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Legendre Wavelet Method for Systems of Volterra Integral Equations

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Abstract

In this paper, a numerical method based on Legendre wavelets has been presented to solve the systems of Volterra integral equations. The Legendre wavelets approximation and Gauss integration formula transforms the given problem to a system of algebraic equations. Then we solve this system of equations by Newton's method to find the unknown coefficients. The proposed method is illustrated by some numerical examples. Comparative error analysis has also been presented.
Key Words: Legendre Wavelets, systems of volterra integral equations, Legendre Wavelets approximation method.

1 Introduction

In recent years, the study of solution of integral equations have attracted the attention of many mathematicians and physicists. Integral equations often arise in electrostatic, electro magnetic scattering problems, low frequency electro magnetic problems and propagation of acoustical and electronic waves [27]. Many scientific and engineering problems are described by integral equations. The Volterra's population growth model, propagation of stocked fish in a new lake, the heat transfer and the heat radiation are among many areas that are described by integral equations [28]. Different analytical and numerical methods have been used to solve systems of Volterra integral equations. The most commonly used methods are Adomian method [2], Bernstein polynomials method [11], Biorthogonal systems [1], modified reproducing kernel method [15], sinc collocation method [6], power series method [7], expansion method [12], He's homotopy perturbation method [3] and block by block method [8]. We introduce a numerical method based on Legendre wavelets for the solution of systems of Volterra integral equations. The main advantage of this technique is that it reduces the given problem to a system of algebraic equations, thus greatly simplifying the problem. In this method Legendre wavelets are used as orthogonal basis functions. There are many orthogonal basis functions such as Walsh functions [20], block pulse functions [21], Laguerre polynomials [22], Legendre polynomials [23], Chebyshev polynomials [24] and Fourier series [25]. But these orthogonal functions are supported on the whole interval $a \leq t \leq b$ [26]. This kind of global support is drawback when the system involve local functions which vanish outside a short interval or when the system has abrupt variations. Wavelets overcome these drawbacks as they have compact support and can approximate functions having abrupt changes [5].

Wavelet transform can be considered as a refinement of Fourier transform. It is useful for aperiodic, transient and intermittent signals. Wavelets can examine signals simultaneously in both time and frequency domain. Wavelets have several useful properties such as orthogonality, compact support, exact representation of polynomials to a certain degree and ability to represent functions at different levels of resolution [4, 5]. Wavelet analysis has applications in the field of data compression, computer graphics, signal processing, numerical analysis, time-frequency analysis, pattern recognition, image processing, data mining and other medical image technology like EEG, ECG etc [4, 31]. The numerical methods based on wavelets have been developed for the solution of differential equations, integral equations, integro-differential equations, partial differential equations, fuzzy integro-differential equations etc [29]. In these numerical methods different kind of wavelet families such as Haar [35], Legendre [9, 14, 34], Chebyhsev [30, 33], Euler wavelets [32] etc. have been used. There are many applications of the Legendre wavelet method in the literature. Legendre wavelets based numerical methods have been applied to solve Fredholom and Volterra integral equations [10], variational problems [13], partial differential equations [9], Volterra integro-differential equations[14, 17].

We have applied Legendre wavelet based numerical method on following system of Volterra integral equations

$$u(x) = v(x) + \int_0^x W(x,t)u(t)dt; \quad 0 \leq x, t \leq 1 \tag{1.1}$$

where $u(x) = [u_1(x), u_2(x), ..., u_p(x)]^T$, $v(x) = [v_1(x), v_2(x), ..., v_p(x)]^T$ and $W(x,t) = [w_{ij}(x,t)]$; $i, j = 1, 2, ..., p$. The kernel $W(x,t)$ is non singular. $u(x)$ is the solution to be determined and functions $W(x,t)$, $v(x)$ are known.

To find $u(x)$ numerically, we will approximate $u(x)$ in terms of Legendre wavelets with unknown coefficients. After approximating unknown function $u(x)$, we will use Gauss integration formula to transform the given problem to a system of algebraic equations. Then we use Newton's iterative method to find the unknown coefficients.

This paper is organized as follows: In section 2, Legendre wavelets and function approximation in terms of Legendre wavelets is given. Section 3 describes the proposed Legendre wavelet based numerical method for the solution of systems of Volterra integral equations. In section 4, numerical examples are given. Conclusion is given in section 6.

2 Legendre Wavelets

Legendre wavelets $\psi_{n,m}(t) = \psi(k,l,m,t)$ have four arguments $l = 2n-1$, $n = 1,2,...,2^{k-1}$, k can assume any positive integer, m is the order of Legendre polynomial and t is the normalized time [13]. We define them on the interval [0,1) as

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}P_m(2^k t - l) & \text{if } \frac{l-1}{2^k} \le t < \frac{l+1}{2^k} \\ 0 & \text{otherwise} \end{cases},$$

where $m = 0,1,2,...,M-1$ and M is a fixed positive integer. The coefficient $\sqrt{m+\frac{1}{2}}$ is used for orthonormality. $P_m(t)$ are Legendre polnomials of order m which are defined on the interval [-1,1] and are given by the following recurrence formulae:

$$P_0(t) = 1, P_1(t) = t,$$
$$(m+1)P_{m+1}(t) = (2m+1)tP_m(t) - mP_{m-1}(t),$$

where $m = 1,2,3,....$

2.1 Function Approximation in terms of Legendre Wavelets

Legendre wavelets approximation of any square integrable function $u(t)$ defined on [0,1) is given by [13]:

$$u(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{n,m}\psi_{n,m}(t), \qquad (2.1)$$

where the wavelet coefficients $a_{n,m}$ are given by

$$a_{n,m} =< u(t), \psi_{n,m}(t) >= \int_0^1 u(t)\psi_{n,m}(t)dt.$$

Now we will truncate the series (2.1) for $n = 1$ to 2^{k-1} and $m = 0$ to $M-1$ then $u(t)$ is approximated as

$$u(t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{n,m}\psi_{n,m}(t) = A^T\psi(t),$$

where $\psi(t)$ and A are matrices given by

$$\psi(t) = [\psi_{1,0}(t), \psi_{1,1}(t), ..., \psi_{1,M-1}(t), \psi_{2,0}(t), \psi_{2,1}(t), ..., \psi_{2,M-1}(t), ..., \psi_{2^{k-1},0}(t), \psi_{2^{k-1},1}(t), ..., \psi_{2^{k-1},M-1}(t)]^T,$$

and

$$A = [a_{1,0}, a_{1,1}, ..., a_{1,M-1}, a_{2,0}, a_{2,1}, ..., a_{2,M-1}, ..., a_{2^{k-1},0}, a_{2^{k-1},1}, ..., a_{2^{k-1},M-1}]^T$$

and are of order $2^{k-1}M \times 1$ [13].

3 Numerical Method

We have consider the systems of equations given by (1.1). This system can also be re-written as

$$u_i(x) = v_i(x) + \int_0^x w_{i,1}(x,t)u_1(t)dt + \int_0^x w_{i,2}(x,t)u_2(t)dt + ... + \int_0^x w_{i,p}(x,t)u_p(t)dt; 0 \le i \le p. \qquad (3.1)$$

To find the unknowns $u_i(x)$ numerically, we will approximate them in terms of Legendre wavelets as

$$u_i(x) \approx A_i^T\psi(x), \qquad (3.2)$$

where $A_i's$ are matrices of unknown wavelets coefficients of order $2^{k-1}M \times 1$ given by

$$A_i = [a^i_{1,0}, a^i_{1,1}, ..., a^i_{1,M-1}, a^i_{2,0}, a^i_{2,1}, ..., a^i_{2,M-1}, ..., a^i_{2^{k-1},0}, a^i_{2^{k-1},1}, ..., a^i_{2^{k-1},M-1}]^T.$$

From (3.1) and (3.2), we have

$$A_i^T\psi(x) = v_i(x) + \int_0^x w_{i,1}(x,t)A_1^T\psi(t)dt + \int_0^x w_{i,2}(x,t)A_2^T\psi(t)dt + ... + \int_0^x w_{i,p}(x,t)A_p^T\psi(t)dt.$$

on substituting collocation points $x_r = \frac{2r-1}{2^k M}$ where r takes values from 1 to $2^{k-1}M$, we get

$$A_i^T\psi(x_r) = v_i(x_r) + \int_0^{x_r} w_{i,1}(x_r,t)A_1^T\psi(t)dt + \int_0^{x_r} w_{i,2}(x_r,t)A_2^T\psi(t)dt + ... + \int_0^{x_r} w_{i,p}(x_r,t)A_p^T\psi(t)dt. \tag{3.3}$$

Now we use Gauss integration formula for equation (3.3). For this we will transform the interval $[0,x_r]$ to [-1,1] by the transformation $\mu = \frac{2}{x_r}t-1$. So above equation can be written as

$$A_i^T\psi(x_r) = v_i(x_r) + \frac{x_r}{2}\int_{-1}^{1} G_{i,1}(x_r, \frac{x_r}{2}(\mu+1))d\mu + \frac{x_r}{2}\int_{-1}^{1} G_{i,2}(x_r, \frac{x_r}{2}(\mu+1))d\mu + ... + \frac{x_r}{2}\int_{-1}^{1} G_{i,p}(x_r, \frac{x_r}{2}(\mu+1))d\mu,$$

where $G_{i,j}(x_r,t) = w_{i,j}(x_r,t)A_j^T\psi(t);\ i,j = 1,2,...,p$.

Using Gauss integration formula for above equations we get the following equations

$$A_i^T\psi(x_r) = v_i(x_r) + \frac{x_r}{2}\Big(\sum_{j=1}^{s_1^i}\nu_{1,j}^i G_{i,1}(x_r, \frac{x_r}{2}(\mu_{1,j}^i + 1)) + \sum_{j=1}^{s_2^i}\nu_{2j}^i G_{i,2}(x_r, \frac{x_r}{2}(\mu_{2,j}^i+1)) + ... + \sum_{j=1}^{s_p^i}\nu_{p,j}^i G_{i,p}(x_r, \frac{x_r}{2}(\mu_{p,j}^i+1))\Big), \tag{3.4}$$

where $\mu_{1,j}^i, \mu_{2,j}^i, ..., \mu_{p,j}^i$ are $s_1^i, s_2^i, ..., s_p^i$ zeros of Legendre polynomials $P_{s_1^i}, P_{s_2^i}$,...,
$P_{s_p^i}$ respectively and $\nu_{1,j}^i, \nu_{2,j}^i, ..., \nu_{p,j}^i$ are the corresponding weights. The Gauss integration formula is exact for polynomials of degree less than or equal to $2s_1^i-1, 2s_2^i-1, ..., 2s_p^i-1$ respectively.

Above system of equations (3.4) gives $2^{k-1}Mp$ nonlinear algebraic equations which can be solved for the unknowns A_i by using Newton's iterative formula. By substituting A_i in equation (3.2) we will find the approximation of the unknown functions $u_i(x)$.

The above method is also applicable for the numerical solution of systems of non-linear Volterra integral equations. The non-linear system contains integral of the form $\int W(x,t)u_i^\alpha(t)u_{i'}^\beta dt$ where $1 \le i,i' \le p$, $\alpha \ge 0$, $\beta \ge 0$ and α and β are not simultaneously equal to zero. In that case integral is approximated as $\int W(x,t)(A_i^T\psi(t))^\alpha(A_{i'}^T\psi(t))^\beta dt$. The other procedure for solving the non-linear system is same as that of linear system.

The Legendre wavelets method for solving systems of Volterra integral equations is general. But for its numerical illustration we have used this method for $p=2$ in section 4.

4 Numerical Examples

In this section, the proposed numerical method has been applied on linear and nonlinear systems of Volterra integral equations. To check accuracy of the present method, we have calculated root mean square error. The results of our method are also compared with the results of existing methods in the literature.

Computations are performed using MATLAB.

Example 5.4.1. Consider the following system of Volterra integral equations

$$u_1(x) = v_1(x) + \int_0^x (x-t)^3u_1(t)dt + \int_0^x (x-t)^2u_2(t)dt,$$
$$u_2(x) = v_2(x) + \int_0^x (x-t)^4u_1(t)dt + \int_0^x (x-t)^3u_2(t)dt,$$

where $v_1(x)$ and $v_2(x)$ are chosen such that the exact solution is $u_1(x) = x^2+1$, $u_2(x) = 1-x^3+x$ [12].

Using Legendre wavelets approximation method for $M = 4$ and $k = 1$, we have the following approximation for $u_1(x)$ and $u_2(x)$

$$u_1(x) \approx 1.333333335 + 0.5000000015(2x-1) + 0.1666666678(6x^2-6x+1) + 0.000000003880100158(20x^3 - 30x^2+12x-1) + 0.00000000272681317 8 (70x^4-140x^3+90x^2-20x+1)$$

and

$$u_2(x) \approx 1.250000001 + 0.05000000078(2x-1) - 0.2499999975(6x^2-6x+1) - 0.04999999586(20x^3 - 30x^2+12x-1) + 0.000000001735621161(70x^4 - 140x^3+90x^2-20x+1).$$

Table 1 shows the absolute error between exact and approximate values of $u_1(x)$ and $u_2(x)$ for different values of x.

For these values of x, the root mean square (RMS) errors obtained by our method are 1.6601e-09 and 1.3183e-09 for $u_1(x)$ and $u_2(x)$ respectively.

In [12], Expansion method has been applied to solve this system of Volterra integral equations and RMS error is 0.0078 for $u_1(x)$, 0.0510 for $u_2(x)$ which is larger than RMS error obtained by our method.

Table 1: Error estimates for Example 5.4.1

x	Absolute error for $u_1(x)$	Absolute error for $u_2(x)$
0	1.4671e-10	3.1562e-10
.1	4.2245e-11	7.9040e-10
.2	1.0963e-09	1.4143e-09
.3	2.1711e-09	1.6635e-09
.4	2.5870e-09	1.3059e-09
.5	2.1226e-09	4.0086e-10
.6	1.0142e-09	7.0054e-10
.7	4.3374e-11	1.3557e-09
.8	1.0262e-10	6.3053e-10
.9	3.0631e-09	2.7008e-09

Example 5.4.2. Consider the following system

$$u_1(x) = v_1(x) + \int_0^x (\sin(x-t)-1)u_1(t)dt + \int_0^x (1-t\cos x)u_2(t)dt,$$
$$u_2(x) = v_2(x) + \int_0^x u_1(t)dt + \int_0^x (x-t)u_2(t)dt,$$

where $v_1(x)$ and $v_2(x)$ are chosen such that the exact solution is $u_1(x) = \cos x$, $u_2(x) = \sin x$ [12].

Using Legendre wavelets approximation method for $M = 4$ and $k = 1$, we have the following approximation for $u_1(x)$ and $u_2(x)$

$$u_1(x) \approx 0.841474309 - 0.2337622174(2x-1) - 0.0718161708(6x^2-6x+1) + 0.003964355901(20x^3-30x^2+12x-1) + 0.0005440002857(70x^4-140x^3 +90x^2-20x+1)$$

and

$$u_2(x) \approx 0.4596912327 + 0.4278978641(2x-1) - 0.03928020042(6x^2-6x+1) - 0.007257986147(20x^3-30x^2+12x-1) + 0.0002288456726(70x^4-140x^3 +90x^2-20x+1).$$

Table 2 shows the absolute error between exact and approximate values of $u_1(x)$ and $u_2(x)$ for different values of x.

For these values of x, the root mean square (RMS) errors obtained by our method are 5.4087e-06 and 1.1075e-05 for $u_1(x)$ and $u_2(x)$ respectively.

In [12], this system of Volterra integral equations is solved by Expansion method and RMS error 0.0156 for $u_1(x)$, 0.0347 for $u_2(x)$ which is larger than RMS error of the present method.

Table 2: Error estimates for Example 5.4.2

x	Absolute error for $u_1(x)$	Absolute error for $u_2(x)$
0	1.5300e-11	4.0002e-13
.1	5.7854e-07	2.0496e-06
.2	2.3692e-06	5.7314e-06
.3	2.2442e-06	5.3590e-06
.4	1.1013e-06	2.5382e-06
.5	3.8326e-06	8.3886e-06
.6	1.5542e-06	3.0514e-06
.7	6.3495e-06	1.3198e-05
.8	1.1498e-05	2.2158e-05
.9	9.5246e-06	2.0238e-05

Example 5.4.3. Consider the following system of Volterra integral equations

$$u_1(x) = \cosh x + x\sin x - \int_0^x e^{x-t}u_1(t)dt - \int_0^x \cos(x-t)u_2(t)dt,$$
$$u_2(x) = 2\sin x + x(\sin^2 x + e^x) - \int_0^x e^{x+t}u_1(t)dt - \int_0^x x\cos(t)u_2(t)dt,$$

with the exact solution $u_1(x) = e^{-x}$, $u_2(x) = 2\sin x$ [3].

Using Legendre wavelets approximation method for $M = 4$ and $k = 1$, we have the following approximation for $u_1(x)$ and $u_2(x)$

$$u_1(x) \approx 0.6321235356 - 0.3109040675(2x-1) + 0.05147374204(6x^2-6x+1) - 0.005099261125(20x^3-30x^2+12x-1) + 0.0003993937287(70x^4-140x^3 +90x^2-20x+1)$$

and

$$u_2(x) \approx 0.9193855862 + 0.8558035725(2x-1) - 0.07855071815(6x^2 - 6x+1) - 0.01450602781(20x^3 - 30x^2 + 12x - 1) + 0.0004626765959(70x^4 - 140x^3 + 90x^2 - 20x + 1).$$

Table 3 shows the absolute error between exact and approximate values of $u_1(x)$ and $u_2(x)$ for different values of x.

In [3] He's homotopy perturbation method is applied to find numerical solution of this system of Volterra integral equations at $x = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. We have calculated the root mean square (RMS) error at these points. The root mean square errors obtained by our method are 3.7502e-06 and 9.8935e-06 for $u_1(x)$ and $u_2(x)$ respectively. The RMS error by He's homotopy perturbation method are 4.1731e-04 for $u_1(x)$, 5.8001e-04 for $u_2(x)$ which is larger than RMS error of the present method.

Table 3: Error estimates for Example 5.4.3

x	Absolute error for $u_1(x)$	Absolute error for $u_2(x)$
0	6.3000e-12	4.4100e-11
.1	2.1751e-06	4.7569e-06
.2	4.5139e-06	1.1810e-05
.3	5.0374e-06	1.1996e-05
.4	7.2280e-07	3.8437e-06
.5	5.7775e-06	1.6628e-05
.6	3.0920e-06	7.3013e-06
.7	8.0247e-06	2.5197e-05
.8	1.6138e-05	4.6917e-05
.9	7.5436e-06	2.6991e-05

Example 5.4.4. Consider the following nonlinear system of Volterra integral equations

$$\begin{aligned} u_1(x) &= \cos x - \frac{1}{2}\sin^2 x + \int_0^x u_1(t)u_2(t)dt, \\ u_2(x) &= \sin x - x + \int_0^x u_1^2(t)dt + \int_0^x u_2^2(t)dt, \end{aligned}$$

with the exact solution $u_1(x) = \cos x$, $u_2(x) = \sin x$ [3].

Using Legendre wavelets approximation method for $M = 4$ and $k = 1$, we have the following approximation for $u_1(x)$ and $u_2(x)$

$$u_1(x) \approx 0.8414748715 - 0.2337610076(2x-1) - 0.0718147384(6x^2 - 6x + 1) + 0.003966354643(20x^3 - 30x^2 + 12x - 1) + 0.000545213953(70x^4 - 140x^3 + 90x^2 - 20x + 1)$$

and

$$u_2(x) \approx 0.4596923035 + 0.4279005011(2x-1) - 0.03927675928(6x^2 - 6x + 1) - 0.007254845653(20x^3 - 30x^2 + 12x - 1) + 0.0002301113014(70x^4 - 140x^3 + 90x^2 - 20x + 1).$$

Table 4 shows the absolute error between exact and approximate values of $u_1(x)$ and $u_2(x)$ for different values of x.

In [3], He's homotopy perturbation method is applied for the numerical solution of this system at the points $x = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. We have calculated the root mean square (RMS) error at these points. The root mean square errors obtained by our method are 2.1679e-06 and 5.0229e-06 for $u_1(x)$ and $u_2(x)$ respectively. The RMS error by He's homotopy perturbation method are 9.3554e-04 for $u_1(x)$, 6.7174e-04 for $u_2(x)$ which is larger than RMS error of present method.

Table 4: Error estimates for Example 5.4.4

x	Absolute error for $u_1(x)$	Absolute error for $u_2(x)$
0	3.0300e-11	4.5700e-11
.1	5.7807e-07	2.0496e-06
.2	2.3699e-06	5.7313e-06
.3	2.2448e-06	5.3589e-06
.4	1.1008e-06	2.5384e-06
.5	3.8321e-06	8.3889e-06
.6	1.5537e-06	3.0517e-06
.7	6.3501e-06	1.3197e-05
.8	1.1499e-05	2.2158e-05
.9	9.5238e-06	2.0238e-05

Example 5.4.5. Consider the following nonlinear system of Volterra integral equations

$$u_1(x) = x - x^2 + \int_0^x (u_1(t) + u_2(t))dt,$$
$$u_2(x) = x - \frac{x^2}{2} - \frac{x^3}{3} + \int_0^x (u_1^2(t) + u_2(t))dt,$$

with the exact solution $u_1(x) = x,\ u_2(x) = x$ [18].
Using Legendre wavelets approximation method for $M = 2$ and $k = 1$, we have the following approximation for $u_1(x)$ and $u_2(x)$

$$u_1(x) \approx 0.5 + 0.4999999999(2x - 1) - (4.273787349e - 11)(6x^2 - 6x + 1)$$

and

$$u_2(x) \approx 0.4999999998 + 0.4999999996(2x - 1) - (0.0000000001698626933)(6x^2 - 6x + 1).$$

Table 5: Error estimates for Example 5.4.5

x	Absolute error for $u_1(x)$	Absolute error for $u_2(x)$
0	5.726213478e-11	3.013726774e-11
.1	6.0340552e-11	4.186308245e-11
.2	5.829048355e-11	3.32054384e-11
.3	5.111183699e-11	4.164224521e-12
.4	3.880468169e-11	4.526040653e-11
.5	2.136890664e-11	1.150686768e-10
.6	1.195377131e-12	2.052604753e-10
.7	2.888811412e-11	3.158356909e-10
.8	6.170952638e-11	4.467944903e-10
.9	9.965939185e-11	5.981367623e-10

Table 5 shows the absolute error between exact and approximate values of $u_1(x)$ and $u_2(x)$ for different values of x.
For these values of x, the root mean square (RMS) error obtained by our method are 5.428497051e-11 and 2.6802e-10 for $u_1(x)$ and $u_2(x)$ respectively.
In [9], this system of Volterra integral equations is solved by homotopy perturbation method and obtained RMS error 0.02105921369 for $u_1(x)$, 0.08113075865 for $u_2(x)$ which is larger than RMS error obtained by our method.

5 Error Analysis

We have applied our method on different kind of systems of Voletrra integral equations with linear, non-linear and non-truncating series solution. We have compared our results with exact solution to show that when the exact solution is linear polynomial we get very small approximation error for smaller values of M and k as discussed in Example 5.5.5. When the exact solution is non-linear polynomial as discussed in Example 5.4.1 or is function of the type $\sin x$, $\cos x$ or $\exp x$ which can be expressed in the form of non-truncating series as discussed in Example 5.4.2, 5.4.3 and 5.4.4 we have taken $M = 4$ and $k = 1$. It is concluded that the proposed numerical method works well for both kind of problems but when the solution is non-linear or non-truncating series larger value of M is required as compared to the other problems having linear solution. So results are verified on different problems with linear, non-linear and non-truncating series solution. From above examples it is concluded that the approximation error depends on the values of M and k. As the values of M and k increases the approximation error decreases.

6 Conclusion

Legendre wavelet method combined with Gauss integration formula is utilized to get numerical solution of systems of Volterra integral equations. Orthogonal basis make numerical calculation easy. Absolute error given in tables and error estimation comparison of our method with existing methods shows that our method performs better than these methods.

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