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Edge Irregularity Strength of Graphs on Mean Labeling

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Abstract

Let $G=(V,E)$ be a connected graph of order $n>1$. Let $f:V\to\{1,2,...,k\}$ be a function and let the weight of an edge $e=uv$ be defined by $\omega(e)=\left\lceil\frac{f(u)+f(v)}{2}\right\rceil$. Then, f is called an edge irregular mean labeling, if all the edge weights are distinct. The edge irregularity strength $s_m(G)$ is the smallest positive integer k such that there is an edge irregular mean labeling $f:V\to\{1,2,...,k\}$. In this paper, we determine the exact value of edge irregularity strength of some classes of graphs.

Keywords: Graph Theory, Graph Labeling, Irregularity Strength of Graphs

1 Introduction

Let $G=(V,E)$ be a connected graph of order $n \geq 2$. Let $f:E\to\{1,2,...,k\}$ be a function and let the weight of a vertex v be defined by $\omega(v)=\sum_{e\in v} f(e)$. Then, f is called an irregular labeling, if all the vertex weights are distinct. The irregularity strength $s(G)$ is the smallest positive integer k such that there is an irregular labeling $f:E\to\{1,2,...,k\}$.

The irregularity strength of a graph was introduced by Chartrand et al. [1], and the irregularity strength of many graphs was determined in [1], e.g., $s(K_n)=3, n\geq 3$. Further, the irregularity strength of a graph was studied by numerous authors see [2, 3, 4, 6, 7, 8, 9]. Two more characteristics, namely total vertex irregularity strength and total edge irregularity strength were introduced by Baca et al. [5]. The mean labeling was introduced by Somasundaram et al. [10]. In this paper, we define a new concept, edge irregularity strength of graphs on mean labeling and we determine the exact value of edge irregularity strength of some classes of graphs.

Definition 1.1. *Let $G=(V,E)$ be a connected graph of order $n>1$. Let $f:V\to\{1,2,...,k\}$ be a function and let the weight of an edge $e=uv$ be defined by $\omega(e)=\left\lceil\frac{f(u)+f(v)}{2}\right\rceil$. Then, f is called an edge* *irregular mean labeling, if all the edge weights are distinct. The edge irregularity strength* $s_m(G)$ *is the smallest positive integer* k *such that there is an edge irregular mean labeling* $f : V \to \{1, 2, ..., k\}$.

Example 1.2. *An optimal edge irregular mean labeling of* P_5 *is shown in Fig.1. Hence* $s_m(P_5) = 4$.

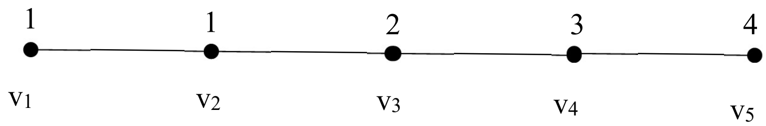

Figure 1

2 Edge irregularity strength of some classes of graphs on mean labeling

Theorem 2.1. *Let* $P_n, n > 1$ *be the path on* n *vertices. Then* $s_m(P_n) = n - 1$.

Proof. Let $P_n = (v_1, e_1, v_2, e_2, ..., e_{n-1}, v_n)$.

Define the labeling $f : V(P_n) \to \{1, 2, ..., n-1\}$ as follows:

$f(v_1) = 1,$

$f(v_i) = i - 1$ for $i = 2, 3, ..., n$.

By the above labeling, $s_m(P_n) \le n - 1$. Since P_n has $n-1$ edges the largest weight of the edge is at least $n-1$ and the incident vertices of that edge must have the label at least n or $n-1$. Hence, the vertices of P_n cannot be labeled with the labels $< n-1$. Therefore, $s_m(P_n) \ge n-1$. Hence, $s_m(P_n) = n-1$.

Theorem 2.2. *Let* $C_n, n > 2$ *be the cycle on* n *vertices. Then,* $s_m(C_n) = \begin{cases} 4 & for \quad n = 3 \\ n & for \quad n \ge 4 \end{cases}$.

Proof. Let $C_n = (v_1, e_1, v_2, e_2, v_3 ..., v_n, e_n, v_1)$.

For $n = 3$, we define the labeling as in the following diagram.

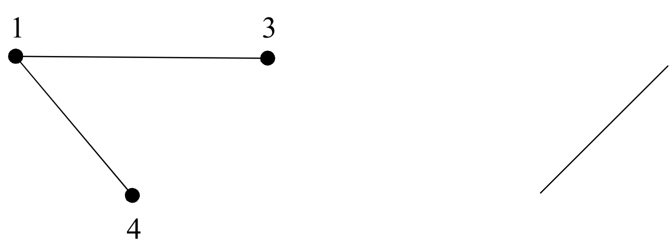

Figure 2

Clearly the weight of the edges of C_3 can be 2, 3 and 4. So the vertices of C_3 cannot be labeled with the labels less than 4: Hence $s_m(C_3) = 4$.

For $n > 3$, we define the labeling $f : V(C_n) \to \{1, 2, ..., n\}$ as follows.

$f(v_1)=1$

If n is odd, then

$$f(v_i)=\begin{cases} i-1 & for \quad 2\le i\le \left\lceil \frac{n}{2}\right\rceil -1 \\ i & for \quad \left\lceil \frac{n}{2}\right\rceil \le i\le n \end{cases}$$

If n is even, then

$$f(v_i)=\begin{cases} i-1 & for \quad 2\le i\le \frac{n}{2} \\ i & for \quad \frac{n}{2}+1\le i\le n \end{cases}$$

By the above labeling, $s_m(C_n)\le n$ for $n\ge 4$. Since C_n has n edges, the largest weight of the edge is at least n and the incident vertices of that edge must have the label at least n or $n-1$. Hence, the vertices of $C_n, n\ge 4$ cannot be labeled with the labels $<n-1$. Therefore, $s_m(C_n)\ge n$. Hence, $s_m(C_n)=n$ for $n\ge 4$

.**Theorem 2.3**. *Let $K_{1,n}, n\ge 1$ be the star graph on $n+1$ vertices. Then, $s_m(K_{1,n})=2n-2$.*

Proof. Let v be the center vertex and $v_i, i=1,2,3,...n$ be the pendant vertices of $K_{1,n}$.

We define the labeling $f:V(K_{1,n})\to\{1,2,3,...,2n-2\}$ as follows.

$f(v)=1, f(v_1)=1$ and

$f(v_i)=2i-2$ for $2\le i\le n$.

By the above labeling, $s_m(K_{1,n})\le 2n-2$. Since $K_{1,n}$ has n edges, the weight of the edges can be $1,2,3,...,n$. Therefore the vertices of $K_{1,n}$ cannot be labeled with the labels $<2n-2$. Therefore, $s_m(K_{1,n})\ge 2n-2$. Hence, $s_m(K_{1,n})=2n-2$.

Definition 2.4. *A bistar graph $B_{m,n}, m,n\ge 1$ is a graph obtained by joining the center vertices of two star graphs $K_{1,m}$ and $K_{1,n}$ by an edge.*

Theorem 2.5. *Let $B_{m,n}, m,n\ge 1$ be the bistar graph on $m+n+2$ vertices.*

Then,

$$s_m(B_{m,n})=\begin{cases} 2m & if \quad m>n \\ 2m+1 & if \quad m=n \end{cases}$$

Proof. Let $B_{m,n}$ be the bistar graph with $m+n+2$ vertices and $m+n+1$ edges.

Let u be the center vertex and $u_i, i=1,2,3,...m$ be the pendant vertices of $K_{1,m}$ and let v be the center vertex and $v_i, i=1,2,3,...n$ be the pendant vertices of $K_{1,n}$.

The edge set of $B_{m,n}$ is $\{uu_1,uu_2,...,uu_m,vv_1,vv_2,...,vv_n,uv\}$.

Without loss of generality let $m\ge n$. If $m<n$, we interchange $K_{1,m}$ and $K_{1,n}$.

We define the labeling $f:V(B_{m,n})\to\{1,2,3...,2m+1\}$ as follows.

$$f(u) = 1, f(u_1) = 1$$

$f(u_i) = 2i - 2$ for $2 \le i \le m$.

$f(v) = 2m$

$f(v_i) = 2i + 1$ for $1 \le i \le n$.

By the above labeling, the maximum labeling is $2m$ if $m > n$ and $2m+1$ if $m = n$. Hence

$$s_m(B_{m,n}) \le \begin{cases} 2m & if \quad m > n \\ 2m+1 & if \quad m = n \end{cases}$$

Since $B_{m,n}$ has $m+n+1$ edges, the weight of the edges can be $1, 2, 3, \ldots m+n+1$. Therefore the vertices of $B_{m,n}$ cannot be labeled with the labels $< 2m+1$.

Therefore,

$$s_m(B_{m,n}) \ge \begin{cases} 2m & if \quad m > n \\ 2m+1 & if \quad m = n \end{cases}$$

Hence,

$$s_m(B_{m,n}) = \begin{cases} 2m & if \quad m > n \\ 2m+1 & if \quad m = n \end{cases}$$

Definition 2.6. *The vertex corona product of two graphs G_1 and G_2 denoted by $G_1 \circ G_2$ is the graph obtained by taking one copy of G_1 which has n vertices and n copies of G_2 and then joining i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.*

Theorem 2.7. *Let P_n be the path on n vertices. Then $s_m(P_n \circ K_1) = 2n-1$.*

Proof. Let $G = P_n \circ K_1$.

Let $V(G) = \{u_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n\}$ and $E(G) = \{e_i = u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i : 1 \le i \le n\}$.

Define the labeling $f : V(G) \to \{1, 2, 3 \ldots, 2n-1\}$ as follows.

For $1 \le i \le n$,

$$f(u_i) = \begin{cases} 2i-1 & if \quad i \quad is\ odd \\ 2i-2 & if \quad i \quad is\ even \end{cases}$$

$$f(v_i) = 2i - 1$$

Since the weights of all the edges of G are distinct and f is an irregular mean labeling, $s_m(G) \le 2n-1$. The graph G has $2n-1$ edges and the weight of the edges are $1, 2, 3 \ldots 2n-1$. So we cannot label the vertices of G with fever than $2n-1$. Therefore $s_m(G) \ge 2n-1$. Hence $s_m(G) = 2n-1$.

Theorem 2.8. *Let P_n be the path on n vertices and $m \ge 2$. Then $s_m(P_n \circ mK_1) = n(m+1) - 1$.*

Proof. Let $G = P_n \circ mK_1$.

Let $V(G) = \{u_i : 1 \le i \le n\} \cup \{v_i^j : 1 \le i \le n, 1 \le j \le m\}$ and $E(G) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i^j : 1 \le j \le m\}$.

Define the labeling $f : V(G) \to \{1, 2, 3 ..., n(m+1)-1\}$ as follows.

Case 1. If n is even

$f(u_1) = 1$

For $2 \le i \le n$,

$$f(u_i) = \begin{cases} (m+1)i - m + 1 & if \quad i \quad is\ odd \\ (m+1)i - 2 & if \quad i \quad is\ even \end{cases}$$

For $1 \le j \le m$ and $i = 1$,

$f(v_1^j) = 2j - 1$

For $1 \le j \le m$ and $2 \le i \le n$,

$$f(v_i^j) = \begin{cases} 2j + (m+1)i - (m+4) & if \quad i \quad is\ odd \\ 2j + (m+1)i - (2m+1) & if \quad i \quad is\ even \end{cases}$$

Case 2. If n is odd and $m = 2, 3$

$f(u_1) = 1$

For $2 \le i \le n$,

$$f(u_i) = \begin{cases} (m+1)i - m + 1 & if \quad i \quad is\ odd \\ (m+1)i - 2 & if \quad i \quad is\ even \end{cases}$$

For $1 \le j \le m$ and $i = 1$,

$f(v_1^j) = 2j - 1$

For $1 \le j \le m$ and $2 \le i \le n$,

$$f(v_i^j) = \begin{cases} 2j + (m+1)i - (m+4) & if \quad i \quad is\ odd \\ 2j + (m+1)i - (2m+1) & if \quad i \quad is\ even \end{cases}$$

Case 3. If n is odd and $m > 3$

$f(u_1) = 1$

For $2 \le i \le n-1$,

$$f(u_i) = \begin{cases} (m+1)i - m + 1 & if \quad i \quad is\ odd \\ (m+1)i - 2 & if \quad i \quad is\ even \end{cases}$$

and

$$f(u_n) = \begin{cases} n(m+1) - 2 & if \quad m \quad is\ odd \\ n(m+1) - 1 & if \quad m \quad is\ even \end{cases}$$

For $1 \le j \le m$ and $i = 1$,

$f(v_1^j) = 2j - 1$

For $1 \le j \le m$ and $2 \le i \le n-1$,

$$f(v_i^j) = \begin{cases} 2j + (m+1)i - (m+4) & if \quad i \quad is\ odd \\ 2j + (m+1)i - (2m+1) & if \quad i \quad is\ even \end{cases}$$

For $i = n$ and if m is odd,

$$f\left(v_n^j\right)=\begin{cases} 2j+n(m+1)-(2m+3) & if \quad j=1,2,...\left\lceil \frac{m-3}{2}\right\rceil \\ 2j+n(m+1)-(2m+1) & if \quad j=\left\lceil \frac{m-3}{2}\right\rceil+1,...m \end{cases}$$

For $i=n$ and m is even,

$$f\left(v_n^j\right)=\begin{cases} 2j+n(m+1)-(2m+4) & if \quad j=1,2,...\left\lceil \frac{m-3}{2}\right\rceil \\ 2j+n(m+1)-(2m+2) & if \quad j=\left\lceil \frac{m-3}{2}\right\rceil+1,...m \end{cases}$$

Since the weights of all the edges of G are distinct and f is an irregular mean labeling, $s(G)\le n(m+1)-1$. The graph G has $n(m+1)-1$ edges and the weight of the edges are $1,2,3,...n(m+1)-1$. So we cannot labels the vertices of G with fever than $n(m+1)-1$. Therefore $s(G)\ge n(m+1)-1$. Hence $s(G)=n(m+1)-1$.

Theorem 2.9. *Let C_n be the cycle on n vertices. Then $s_m\left(C_n\circ K_1\right)=2n$.*

Proof. Let $G=C_n\circ K_1$.

Let $V(G)=\{u_i:1\le i\le n\}\cup\{v_i:1\le i\le n\}$ and $E(G)=\{e_i=u_iu_{i+1}:1\le i\le n-1\}\cup\{e_n=u_nu_1\}\cup\{u_iv_i:1\le i\le n\}$.

Define the labeling $f:V(G)\to\{1,2,3...,2n\}$ as follows.

Case 1. If n is odd,

$f\left(u_1\right)=1$

$$f\left(u_i\right)=\begin{cases} 2i-1 & for \quad 2\le i\le \left\lceil \frac{n}{2}\right\rceil \\ 2i & for \quad \left\lceil \frac{n}{2}\right\rceil+1\le i\le n \end{cases}$$

$f\left(v_1\right)=1$

$$f\left(v_i\right)=\begin{cases} 2i & for \quad 2\le i\le \left\lceil \frac{n}{2}\right\rceil \\ 2i-1 & for \quad \left\lceil \frac{n}{2}\right\rceil+1\le i\le n \end{cases}$$

Case 2. If n is even,

$f\left(u_1\right)=1$

$$f\left(u_i\right)=\begin{cases} 2i-1 & for \quad 2\le i\le \left\lceil \frac{n}{2}\right\rceil+1 \\ 2i & for \quad \left\lceil \frac{n}{2}\right\rceil+2\le i\le n \end{cases}$$

$$f(v_1) = 1$$

$$f(v_i) = \begin{cases} 2i \quad for \quad 2 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ n+2 \quad for \quad i = \left\lceil \frac{n}{2} \right\rceil + 1 \\ 2i-1 \quad for \quad \left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n \end{cases}$$

Since the weights of all the edges of G are distinct and f is an irregular mean labeling, $s(G) \le 2n$. The graph G has $2n$ edges and the weight of the edges are $1, 2, 3, \ldots 2n$. So we cannot labels the vertices of G with fever than $2n$. Therefore $s(G) \ge 2n$. Hence $s(G) = 2n$.

Definition 2.10. *The edge corona product of two graphs G_1 and G_2 denoted by $G_1 \diamond G_2$ is the graph obtained by taking one copy of G_1 which has m edges and m copies of G_2 and then joining two end vertices of the i^{th} edge of G_1 to every vertex in the i^{th} copy of G_2*

Theorem 2.11. *Let P_n be the path on n vertices. Then $s_m(P_n \diamond K_1) = 3n-2$.*

Proof. Let $G = P_n \diamond K_1$ for $n \ge 2$.

Let $V(G) = \{u_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n-1\}$ and

$E(G) = \{e_i = u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_i v_i : 1 \le i \le n-1\} \cup \{u_i v_{i-1} : 2 \le i \le n\}$.

Define the labeling $f : V(G) \to \{1, 2, 3 \ldots, 3n-2\}$ as follows.

$f(u_1) = 1$

$f(u_i) = 3i-2$ for $1 < i \le n$

$f(v_i) = 3i$ for $1 \le i \le n-1$

By above labeling we have $s_m(P_n \diamond K_1) \le 3n-2$. Since there are $3n-3$ edges and the weight of the edges are $2, 3, 4 \ldots 3n-2$, we cannot label the vertices of G with fever than $3n-2$. Hence $s_m(P_n \diamond K_1) = 3n-2$

.**Conclusion:** In this paper we introduced the new concept irregularity strength on mean labeling and we found the irregularity strength on mean labeling of some families of graphs. Finding the irregularity strength on mean labeling of other family of graphs is left to the reader.

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