_f(2) \Rightarrow |v_f(1) - v_f(2)| = 0$.
Also, $e_f(1) = n$ and $e_f(2) = n - 1 \Rightarrow |e_f(1) - e_f(2)| = 1$.
Hence, the alternate zig-zag triangular snake graph $AZ(T_n)$; $n \geq 7$ admits Harmonic Mean Cordial labeling for $n \equiv 0(mod2), n \equiv 2(mod3)$ and $t \equiv 0(mod2)$.

Illustration 4.7.2 HMC labeling of $AZ(T_8)$ is shown in the following figures.



$AZ(T_8)$



HMC labeling of $AZ(T_8)$

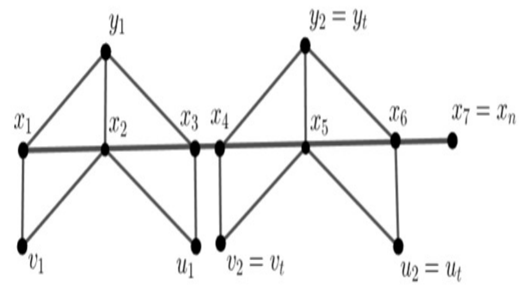
Case 3 : $n \equiv 1(mod2), n \equiv 1(mod3)$ and $t \equiv 0(mod2)$
$f(x_i) = \{2\ if\ \ 1 \le i \le \frac{n+1}{2}\ 1\ if\ \ \frac{n+1}{2} < i \le n$

$f(y_i) = \{2\ if\ \ 1 \le i \le \frac{t}{2}\ 1\ if\ \ \frac{t}{2} < i \le t$

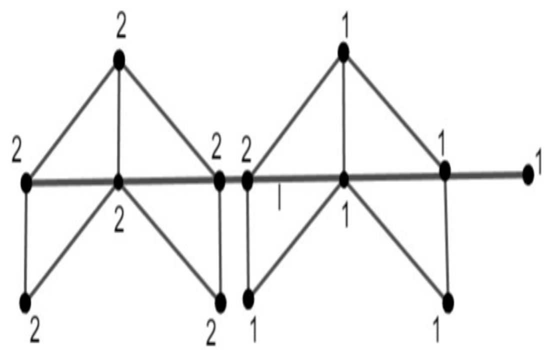
$f(v_i) = f(u_i) = \{2\ if\ \ 1 \le i \le \frac{t}{2}\ 1\ if\ \ \frac{t}{2} < i \le t$

Then, we get, $v_f(1) = n - 1$ and $v_f(2) = n \Longrightarrow |v_f(1) - v_f(2)| = 1$.
Also, $e_f(1) = 5t = e_f(2) \Longrightarrow |e_f(1) - e_f(2)| = 0$.
Hence, the alternate zig-zag triangular snake graph $AZ(T_n)$; $n \geq 7$ admits Harmonic Mean Cordial labeling for $n \equiv 1(mod2), n \equiv 1(mod3)$ and $t \equiv 0(mod2)$.

Illustration 4.7.3 HMC labeling of $AZ(T_7)$ is shown in the following figures.



$AZ(T_7)$



HMC labeling of $AZ(T_7)$

Now, we prove for the case when the alternate zig-zag triangular snake graph $AZ(T_n)$; $n \geq 7$ does not admit harmonic mean cordial labeling for $n \equiv 0(mod3)$.

We have here $|V(G)| = n + 3t$ which means we will have to label $\frac{n+3t}{2}$ number of vertices as 1 and the rest vertices as 2. This satisfies the vertex condition for $AZ(T_n)$; $n \geq 7$ to be HMC since, we get, $|v_f(1) - v_f(2)| = 0$. But, for this case, we have, $|E(G)| = 10t - 1$ which means we must have $\frac{10t-2}{2} = 5t - 1$ number of edges labeled as 2 and $\frac{10t}{2} = 5t$ number of edges labeled as 1 so as to minimize $|e_f(1) - e_f(2)|$ for the best possible situation for the graph to be HMC. But, what we get instead here is, $e_f(1) = 5t + 1$ and $e_f(2) = 5t - 2 \Longrightarrow |e_f(1) - e_f(2)| = 3 > 1$.

Hence, the alternate zig-zag triangular snake graph $AZ(T_n)$; $n \geq 7$ does not admit Harmonic Mean Cordial labeling for $n \equiv 0(mod3)$.

Thus, the Alternate Zig-Zag Triangular Snake graph $AZ(T_n)$; $n \geq 7$ admits Harmonic Mean Cordial labeling for the following cases except for the case when $n \equiv 0(mod3)$:

1) $n \equiv 1(mod2), n \equiv 2(mod3)$ and $t \equiv 1(mod2)$
2) $n \equiv 0(mod2), n \equiv 2(mod3)$ and $t \equiv 0(mod2)$
3) $n \equiv 1(mod2), n \equiv 1(mod3)$ and $t \equiv 0(mod2)$.

5. Conclusion

Within this manuscript, we showcased the Harmonic Mean Cordial labeling of multiple families of triangular and quadrilateral snake graphs, including double triangular, alternate and double alternate snake graphs, spiked and alternate zig-zag triangular versions. We have given illustrations of all HMC labeling outcomes for better understanding.

References

[1] Parejiya J., Jani D. B. and Hathi Y. M., *Harmonic Mean Cordial labeling of some graphs*, TWMS Journal Of Applied And Engineering Mathematics, 14(1), (2024), 284-297.
[2] J. Gowri and J. Jayapriya, *Hmc Labeling Of Certain Types Of Graphs*, 12(10), (2021), 3913-3915.
[3] S. Meena and M. Sivasakthi, *New results on harmonic mean graphs*, Malaya Journal of Matematik, Vol. S, No. 1, (2020), 482-486.
[4] Yesha M. Hathi, *et.al., HARMONIC MEAN CORDIAL LABELING OF SOME CYCLIC AND PATH RELATED GRAPHS*, Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition), Vol. 43, No.9, (2024), 26-45.
[5] K. Chitra Lakshmi, K. Nagarjan, *Geometric Mean Cordial Labeling of Graphs*, International Journal of Mathematics and Soft Computing, 7(1), (2017), 75-87.
[6] K.M. Baby Smitha and K. Thirusangu, *Distance Two Labeling of Certain Snake Graphs*,International Mathematical Forum,, 11(11), (2016), 503-502.
[7] I. Cahit, *Cordial graphs : A weaker version of Graceful and Harmonious Graphs*, Ars Combinatorial, 23(3), 1987, 201-207.
[8] 15. A. Rosa, *On certain valuations of the vertices of theory of graphs*, (Internat. Symposium, Rome, July 1966) Gordon and Breach, N. Y. and Dunod Paris (1967), 349-355.
[9] J. A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, 19, #DS6, 2012.
[10] J. Gross and J. Yellen, Graph Theory and its Applications (Second edition), Boca Raton : CRC press, 2006.
[11] S. K. Vaidya and N. B. Vyas, *Product cordial labeling for alternate snake graphs*,Malaya Journal of Matematik,,2(3), (2014), 188-196.
[12] R. Ponraj and M. Sivakumar, M. Sundaram, *Mean Cordial Labeling of Graphs*, Open Journal of Discrete Mathematics, 2(4), 145-148.
[13] A. Sugumaran and V. Mohan, *Difference cordial labeling of some graphs and related to fan graphs*, 12(04), (2019), 57-60.
[14] A.Sugumaran and V.Mohan, *Further results on prime cordial labeling*, Annals of Pure and Applied Mathematics, 14(3), (2017), 489 - 496.
[15] A. Sugumaran and V. Mohan, *Prime Cordial labeling of Some graphs related to H-Graphs*, Annals of Pure and Applied Mathematics, 16(1), 2018, 171-176.
[16] Sinjo James Varghese and Jeena B Edayadayil, *Dynamic Response of Radial, Fan and Harp Type of Cable Stayed Bridges Due to Moving Load and Earth Quake Load*, 3(29), 2015.
[17] J. Gowri and J. Jayapriya, *Hmc labeling of Some Snake Graphs*, International Journal of Special Education, 37(03), (2022), 4212-4219.
[18] Chitharalakshmi.K and Nagarajan.K, *Geometric mean cordial labeling of graphs*, International Journal of mathematics and soft computing, 07(01), (2017), 75–87.
[19] S.K. Vaidya and N.B. Vyas, *Product Cordial Labeling for alternate snake graphs*, Malaya Journal of Matematik, 02(03), (2014), 188-196.
[20] J.D Horton and I.Z Bouwer, *Symmetric Y-Graphs and H-Graphs*, Journal of Combinatorial theory, 53(01), September 1991, 114-129.
[21] Marta Hidalgo and Robert Joan-Arinyo, *h-graphs : A new representation of tree decompositions of graphs*, Computer-Aided, Design, 67(68), October 2015, 38-47.
[22] S. S. Sandhya, S. Somasundaram and R. Ponraj, *Some Results on Harmonic Mean Graphs*, Int. J. Contemp. Math. Sciences, 7(04), 2012, 197 - 208.
[23] Ali N. A. Koam and Ali Ahmad, *Barycentric subdivision of Cayley graphs and Constant Edge Metric Dimension*, IEEE Access, Vol. 8, (2020), 80624-80628.
[24] S. K. Vaidya and N. H. Shah,*Some Star and Bistar related Divisor Cordial Graphs*, 3(01), (2013), 67-77.
[25] V. Yegnanaryanan and P. Vaidhyanathan, *Some interesting applications of graph labellings*, J. Math. Comput. Sci., 2(5), (2012), 1522-1531.
[26] S. K. Vaidya and N. H. Shah, *Prime cordial labeling of some graphs*, Open Journal of Discrete Mathematics, 2, (2012), 11-16.
[27] S. K. Vaidya and P. L. Vihol, *Prime cordial labeling for some graphs*, Modern Applied Science, 4(8), (2010), 119-126.
[28] Raja Ponraj, Muthirulan Sivakumar and Murugesan Sundaram, *Mean Cordial Labeling of Graphs*, 2, 2012, 145-148.
[29] R. Ponraj and M. Sivakumar, *On MEan Cordial Graphs*, Math. Combin. Book, 3, (2013), 78-84.
[30] S. Meena and M. Sivasakthi, *Harmonic Mean Labeling Of Subdivision Graphs*, 6(1), 196-204.

[31] S.S. Sandhya, S.Somasundaram and R.Ponraj ,*Some Results On Harmonic Mean graphs*,Internation Journal Of Comtemporary Mathematical Sciences, 7(4),(2012), 197-208.
[32] S.S.Sandhya, C. Jayasekaran and C. David raj ,*Some New Families Of Harmonic Mean Graphs*, International Journal Of Mathematical Research ,5(1), (2013), 223-232.
[33] Dharamvirsinh Parmar and Shah Pratik,*I - Cordial labeling of Triangular Snake Graphs*, An International referred and Peer Reviewed Journal of Multidisciplinary Research ,6(2), (2019), 49-59.
[34] Dr. S. Meena and K.Vaithilingam,*Prime Labeling For Some Fan Related Graphs*, International Journal of Engineering Research and Technology (IJERT) ,1(9), (2012), 1-19.
[35] Juan Manuel García-Guerrero and Juan José Jorquera-Lucerga,*Improving the Structural Behavior of Tied-ArchBridges by Doubling the Set of Hangers*, Applied Sciences, (2020), 1-23.
[36] J. Jeba Jesintha, D. Devakirubanithi and Jasma Pershi. E. A,*New Classes of Difference Cordial Graphs*, Journal of Information and Computational Science, (2023), 636-643.
[37] S. K. Vaidya , N. A. Dani , K. K. Kanani and P. L. Vihol,*Cordial and 3-equitable Labeling for Some Wheel Related Graphs*, IAENG International Journal of Applied Mathematics, 41(2), (2011).