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Parametric Set Mathematical Programming Problem Over Cone Using P-K-Locally Connected Set Maps

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Abstract. In this paper, we investigate a set-valued mathematical programming problem with parameters (PSVP), where set-valued functions are used for the constraints and objective functions. Set valued ρ-locally connected maps over cone (ρ-k-LC) are introduced and using these maps and contingent epiderivatives, the Karush-Kuhn-Tucker criteria of sufficiency is investigated for the above-mentioned problem. Lastly, using the same presumptions, the Mond-Weir dual is developed and duality theorems are demonstrated.
Primary 26B25; Secondary 49N15

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1. **Introduction**

Parametric optimization is a fascinating area of study. It involves optimization problems that depend on a parameter, and researchers have explored various aspects of these problems, including how the feasible set, value function, and minimizers change with the parameter. In the context of set order relations, researchers have delved into several key areas as seen in the referred papers [2, 4, 6, 7, 8].

Locally connected functions and their generalizations play a significant role in optimization theory. These functions [5] were introduced in 1988. Later researchers [9] extended these functions to cone locally connected set-valued maps and their generalisations. They also proved the optimality results under these maps.

Recently Das and Samei [2] have worked on set-valued ρ cone arc-wise connected maps and proved sufficiency conditions for parametric set mathematical problems under contingent epiderivative and the said function. The main objective of this paper is to introduce and employ a set-valued ρ-cone locally connected map to further generalise optimality and duality results for (PSVP) under contingent epiderivative assumption.

This paper reviews some basic definitions and concepts concerned with our study. We define a set-valued ρ-locally connected map over a cone called a set-valued ρ-K locally connected (ρ-K-LC) map. An example illustrates a case where a set map is ρ-K-LC however it is non-K-LC. Further, one property of ρ-K-LC is proved which is very useful to demonstrate the optimality and duality results in the paper. In
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the next Section, we proposed a set-valued parametric mathematical programming over cone (PSVP). Next is the Karush-Kuhn-Tucker criteria of sufficiency under ρ-K-LC and contingent epiderivative assumptions for (PSVP). The last section is devoted to duality models of Mond-Weir kind. We investigate the duality result under the same assumptions.

2. Background and Definitions

Suppose K is a closed convex pointed cone in Y with a non-empty interior where Y is a real normed space. The positive dual cone K^+ is the set of all those points y of Y for which the inner product of y with x is non-negative for all x of K. If this product is positive for all x of $K \setminus \{0\}$ then it is called a strict positive dual cone K^{s+} of K. The interior of K is denoted by int(K).

Let $\langle y,V\rangle \subseteq R$ is the union of all the sets of the form $\langle y,v\rangle$ where v is an element of V and V is a proper subset of Y . Let W_1 and W_2 be any two subsets of Y . For $y^*\in Y$, the following notations will be used:

(i) $\langle y^*, W_1\rangle \geq 0$ (or $0 \leq \langle y^*, W_1\rangle$) $\Leftrightarrow \langle y^*,w\rangle \geq 0,\ \forall w \in W_1$;
(ii) $\langle y^*, W_1\rangle \geq \langle y^*, W_2\rangle$ (or $\langle y^*, W_2\rangle \leq \langle y^*, W_1\rangle$) $\Leftrightarrow \langle y^*,w_1\rangle \geq \langle y^*,w_2\rangle,\ \forall w_1 \in W_1$ and $\forall w_2 \in W_2$,
(iii) $W_1 \leq 0_Y \Leftrightarrow w \leq 0_Y, \forall w \in W_1$
(iv) $W_1 < 0_Y \Leftrightarrow w < 0_Y, \forall w \in W_1$
(v) $W_1 \leq W_2 \Leftrightarrow w_1 \leq w_2,\ \forall w_1 \in W_1,\ w_2 \in W_2$

The (v) inequality above still holds if there is ’<’ instead of ’≤’.
For a closed convex pointed cone K in Y , we have the following two types of cone orderings in Y .

(i) $y_1 \quad y_2 \Leftrightarrow y_2 - y_1 \in \quad , \quad y_1, y_2 \in$
(ii) $y_1 < y_2 \Leftrightarrow y_2 - y_1 \in \operatorname{int} K,\ \forall\ y_1, y_2 \in Y$
The next definition is of contingent cone given by Aubin [1].

Definition 2.1. [1] *If $C \subseteq Y$ is non-empty,then$T(C,z^*)$ represents the contingent cone to C at a point $z^*$of C given as*
$T(C,z^*) = \{z \in Y : \exists z_n \to z^*,\ z_n \in C,\ p_n > 0,\ n \to \infty$ suchthat $p_n(z_n \to z^*) \to z\}$.
Suppose $F : X \to 2^Y$ where X and Y are real normed spaces and K having a nonempty interior is a pointed closed and convex cone in Y .
The definitions mentioned below will be used in this paper.
Let *dom F* be the set of all those points x in X for which $F(x)$ is non-empty, gr F is the set of points (x,y) where x belonging to X and y belonging to $F(x)$ and epi(F) is the set of points (x,y) where x belonging to X and y belonging to $F(x)$ +K.
Here *domF* stands for domain of F, gr$F$ stands for graph of F and epi$F$ stands for epigraph of F.

Definition 2.2. [3] *For a point (x^*,y^*) that is part of grF. Contingent epiderivative of F at (x^*,y^*) is the map $DF(x^*,y^*) : X \to Y$ whose epigraph is the contingent cone to epiF at (x^*,y^*), DF is a contingent epiderivative if the contingent epiderivative of F exist at every point in grF.*
We now concentrate on the idea of set-valued maps over cones that are locally connected.

Definition 2.3. *A locally connected (LC) set is a subset $S \subseteq X$ where for each $x,x^* \in S$, we have $H_{x^*,x}(\lambda)$ from $[0,1]$ to S with a maximum number $a(x^*,x)$ belongs to (0,1], so that $H_{x^*,x}(\lambda)$ belongs to S where λ lies between 0 and $a(x^*,x)$. In addition $H_{x^*,x}(\lambda)$ is continuous in $(0,a(x^*,x))$, $H_{x^*,x}(\lambda)$ is equal to $x^*$when $\lambda = 0$ and equal to x when $\lambda = 1$.*

Definition 2.4. [9] *If $\exists$a positive number $b(x^*,x) \leq a(x^*x)$ satisfying the following condition for each $x \in S$ (having an associated function $H_{x^*,x}(\lambda)$ along with a maximum number $a(x^*,x)$ as in definition (2.3)) then map F is called as K-locally connected (K-LC) at $x^*\in S$.*
$(1 - \lambda)F(x^*) + \lambda F(x) \subseteq F(H_{x^*,x}(\lambda)) + K,\ 0 < \lambda < b(x^*,x)$
F is called a K-locally connected set-valued map (K-LC) on S if $S \subseteq X$ is LC and at each $x^*\in S$, the map F is K-LC.
We now introduce ρ-K-locally connected set-valued map.

Definition 2.5. *If $\exists$a positive number $b(x^*,x) \leq a(x^*x)$ and $\rho \in R$ satisfying the below condition for each $x \in S$ (having an associated function $H_{x^*,x}(\lambda)$ along with a maximum number $a(x^*,x)$ as in definition (2.3)) then map F is called as $\rho - K$-locally connected $(\rho - K - LC)$ at $x^*\in S$ pertaining to γ.*
$(1-\lambda)F(x^*)+\lambda F(x) \subseteq F(H_{x^*,x}(\lambda))+\rho\lambda(1-\lambda)\|x - x^*\|^2\gamma+K,\ 0 < \lambda < b(x^*,x)$, *where* $\gamma \in \operatorname{int}(K)$.
When F is ρ-K-locally connected (ρ-K-LC) map at each $x^*\in S$ and S is LC subset of X, F is referred to be ρ-K-LC set-valued map pertaining to γ on S.

Remark. (i) *As defined in* [2], *F becomes ρ-K-arc-wise connected set map pertaining to γ on S If* $b(x^*,x) = a(x^*,x) = 1$ *at all $x^*,x \in S$.*
(ii) *If ρ = 0, then F becomes K-locally connected set-valued map as defined in*
[9].
(iii) *F is considered strongly ρ-K-locally connected if ρ| > 0, and weakly ρ-Klocally connected if ρ < 0. Every strongly ρ-K-locally connected map is weakly ρ-K − locallyconnectedbecauseeverystronglyρ-k-locally connected is k-locally connected and every k-locally connected is weakly ρ-k locally connected.*
Next, we take the case where a set map is ρ-K-LC however, it is non K-LC.

Example 2.6. *Let $X = \{(x_1,x_2) : x_1 > 0,\ x_2 > 0\}$, Y is taken as R, K is taken as R_+ and* $S = \{(x_1, x_2) : x_1^2 + x_2^2 \geq 1,\ x_1 > 0,\ x_2 > 0,\ x_1 \neq x_2\}$.
$H_{x^*,x}$is expressed as
$H_{x^*,x}(\lambda) = ((1 - \lambda){x_1^*}^2 + \lambda x_1^2)^{\frac{1}{2}}, ((1 - \lambda){x_2^*}^2 + \lambda x_2^2)^{\frac{1}{2}})$,

where $x^* = (x_1^*, x_2^*),\ x = (x_1, x_2)$. *Clearly, S ⊆X will be LC.*
Define F :X → 2^Y as

$$F(x_1,x_2) = \begin{cases} [0,2], & \textit{if } (x_1,x_2) \in S,\ x_1 > x_2 \\ [2,3], & \textit{if } (x_1,x_2) \in S,\ x_1 < x_2 \\ [1,4], & \textit{if } (x_1,x_2) \notin S. \end{cases}$$

Obviously F is not K-locally connected on S because for $x^*= (1,2)$, $x = (2,1)$, $\lambda = 1/2$

$$(1-\lambda)F(x^*) + \lambda F(x) = \left[1, \frac{5}{2}\right]$$

$$H_{x^*,x}(\lambda) = \left(\frac{5^{\frac{1}{2}}}{2}, \frac{5^{\frac{1}{2}}}{2}\right)$$

$$F(H_{x^*,x}(\lambda)) = [1,4]$$

Therefore
$(1-\lambda)F(x^*) + \lambda F(x) \nsubseteq F(H_{x^*,x}(\lambda)) + R_+$.
So F is not K-LC for (1,2) *and* (2,1).
But by taking ρ *=-1 and* $\gamma = 3$, *F is ρ-K-LC corresponding to* $\gamma = 3$ *on S for* (1,2) *and* (2,1).

Theorem 2.7. *Suppose F is ρ-K-LC on a LC subset S pertaining to γ,* $x^* \in S$ *and* $y^* \in F(x^*)$. *Prove that*
$$F(x) - y^* \subseteq DF(x^*,y^*)(H^*_{x^*,x}(0^+) + \rho\|x - x^*\|^2\gamma + K, \quad \forall\ x \in S,$$
where

$$H^*_{x^*,x}(0^+) = \lim_{\lambda\to 0^+} \frac{H_{x^*,x}(\lambda) - H_{x^*,x}(0)}{\lambda}$$

$\gamma \in$ int*K and assuming that for all* $x, x^* \in S,\ H^*_{x^*,x}(0^+)$ *holds.*

Proof. As F is ρ -K-LC pertaining to γ, so ∃ a positive number $b(x^*,x) \le a(x^*,x)$ and $\rho \in R$ satisfying the below condition for each $x \in S$ (having a associated function $H_{x^*,x}(\lambda)$ along with maximum number $a(x^*,x)$ as in definition (2.3))
$(1-\lambda)F(x^*)+\lambda F(x) \subseteq F(H_{x^*,x}(\lambda))+\rho\lambda(1-\lambda)\|x - x^*\|^2\gamma+K,$ for $0 < \lambda < b(x^*,x)$.
Let y be an element of $F(x)$. Let us take a sequence $\{\lambda_n\}$ in R where λ_n belongs to open interval 0 to 1, $n \in N$ and λ_n approaches to 0+ when $n \to \infty$. Taking $x_n = H_{x^*,x}(\lambda_n)$ and $y_n = (1-\lambda_n)y^* + \lambda_n y - \rho\lambda_n(1-\lambda_n)\|x - x^*\|^2\gamma$, where $y^* \in F(x^*)$.
Therefore $y_n \in F(x_n) + K$. Also
$xn = Hx*,x(\lambda n) \to Hx*,x(0) = x*$
and $y_n \to y^*$, when $n \to \infty$.
Now
$$\frac{x_n - x^*}{\lambda_n} = \frac{H_{x^*,x}(_n) - H_{x^*,x}(o)}{\lambda_n} \to H_{x^*,x}(0^+),$$
when $\lambda_n \to 0+$, and
$$\frac{y_n - y^*}{\lambda_n} = y - y^* - \rho(1-\lambda_n)\|x - x^*\|^2\gamma$$
$$\to y - y^* - \rho\|x - x^*\|^2\gamma, \qquad \text{when } \lambda_n \to 0+.$$
Thus
$$(H^*_{x^*,x}(0^+), y - y^* - \rho\|x - x^*\|^2\gamma) \in T(epi(F),(x^*,y^*)) = epi(DF(x^*,y^*)).$$
As a result
$$y - y^* - \rho\|x - x^*\|^2\gamma \in DF(x^*,y^*)(H^*_{x^*,x}(0^+)) + K.$$
It is valid at every point y belonging to $F(x)$. Hence
$$F(x) - y^* \subseteq DF(x^*,y^*)(H^*_{x^*,x}(0^+)) + \rho\|x - x^*\|^2\gamma + K, \quad \forall\ x \in S. \square$$

3. Optimality Conditions
Below-mentioned parametric set-valued optimization problem (PSVP) is examined in this paper:
$$\min_{(x,a\ \in S\times A)} F(x,a)$$
having Inequality constraint: $G(x,a) \cap (-K_2) \ne \phi$
and equality constraint $p(x,a) = 0$
Here $S \subseteq X$ is non-empty, X is a real normed space, x is the variable and A is an arbitrary set of parameters where $a \in A$. $F : X \times A \to 2^{Y_1}$, $G : X \times A \to 2^{Y_2}$ and $p : X \times A \to Y_3$ with $S \times A \subseteq \text{dom}(F) \cap \text{dom}(G)$. Y_1, Y_2 and Y_3 are taken as real normed spaces and K_1, K_2 and K_3 as closed pointed convex cones in Y_1, Y_2 and Y_3, respectively.
$X^* = \{(x,a) \in S \times A : G(x,a) \cap (-K_2) \ne \phi \text{ and } p(x,a) = 0\}$.
is the feasible set of the above (PSVP) problem.

The following types of solutions to the above problem are examined in this paper.

Definition 3.1. *A point* $(x^*, a^*, y_1^*) \in X \times A \times Y_1$*, with* $(x^*,a^*) \in X^*$ *and* $y_1^* \in F(x^*,a^*)$ *is called minimal of (PSVP) on a condition that point* $(x,a,y_1) \in X \times A \times Y_1$ *does not exist where* $(x,a) \in X^*$ *and* $y_1 \in F(x,a)$*, for which*
$y_1 - y_1^* \in K_1 \setminus \{0_{Y_1}\}$
and is called weak minimal of (PSVP), on a condition that a point $(x,a,y_1) \in X \times A \times Y_1$ *does not exist where* $(x,a) \in X^*$ *and* $y_1 \in F(x,a)$*, for which*
$y_1 - y_1^* \in -\mathrm{int}K_1$.
In this work, the below presumptions are valid:
If $(x,a), (x^0,a), (x,a^0), (x^0,a^0) \in X \times A$, $y_1^0 \in F(x^0,a^0)$ and $y_2^0 \in G(x^0,a^0)$, then

$$\left.\begin{array}{ll} \text{(i)} & F(x,a) - F(x^0,a) \subseteq K_1, \\ \text{(ii)} & G(x,a) - G(x^0,a) \subseteq K_2, \\ \text{(iii)} & F(x,a^0) - y_1^0 \subseteq -K_1, \\ \text{(iv)} & G(x,a^0) - y_2^0 \subseteq -K_2, \\ \text{(v)} & p(x,a^0) + p(x^0,a) \in -K_3. \end{array}\right\} \tag{3.1}$$

Our next objective is to obtain KKT optimality sufficiency results for the problem (PSVP).
In the following results $p'(\cdot,a^0)(x^0)$ denotes the Gateaux derivative.

Lemma 3.2. *Suppose* $S \subseteq X$ *is LC together with* $(x^*,a^*) \in X \times A$ *and* $y_1^0 \in F(x^0,a^0)$, $y_2^0 \in G(x^0,a^0)$ *and* $p(x^0,a^0) \geq 0$. *Let* $e_1 \in \mathrm{int}K_1$, $e_2 \in \mathrm{int}K_2$ *and* $e_3 \in \mathrm{int}K_3$. *Suppose that* $F(\cdot,a^0) : X \to 2^{Y_1}$ *is* ρ_1*-*K_1*-LC with respect to* e_1, $G(\cdot,a^0) : X \to 2^{Y_2}$ *is* ρ_2*-*K_2*-LC with respect to* e_2, *and* $p(\cdot,a^0) : X \to Y_3$ *is* ρ_3*-*K_3*LC with respect to* e_3 *on S.*
Assume that contingent epiderivatives $DF(\cdot,a^0)(x^0,y_1^0)$ *and* $DG(\cdot,a^0)(x^0,y_2^0)$ *exist together with* $p'(\cdot,a^0)(x^0)$. *In case* (3.1) *is met, the below inequality is valid*

$$\begin{aligned} &\langle y_1^*, F(x,a) - y_1^0\rangle + \langle y_2^*, G(x,a) - y_2^0\rangle \\ &\quad \geq \langle y_1^*, DF(\cdot,a^0)(x^0,y_1^0)(H^*_{x^0,x}(0^+)) + F(x^0,a) - y_1^0\rangle \\ &\qquad + \langle y_2^* DG(\cdot,a^0)(x^0,y_2^0)(H^*_{x^0,x}(0^+) + G(x^0,a) - y_2^0\rangle \\ &\qquad + \langle y_3^*, p'(\cdot,a^0)(x^0)(H^*_{x^0,x}(0^+)) + p(x^0,a)\rangle \\ &\qquad + \|x - x^0\|^2(\rho_1\langle y_1^*, e_1\rangle + \rho_2\langle y_2^*, e_2\rangle + \rho_3\langle y_3^*, e_2\rangle), \quad \forall\ (x,A) \in S \times A. \quad (3.2) \end{aligned}$$

Proof. Let $(x,a) \in S \times A$. As $F(\cdot,a^0) : X \to 2^{Y_1}$ is ρ_1-K-LC with respect to e_1 on S and $y_1^0 \in F(x^0,a^0)$, so we obtain the following from Theorem 2.7

$$F(x,a^0) - y_1^0 \subseteq DF(\cdot,a^0)(x^0,y_1^0)(H_{x^*0,x}(0^+)) + \rho_1\|x - x^0\|^2 e_1 + K_1, \qquad \forall x \in S. (3.3)$$

Again as $G(\cdot,a^0) : X \to 2^{Y_2}$ is ρ_2-K-LC with respect to e_2 on S and $y_2^0 \in G(x^0,a^0)$,
therefore by Theorem 2.7, we have

$$G(x,a^0) - y_2^0 \subseteq DG(\cdot,a^0)(x^0,y_2^0)(H_{x^*0,x}(0^+) + \rho_2\|x - x^0\|^2 e_2 + K_2, \qquad \forall x \in S. (3.4)$$

Again as $p(\cdot,a^0) : X \to Y_3$ is $P_3 - K - LC$ with respect to e_3 on S, we have

$$p(x,a^0) - p(x^0,a^0) \in p'(\cdot,a^0)(x^0)(H^*_{x^0,x}(0^+) + \rho_3\|x - x^0\|^2 e_3 + K_3. \tag{3.5}$$

From equation(3.3), we have $y_1^*(F(x,a^0) - y_1^0) + y_1^*(F(x^0,a) - y_1^0) - y_1^*(F(x^0,a) - y_1^0)$
$- y_1^* DF(\cdot,a^0)(x^0,y_1^0)(H_{x^*0,x}(0^+) - y_1^*\rho_1\|x - x^0\|^2 e_1 \geq 0,$ where $y_1^* \in K_1^+$.

From equation (3.4), we have $y_2^*(G(x,a^0) - y_2^0) + y_2^*(G(x^0,a) - y_2^0) - y_2^*(G(x^0,a) - y_2^0)$
$- y_2^* DG(\cdot,a^0)(x^0,y_2^0)(H_{x^*0,x}(0^+) - y_2^*\rho_2\|x - x^0\|^2 e_2 \geq 0,$ where $y_2^* \in K_2^+$.

From equation (3.5), we have
$y_3^*(p(x,a^0) - p(x^0,a^0) + p(x^0,a)) - y_3^*(p'(\cdot,a^0)(x^0)(H_{x^0,x}(0^+) + p(x^0,a))$
$- y_3^*\rho_3\|x - x^0\|^2 e_3 \geq 0$, for some $y_3^* \in K_3^+$.

That is
$\langle y_1^*, F(x,a^0) - y_1^0 + F(x^0,a) - y_1^0\rangle + \langle y_2^*, G(x,a^0) - y_2^0$
$+ G(x^0,a) - y_2^0\rangle + \langle y_3^*, p(x,a^0) - p(x^0,a^0) + p(x^0,a)\rangle \geq \langle y_1^*, DF(\cdot,a^0)(x^0,y_1^0)(H_{x^*0,x}(0^+) + F(x^0,a) - y_1^0\rangle$
$+ \langle y_2^*, DG(\cdot,a^0)(x^0,y_2^0)(H_{x^*0,x}(0^+) + G(x^0,a) - y_2^0\rangle$
$+ \langle y_3^*, p'(\cdot,a^0)(x^0)(H_x0_{x,}(0^+) + p(x^0,a)\rangle$
$+ \|x - x^0\|^2(\rho_1\langle y_1^*, e_1\rangle + \rho_2\langle y_2^*, e_2\rangle + \rho_3\langle y_3^*, e_3\rangle)$.(3.6)

By equations (3.1)(i),
$y_1^*(F(x,a) - F(x^0,a)) \geq 0$
$\Rightarrow \quad y_1^*F(x,a) \geq y_1^*F(x^0,a)$
$\Rightarrow \quad y_1^*(F(x,a) - y_1^0) \geq y_1^*(F(x^0,a) - y_1^0)$
Again from equation (3.1)(i)
$y_2^*(G(x,a) - y_2^0) \geq y_2^0(G(x^0,a) - y_2^0)$.
From equation (3.1)(iii), we have
$y_1^*(F(x,a^0) - y_1^0) \leq 0$
and
$y_2^*(G(x,a^0) - y_2^0) \leq 0$.
From equations (3.1)(v), we have
$y_3^*(p(x,a^0) + p(x^0,a)) \leq 0$.
Also, we have $p(x^0,a^0) \geq 0$. Adding all these inequalities, we get $y_1^*(F(x,a) - y_1^0) + y_2^*(G(x,a) - y_2^0)$
$\geq y_1^*(F(x^0,a) - y_1^0) + y_2^*(G(x^0,a) - y_2^0) + y_1^*(F(x,a^0) - y_1^0) + y_2^*(G(x,a^0) - y_2^0) + y_3^*(p(x,a^0) + p(x^0,a) - p(x^0,a^0))$.

That is
$\langle y_1^*, F(x,a) - y_1^0\rangle + \langle y_2^*, G(x,a) - y_2^0\rangle$
$\geq \langle y_1^*, F(x,a^0) - y_1^0 + F(x^0,a) - y_1^0\rangle$
$+ \langle y_2^*, G(x,a^0) - y_2^0 + G(x^0,a) - y_2^0\rangle$
$+ \langle y_3^*, p(x,a^0) - p(x^0,a^0) + p(x^0,a)\rangle$ (3.7)

(3.6) and (3.7) give
$\langle y_1^*, F(x,a) - y_1^0\rangle + \langle y_2^*, G(x,a) - y_2^0\rangle$
$\geq \langle y_1^*, DF(\cdot,a^0)(x^0,y_1^0)(H_{x^*0,x}(0^+) + F(x^0,a) - y_1^0\rangle$
$+ \langle y_2^*, DG(\cdot,a^0)(x^0,y_2^0)(H_{x^*0,x}(0^+) + G(x^0,a) - y_2^0\rangle$
$+ \langle y_3^*, p'(\cdot,a^0)(x^0)(H_{x^*0,x}(0^+)) + p(x^0,a)\rangle$
$+ \|x - x^0\|^2(\rho_1\langle y_1^*, e_1\rangle + \rho_2\langle y_2^*, e_2\rangle + \rho_3\langle y_3^*, e_3\rangle)$.
Hence proved. □

We now demonstrate (PSVP) problem's sufficiency conditions beneath the ρ-KLC and contingent epi-derivatives presumptions.

Theorem 3.3. *Suppose $S \subseteq X$ is LC and $(x^0,a^0) \in X \times A$, with $(x^0,a^0) \in X^*$, $y_1^0 \in F(x^0,x^0)$, $y_2^0 \in G(x^0,a^0) \cap (-K_2)$ and $p(x^0,a^0) \geq 0$. Let $e_1 \in int(K_1)$, $e_2 \in int(K_2)$, and $e_3 \in int(K_3)$. Suppose that $F(\cdot,a^0) : X \to 2^{Y1}$ is ρ_1-K_1-LC with respect to e_1, $G(\cdot,a^0) : X \to 2^{Y2}$ is ρ_2-K_2-LC with respect to e_2, and $p(\cdot,a^0) : X \to Y_3$ is ρ_3-K_3-LC with respect to e_3 on S. Also, suppose that $DF(\cdot,a^0)(x^0,y_1^0)$ and $DG(\cdot,a^0)(x^0,y_2^0)$ exist together with $p'(\cdot,a^0)(x^0)$. Let's assume that the condition of Lemma (3.3) is valid at $(x^0,a^0,y_1^0,y_2^0,y_3^0,y_1^*,y_2^*,y_3^*)$ for some $(y_1^*,y_2^*,y_3^*) \in K_1^+ \times K_2^+ \times K_3^+$ with $y_1^* \neq 0_{Y1}$ such that*
$\langle y_1^*, DF(\cdot,a^0)(x^0,y_1^0)(H_{x^*0,x}(0^+)) + F(x^0,a) - y_1^0\rangle$
$+ \langle y_2^*, DG(\cdot,a^0)(x^0,y_2^0)(H_{x^*0,x}(0^+)) + G(x^0,a) - y_2^0\rangle$
$+ \langle y_3^*, p'(\cdot,a^0)(x^0)(H_{x^*0,x}(0^+)) + p(x^0,a)\rangle \geq 0, \quad \forall (\cdot,a) \in S \times A$ (3.8) *and*
$\langle y_2^*, y_2^0\rangle = 0$, (3.9)
then (x^0,a^0,y_1^0) is a (PSVP's) weak minimal provided
$\rho_1\langle y_1^*, e_1\rangle + \rho_2\langle y_2^*, e_2\rangle + \rho_3\langle y_3^*, e_3\rangle \geq 0$.(3.10)

Proof. Let if possible the problem (PSVP) doesn't have (x^0,a^0,y_1^0) as its weak minimal. This give $(x,a) \in X^*$and $y_1 \in F(x,a)$ so that $y_1 - y_1^0 \in -intK_1$.
As $y_1^* \in K_1^+/\{0_{Y_1}\}$, so
$\langle y_1^*, y_1 - y_1^0\rangle < 0$.
As $(x,a) \in X^*$, therefore there exist $y_2 \in G(x,a) \cap (-K_2)$, so
$\langle y_2^*, y_2\rangle \leq 0$, as $y_2^* \in K_2^+$.
Now
$\langle y2*, y2 - y20\rangle = \langle y2*, y2\rangle - \langle y2*, y20\rangle$

$= \langle y_2^*, y_2\rangle$ (as $\langle y_2^*, y_2^0\rangle = 0) \leq 0$.
Therefore,
$\langle y_1^*, y_1 - y_1^0\rangle + \langle y_2^*, y_2 - y_2^0\rangle < 0$. (3.11)
Since at ($x^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*$) Lemma 3.2 is valid, therefore the equations (3.2), (3.9) and (3.10) give
$\langle y_1^*, F(x,a) - y_1^0\rangle + \langle y_2^*, G(x,a) - y_2^0\rangle \geq 0$.
Hence
$\langle y_1^*, y_1 - y_1^0\rangle + \langle y_2^*, y_2 - y_2^0\rangle \geq 0$,
which runs counter to (3.11). Hence (PSVP) has a weak minimal $\langle x^0, a^0, y_1^0\rangle$. □ 4. Duality
The above study leads us to correlate the primal problem (PSVP) with a belowmentioned dual of Mond-Weir type.
(PD) $\max y_1^0$,
w.r.t$\langle y_1^*, DF(\cdot,a^0)(u^0,y_1^0)(H_{u^*0,u}(0^+)) + F(u^0,a) - y_1^0\rangle$
$+ \langle y_2^*, DG(\cdot,a^0)(u^0,y_2^0)(H_{u^*0,u}(0^+) + G(u^0,a) - y_2^0\rangle + \langle y_3^*, p'(\cdot,a^0)(u^0)(H_{u^*0,u}(0^+)) + p(u^0,a)\rangle \geq 0$,
$\forall\ (u^0,u) \in S \times A$, (4.1)
$\langle y_2^*, y_2^0\rangle \geq 0$, (4.2)
where $u^0 \in S,\ a^0 \in A,\ y_1^0 \in F(u^0,a^0),\ y_2^0 \in G(u^0,a^0),\ p(u^0,a^0) \geq 0,\ (y_1^*, y_2^*, y_3^*) \in K_1^+ \times K_2^+ + K_3^+$ and $\langle y_1^*, e_1\rangle = 1$.
A problem (PD) has ($u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*$) as a feasible point if it is true for all the constraints of (PD).

Definition 4.1. *if there is no feasible point* $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$ *of (PD) such that* $y_1 - y_1^0 \in \operatorname{int} K_1$ *then a feasible point* $(u,a,y_1,y_2,y_1,y_2,y_3)$ *of the problem (PD) is referred to as the weak maximal of (PD).*

Theorem 4.2 (Weak duality)**.** *Suppose* $S \subseteq X$ *is LC. Let* $(u^*,a^*) \in X^*$ *and the problem (PD) has* $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$ *as its feasible point. Suppose that*
$F(\cdot,a^0) : X \to 2^{Y_1}$ *is* ρ_1-K_1-*LC concerning* e_1, $G(\cdot,a^0) : X \to 2^{Y_2}$ *is* ρ_2-K_2-*LC with respect to* e_2 *and* $p(\cdot,a^0) : X \to Y_3$ *is* ρ_3-K_3-*LC with regards to* e_3 *on S.*
Further suppose that the $DF(\cdot,a^0)(u^0,y_1^0)$ *and* $DG(\cdot,a^0)(u^0,y_2^0)$ *exist together with* $p'(\cdot,a^0)(u^0)$. *Assume further that the circumstances stated in Lemma 3.2 are true at* $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$. *Then*
$F(u^*,a^*) - y_1^0 \subseteq Y_1 \setminus -\mathrm{int}K_1$,
provided
$\rho_1(y_1^*, e_1) + \rho_2(y_2^*, e_2) + \rho_3(y_3^*, e_3) \geq 0$. (4.3)
Proof. Let if possible for some $y \in F(u^*,a^*),\ y - y_1^0 \in -\mathrm{int}K_1$.
Then $\langle y_1^*, y - y_1^0\rangle < 0$ (as $0_{Y_1} \neq y_1^* \in K_1^+$).
Again, since $(u^*,a^*) \in X^*$, we have $G(u^*,a^*) \cap (-K_2) \neq \phi$.
Let $y_2^1 \in G(u^*,a^*) \cap (-K_2)$.
Then $\langle y_2^*, y_2^1\rangle \leq 0$ (as $y_2^* \in K_2^+$).
Therefore, the above equation together with (4.2) gives
$\langle y_2^*, y_2^1 - y_2^0\rangle = \langle y_2^*, y_2^1\rangle - \langle y_2^*, y_2^0\rangle \leq 0$.
Hence
$\langle y_1^*, y - y_1^0\rangle + \langle y_2^*, y_2^1 - y_2^0\rangle < 0$. (4.4)
Since at ($u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*$), Lemma 3.3 is valid, therefore from equations (3.2), (4.1) and (4.3) we have
$\langle y_1^*, F(u^*,a^*) - y_1^0\rangle + \langle y_2^*, G(u^*,a^*) - y_2^0\rangle \geq 0$.
Hence $\langle y_1^*, y - y_1^0\rangle + \langle y_2^*, y_2^1 - y_2^0\rangle \geq 0$, which contradicts (4.4). Thus
$F(u^*,a^*) - y_1^0 \subseteq Y_1 \setminus -\mathrm{int}(K_1)$.
□
Theorem 4.3 (Strong Duality)**.** *Suppose* $S \subseteq X$ *is a LC, let* (u^0, a^0, y_1^0) *be a weak minimal of (PSVP) and* $y_2^0 \in G(u^0,a^0) \cap (-K_2)$. *Assume that for some*
$(y_1^*, y_2^*, y_3^*) \in K_1^+ \times K_2^+ \times K_3^+$,
with $\langle y_1^*, e_1\rangle = 1$. *Equations* (3.8) *and* (3.9) *are satisfied at the point* $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$ *Then (PD) has a a feasible solution* $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^0)$. *Moreover, if for each feasible point of (PD), the hypothesis of weak Duality Theorem holds, then (PD) has a weak maximal* $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$.
Proof. As at the point ($u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*$), the equations (3.8) and (3.9) are true, therefore
$\langle y_1^*, DF(\cdot,a^0)(u^0,y_1^0)(H_{u^*0,u}(0^+)) + F(u^0,a) - y_1^0\rangle$
$+ \langle y_2^*, DG(\cdot,a^0)(u^0,y_2^0)(H_{u^*0,u}(0^+) + G(u^0,a) - y_2^0\rangle$

$+ \langle y_3^*, p'(\cdot,a^0)(u^0)(H_{u^*0,u}(0^+) + p(u^0,a)\rangle \geq 0, \quad \forall (u,a) \in S \times A,$ and
$\langle y_2^*, y_2^0 \rangle = 0$.

Also $p(u^0,a^0) = 0$ as $(u^0,a^0) \in X^*$. Hence the equality (4.1) has the feasible solution $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$. Let's presume that the weak duality Theorem 4.2 is satisfied but (PD) doesn't have the point $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$ as weak maximal. Let $(\overline{u}, \overline{a}, \overline{y}_1, y_2, y_1^*, y_2^*, y_3^*)$ satisfy all the constraints of (PD) such that $y_1^0 - y_1 \in -int\mathrm{K}_1$.

The weak duality theorem is in conflict with it. As a result (PD) has a weak maximal $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$. □

Theorem 4.4 (Converse duality)**.** *Let $S \subseteq X$ be a LC, $p(u^0,a^0) \geq 0$ and $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$ satisfy all the constraints of (PD). Suppose $e_1 \in \mathrm{int}K_1$, $e_2 \in \mathrm{int}K_2$ and $e_3 \in \mathrm{int}K_3$. Suppose that $F(\cdot,a^0) : X \to 2^{Y1}$ is ρ_1-K_1-LC with respect to e_1, $G(\cdot,a^0) : X \to 2^{Y2}$ is ρ_2-K_2-LC with respect to e_2, and $p(\cdot,a^0) : X \to Y_3$ is ρ_3-K_3-LC with regards to e_3 on S. Assume additionally that the $p'(\cdot,a^0)(u^0)$ exists together with $DF(\cdot,a^0)(u^0,y_1^0)$ and $DG(\cdot,a^0)(u^0,y_2^0)$. Further presume that the Lemma 3.2 is true at $(u^0, a^0, y_1^0, y_2^0,$
$y_1^*, y_2^*, y_3^*)$ and (4.3) is satisfied. Then (PSVP) has a weak minimal (u^0, a^0, y_1^0) if $(u^0,a^0) \in X^*$.*

Proof. Let if possible (u^0, a^0, y_1^0) be a non weak minimal of (PSVP). This implies that the $(u,a) \in X^*$along with $y_1 \in F(u,a)$ give$y_1 < y_1^0$.

Now,
$\langle y_1^*, y_1 - y_1^0 \rangle < 0, \text{ as } y_1^* \in K_1^+ \setminus \{0_{Y1}\}$ (4.5)

Also as $(u^0,a^0) \in X^*$so there exists $y_2 \in G(u,a) \cap (-K_2)$.

This gives$\langle y_2^*, y_2 \rangle \leq 0$ as $y_2^* \in K_2^+$.

Using the dual feasibility of $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$, we have
$\langle y_2^*, y_2^0 \rangle \geq 0$.

Therefore
$\langle y_2^*, y_2 - y_2^0 \rangle = \langle y_2^*, y_2 \rangle - \langle y_2^*, y_2^0 \rangle \leq 0$. (4.6)

Adding (4.5) and (4.6), we get
$\langle y_1^*, y_1 - y_1^0 \rangle + \langle y_2^*, y_2 - y_2^0 \rangle < 0$. (4.7)

As the Lemma 3.2 is staisfied at $(u^0, a^0, y_1^0, y_2^0, y_1^*, y_2^*, y_3^*)$.

So from (3.2), (4.3) and dual feasibility of this point, we get
$\langle y_1^*, F(u,a) - y_1^0 \rangle + \langle y_2^*, G(u,a) - y_2^0 \rangle \geq 0$.

Thus
$\langle y_1^*, y_1 - y_1^0 \rangle + \langle y_2^*, y_2 - y_2^0 \rangle \geq 0$,

which does not satisfy (4.7). As a result (PSVP) has a weak minimal (u^0, a^0, y_1^0) .□

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