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FIXED POINTS FROM WEAKLY C_S^'-CONTRACTION IN COMPLETE S-METRIC SPACE

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ABSTRACT

In this paper, we generalize the concept of Binayak S. Choudhury in complete S- metric space. Also, we give examples which strengthen our result.

KEYWORDS

Weak C-contraction, fixed point, S-metric space.

SUBJECT CLASSIFICATION CODE

54H25, 47H10

INTRODUCTION

Today lot of generalization of Banach contraction which is known as pivotal result in fixed point Theory. Also, weak contraction condition introduced by Alber. Ya. I and Guerre -Delabrieer in Hilbert spaces. In this way, Binayak S. Choudhury introduced the concept weak C-contraction and also he established that every weak C-contraction in complete metric space has a unique invariant point. Here we generalize this condition in complete S-metric space.

1. PRELIMINARIES

Definition 2.1 [4]

Let X be a non empty set. An S-metric on X is a function $S: X \times X \times X \to R^{+}$ that satisfies the following conditions for each $\zeta, \varpi, w, t \in X$,

(S1) $S(\zeta, \varpi, w) \geq 0$.

(S2) $S(\zeta, \varpi, w) = 0$ if and only if $\zeta = \varpi = w$.

(S3) $S(\zeta, \varpi, w) \leq S(\zeta, \zeta, t) + S(\varpi, \varpi, t) + S(w, w, t)$

Then the pair (X, S) is called an S-metric space.

Example 2.2 [4]

Define $S: R^3 \to R^{+}$ by $S(\zeta, \varpi, w) = |\zeta - w| + |\varpi - w| \; \forall \zeta, \varpi, w \in R$, then (R, S) is S-metric space.

Definition 2.3 [4]

If $S(\eta_n, \eta_n, \eta) \to 0$ as $n \to \infty$, then we say that $\{\eta_n\}$ in X converges to $\eta \in X$. (i.e.) for arbitrary $\varepsilon > 0$, there exists a natural number m such that $S(\eta_n, \eta_n, \eta) < \varepsilon \; \forall \, n \geq m$. Also, we write it by $\lim_{n\to\infty} \eta_n = \eta$.

Definition 2.4 [4]

A sequence $\{\eta_n\}$ in S-metric space (X, S) is said to be Cauchy sequence if for any

$\varepsilon > 0, m \in N$ such that $S(\eta_n, \eta_n, \eta_m) < \varepsilon \; \forall \; n, m \geq n_0$.

Definition 2.5 [4]

A S-metric space (X, S) is said to be complete if every Cauchy sequence in X is converges to a limit in X.

Definition 2.6 [4]

Let (X, S) be S-metric space. A mapping $h: X \to X$ is said to be S-contraction if $(h\eta, h\eta, hv) \leq tS(\eta, \eta, v) \; \forall \; \eta, v \in X \; \& \; 0 \leq t < 1$.

Lemma 2.7 [2]

In the S-metric space $S(\zeta, \zeta, \vartheta) = S(\vartheta, \vartheta, \zeta)$.

Lemma 2.8 [2]

In the S-metric space,

$$S(\zeta, \zeta, \vartheta) \leq 2S(\zeta, \zeta, w) + S(\vartheta, \vartheta, w) \text{ and}$$
$$S(\zeta, \zeta, \vartheta) \leq 2S(\zeta, \zeta, w) + S(w, w, \vartheta)$$

Definition 2.9 [6]

Let (X, d) be a metric space. Any self mapping C on X is called C-contraction if it satisfies the following inequality

$$d(C\zeta, C\vartheta) \leq \rho[d(\zeta, C\vartheta) + d(\vartheta, C\zeta)]$$

with $\rho \in \left(0, \frac{1}{2}\right)$ and for all $\zeta, \vartheta \in X$.

This concept was introduced by Chatterjee [6].

Theorem 2.10 [6]

Any C-contraction mapping on a complete metric space has a unique fixed point.

Definition 2.11 [1]

Let (X, d) be a complete metric space. A self mapping C on X is said to be weak C-contractive if

$$d(C\zeta, C\vartheta) \leq \frac{1}{2}[d(\zeta, T\vartheta) + d(\vartheta, T\zeta)] - \rho[d(\zeta, C\vartheta), d(\vartheta, C\zeta)]$$

where $\rho: {R^+}^2 \to R^+$ and also a continuous mapping such that $\rho(\zeta, \vartheta) = 0$ iff $\zeta = \vartheta = 0$.

Theorem 2.12 [1]

Any weak C-contraction mapping on a complete metric space has a unique fixed point.

This statement was stated by Binayak S. Choudhury.

In the next section, we generalize this concept in complete S-metric space.

MAIN RESULTS

Definition 3.1

A self mapping C on a complete S-metric space X is said to be weakly C_s'-contraction if

$$S(Cu, Cu, C\vartheta) \leq \frac{1}{3}\left[S(u, u, C\vartheta) + S(\vartheta, \vartheta, C\zeta) - \rho\big(S(u, u, C\vartheta), S(\vartheta, \vartheta, Cu), S(u, u, \vartheta)\big)\right] \qquad (3.1.1)$$

where $\rho: {R^+}^3 \to R^+$ is a continuous function satisfying the condition that $\rho(u, \vartheta, w) = 0$ iff one of u, ϑ, w is zero.

Example 3.2

Let $X = [0,1]$. Define the metric S on X as $S(u, \vartheta, w) = |u - w| + |\vartheta - w| \; \forall \; u, \vartheta, w \in X$.

Define $C: [0,1] \to [0,1]$ by $Cu = \begin{cases} \frac{u}{2}, u \in \left[0, \frac{1}{2}\right] \\ \frac{1}{4}, u \in \left(\frac{1}{2}, 1\right] \end{cases}$

Solution.

Define $\rho(u, v, w) = min\{u, v, w\}$

Case (i)

Let $u = \frac{1}{3} \in \left[0, \frac{1}{2}\right], v = \frac{3}{4} \in \left(\frac{1}{2}, 1\right]$, Then $Cu = \frac{1}{6}$ and $Cv = \frac{1}{4}$

$$S(Cu, Cu, Cv) = S\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{4}\right) = 2\left|\frac{1}{6} - \frac{1}{4}\right| = \frac{2 \times 2}{24} = \frac{1}{6}$$

$$S(u,u,Cv)=S\left(\frac{1}{3},\frac{1}{3},\frac{1}{4}\right)=2\left|\frac{1}{3}-\frac{1}{4}\right|=\frac{2}{12}=\frac{1}{6}$$
$$S(v,v,Cu)=S\left(\frac{3}{4},\frac{3}{4},\frac{1}{6}\right)=2\left|\frac{3}{4}-\frac{1}{6}\right|=2\times\frac{7}{12}=\frac{7}{6}$$
$$S(u,u,v)=S\left(\frac{1}{3},\frac{1}{3},\frac{3}{4}\right)=2\left|\frac{1}{3}-\frac{3}{4}\right|=2\times\frac{5}{12}=\frac{5}{6}$$
$$\frac{1}{3}\left[S(u,u,Cv)+S(v,v,Cu)-\rho\left(S(u,u,Cv),S(v,v,Cu),S(u,u,v)\right)\right]$$
$$=\frac{1}{3}\left[\frac{1}{6}+\frac{7}{6}-\rho\left(\frac{1}{6},\frac{7}{6},\frac{5}{6}\right)\right]$$
$$=\frac{1}{3}\left[\frac{8}{6}-\frac{1}{6}\right]$$
$$=\frac{1}{3}\left[\frac{7}{6}\right]$$
$$=\frac{7}{18}$$
$$>\frac{1}{6}$$

∴ Inequality (3.1.1) is satisfied.

Subcase

Let $u=0\in\left[0,\frac{1}{2}\right]$, $v=1\in\left(\frac{1}{2},1\right]$, then $Cu=0, Cv=\frac{1}{4}$

$$S(Cu,Cu,Cv)=S\left(0,0,\frac{1}{4}\right)=2\left|0-\frac{1}{4}\right|=\frac{1}{2}$$
$$S(u,u,Cv)=S\left(0,0,\frac{1}{4}\right)=\frac{1}{2}$$
$$S(v,v,Cu)=S(1,1,0)=2$$
$$S(u,u,v)=S(0,0,1)=2$$
$$\frac{1}{3}\left[S(u,u,Cv)+S(v,v,Cu)-\rho\left(S(u,u,Cv),S(v,v,Cu),S(u,u,v)\right)\right]$$
$$=\frac{1}{3}\left[\frac{1}{2}+2-\rho\left(\frac{1}{2},2,2\right)\right]$$
$$=\frac{1}{3}\left[\frac{5}{2}-\frac{1}{2}\right]$$
$$=\frac{2}{3}$$
$$>\frac{1}{2}$$

∴ Inequality (3.1.1) is satisfied.

Case (ii)

$u=\frac{1}{5}\in\left[0,\frac{1}{2}\right]$ and $v=\frac{1}{3}\in\left[0,\frac{1}{2}\right]$,Then $Cu=\frac{1}{10}, Cv=\frac{1}{6}$

$$S(Cu,Cu,Cv)=S\left(\frac{1}{10},\frac{1}{10},\frac{1}{6}\right)=2\left|\frac{1}{10}-\frac{1}{6}\right|=\frac{2\times 4}{60}=\frac{2}{15}$$
$$S(u,u,Cv)=S\left(\frac{1}{5},\frac{1}{5},\frac{1}{6}\right)=2\left|\frac{1}{5}-\frac{1}{6}\right|=\frac{2}{30}=\frac{1}{15}$$
$$S(v,v,Cu)=S\left(\frac{1}{3},\frac{1}{3},\frac{1}{10}\right)=2\left|\frac{1}{3}-\frac{1}{10}\right|=2\times\frac{7}{30}=\frac{7}{15}$$
$$S(u,u,v)=S\left(\frac{1}{5},\frac{1}{5},\frac{1}{3}\right)=2\left|\frac{1}{5}-\frac{1}{3}\right|=2\times\frac{2}{15}=\frac{4}{15}$$
$$\frac{1}{3}\left[S(u,u,Cv)+S(v,v,Cu)-\rho\left(S(u,u,Cv)+S(v,v,Cu),S(u,u,v)\right)\right]$$
$$=\frac{1}{3}\left[\frac{1}{15}+\frac{7}{15}-\rho\left(\frac{1}{15},\frac{7}{15},\frac{4}{15}\right)\right]$$
$$=\frac{1}{3}\left[\frac{8}{15}-\frac{1}{15}\right]$$

$$= \frac{1}{3}\left[\frac{7}{15}\right]$$
$$= \frac{7}{45}$$
$$> \frac{2}{15}$$

∴ Inequality (3.1.1) is satisfied.

Subcase

$u = v = \frac{1}{4} \in \left[0, \frac{1}{2}\right]$, then $Cu = \frac{1}{8}, Cv = \frac{1}{8}$.

$$S(Cu, Cu, Cv) = S\left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right) = 0$$
$$S(u, u, Cv) = S\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right) = \frac{1}{4}$$
$$S(v, v, Cu) = S\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right) = \frac{1}{4}$$
$$S(u, u, v) = S\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = 0$$
$$\frac{1}{3}\left[S(u, u, Cv) + S(v, v, Cu) - \rho\left(S(u, u, Cv), S(v, v, Cu), S(u, u, v)\right)\right]$$
$$= \frac{1}{3}\left[\frac{1}{4} + \frac{1}{4} - \rho\left(\frac{1}{4}, \frac{1}{4}, 0\right)\right]$$
$$= \frac{1}{3} \times \frac{2}{4}$$
$$= \frac{1}{6} > 0$$

∴ Inequality (3.1.1) is satisfied.

Case (iii)

Let $u, v \in \left(\frac{1}{2}, 1\right]$, $u = \frac{3}{4}, v = \frac{2}{3}$, then $Cu = \frac{1}{4}, Cv = \frac{1}{4}$

$$S(Cu, Cu, Cv) = S\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = 0$$
$$S(u, u, Cv) = S\left(\frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right) = 2\left|\frac{3}{4} - \frac{1}{4}\right| = 1$$
$$S(v, v, Cu) = S\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{4}\right) = 2\left|\frac{2}{3} - \frac{1}{4}\right| = \frac{5}{6}$$
$$S(u, u, v) = S\left(\frac{3}{4}, \frac{3}{4}, \frac{2}{3}\right) = 2\left|\frac{3}{4} - \frac{2}{3}\right| = \frac{1}{6}$$
$$\frac{1}{3}\left[S(u, u, Cv) + S(v, v, Cu) - \rho\left(S(u, u, Cv), S(v, v, Cu), S(u, u, v)\right)\right]$$
$$= \frac{1}{3}\left[1 + \frac{5}{6} - \rho\left(1, \frac{5}{6}, \frac{1}{6}\right)\right]$$
$$= \frac{1}{3}\left[\frac{11}{6} - \frac{1}{6}\right]$$
$$= \frac{1}{3}\left[\frac{10}{6}\right]$$
$$= \frac{5}{9} > 0$$

∴ Inequality (3.1.1) is satisfied.

Subcase:

Suppose $u = v = \frac{2}{3} \in \left(\frac{1}{2}, 1\right]$,

then $Cu = Cv = \frac{1}{4}$

$$S(Cu, Cu, Cv) = 0$$
$$S(u, u, Cv) = S\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{4}\right) = 2\left|\frac{2}{3} - \frac{1}{4}\right| = \frac{5}{6}$$

$$S(v,v,Cu) = S\left(\frac{2}{3},\frac{2}{3},\frac{1}{4}\right) = \frac{5}{6}$$
$$S(u,u,v) = S\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right) = 0$$
$$\frac{1}{3}\left[S(u,u,Cv) + S(v,v,Cu) - \rho\left(S(u,u,Cv), S(v,v,Cu), S(u,u,v)\right)\right]$$
$$= \frac{1}{3}\left[\frac{5}{6} + \frac{5}{6} - \rho\left(\frac{5}{6},\frac{5}{6},0\right)\right]$$
$$= \frac{1}{3}\left[\frac{10}{6} - 0\right]$$
$$= \frac{1}{3}\left[\frac{10}{6}\right]$$
$$= \frac{5}{9} > 0$$

∴ Inequality (3.1.1) is satisfied.

Thus, in all the cases, the self-mapping C satisfies the inequality (3.1.1).

Hence, C is the weakly C_s'-contraction for all values of $u, v \in X$.

Theorem 3.3

Let $C: X \to X$, where (X,S) is a complete S-metric space be a weakly C_s'-contraction. Then C has a unique fixed point.

Proof.

Let $u_0 \in X$ and $Cu_n = u_{n+1} \forall n \geq 1$.

Suppose $u_n = u_{n+1} = Cu_n$, then u_n is a fixed point of C.

Let us assume that $u_n \neq u_{n+1}$. Put $u = u_{n-1}, v = u_n$ in (3.1.1).

Then we have,

$$S(Cu_{n-1}, Cu_{n-1}, Cu_n) = S(u_n, u_n, u_{n+1})$$

Again by (3.1.1)

$$S(Cu_{n-1}, Cu_{n-1}, Cu_n) \leq \frac{1}{3}[S(u_{n-1}, u_{n-1}, Cu_n) + S(u_n, u_n, Cu_{n-1})$$
$$-\rho\left(S(u_{n-1}, u_{n-1}, Cu_n), S(u_n, u_n, Cu_{n-1}), S(u_{n-1}, u_{n-1}, u_n)\right)]$$
$$\Rightarrow S(u_n, u_n, u_{n+1}) \leq \frac{1}{3}[S(u_{n-1}, u_{n-1}, u_{n+1}) + S(u_n, u_n, u_n)$$
$$-\rho\left(S(u_{n-1}, u_{n-1}, u_{n+1}), S(u_n, u_n, u_n), S(u_{n-1}, u_{n-1}, u_n)\right)]$$
$$\leq \frac{1}{3}[2S(u_{n-1}, u_{n-1}, u_n) + S(u_n, u_n, u_{n+1})$$
$$-\rho\left(S(u_{n-1}, u_{n-1}, u_{n+1}), 0, S(u_{n-1}, u_{n-1}, u_n)\right)]$$

[∵ by Lemma 2.8, $S(u_{n-1}, u_{n-1}, u_{n+1}) \leq 2S(u_{n-1}, u_{n-1}, u_n) + S(u_n, u_n, u_{n+1})$]

$$\leq \frac{1}{3}[2S(u_{n-1}, u_{n-1}, u_n) + S(u_n, u_n, u_{n+1})]$$
$$= \frac{2}{3}S(u_{n-1}, u_{n-1}, u_n) + \frac{1}{3}S(u_n, u_n, u_{n+1})$$
$$\Longrightarrow S(u_n, u_n, u_{n+1}) - \frac{1}{3}S(u_n, u_n, u_{n+1}) \leq \frac{2}{3}S(u_{n-1}, u_{n-1}, u_n)$$
$$\frac{2}{3}S(u_n, u_n, u_{n+1}) \leq \frac{2}{3}S(u_{n-1}, u_{n-1}, u_n)$$
$$S(u_n, u_n, u_{n+1}) \leq S(u_{n-1}, u_{n-1}, u_n)$$

$\Longrightarrow \{S(u_n, u_n, u_{n+1})\}$ is a monotonic decreasing sequence whose terms are non-negative real numbers and so is convergent.

Let $\lim_{n\to\infty} S(u_n, u_n, u_{n+1}) = a$

To prove $a = 0$.

Suppose not. Then $a > 0$. Now,

$$S(Cu_{n-1}, Cu_{n-1}, Cu_n) \leq \frac{1}{3}[S(u_{n-1}, u_{n-1}, Cu_n) + S(u_n, u_n, Cu_{n-1})$$

$$-\boldsymbol{\rho}\left(\boldsymbol{S}(\boldsymbol{u}_{n-1},\boldsymbol{u}_{n-1},\boldsymbol{C}\boldsymbol{u}_{n}),\boldsymbol{S}(\boldsymbol{u}_{n},\boldsymbol{u}_{n},\boldsymbol{C}\boldsymbol{u}_{n-1}),\boldsymbol{S}(\boldsymbol{u}_{n-1},\boldsymbol{u}_{n-1},\boldsymbol{u}_{n})\right)]$$

$$\begin{aligned}S(u_n,u_n,u_{n+1}) &\leq \frac{1}{3}[S(u_{n-1},u_{n-1},u_{n+1})+S(u_n,u_n,u_n)\\ &\quad -\rho\left(S(u_{n-1},u_{n-1},Cu_{n+1}),S(u_n,u_n,u_n),S(u_{n-1},u_{n-1},u_n)\right)\\ &=\frac{1}{3}[S(u_{n-1},u_{n-1},u_{n+1})+0\\ &\quad -\rho\left(S(u_{n-1},u_{n-1},Cu_{n+1}),0,S(u_{n-1},u_{n-1},u_n)\right)\\ &=\frac{1}{3}[S(u_{n-1},u_{n-1},u_{n+1})]\end{aligned}$$

$$(i.e)\; S(u_n,u_n,u_{n+1}) \leq \frac{1}{3}[S(u_{n-1},u_{n-1},u_{n+1})]$$

Taking $n\to\infty$, we get, $a \leq \frac{1}{3}a$.

which is a contradiction.

$$\therefore a = 0.$$

Hence, $S(u_n,u_n,u_{n+1}) \to 0$ as $n\to\infty$

Next to prove that $\{u_n\}$ is Cauchy.

Suppose not, then given $\epsilon > 0$, there exists sequence of integers $\{s_i\}$ and $\{t_i\}$, which is increasing such that for all $i, i \in Z^+, n(i) > m(i) > i$,

$S\left(u_{m(i)},u_{m(i)},u_{n(i)}\right) \geq \varepsilon \geq \frac{\varepsilon}{2}$ and $S\left(u_{m(i)},u_{m(i)},u_{n(i)-1}\right) < \varepsilon$

Thus, we have,

$$\begin{aligned}\varepsilon &\leq S\left(u_{m(i)},u_{m(i)},u_{n(i)}\right)\\ &= S\left(Cu_{m(i)-1},Cu_{m(i)-1},Cu_{n(i)-1}\right)\\ &\leq \frac{1}{3}\Big[S\left(u_{m(i)-1},u_{m(i)-1},Cu_{n(i)-1}\right)+S\left(u_{n(i)-1},u_{n(i)-1},Cu_{m(i)-1}\right)\\ &\quad -\rho\left(S\left(u_{m(i)-1},u_{m(i)-1},Cu_{n(i)-1}\right),S\left(u_{n(i)-1},u_{n(i)-1},Cu_{m(i)-1}\right),S\left(u_{m(i)-1},u_{m(i)-1}u_{n(i)-1}\right)\right)\Big]\\ &=\frac{1}{3}\Big[S\left(u_{m(i)-1},u_{m(i)-1},u_{n(i)}\right)+S\left(u_{n(i)-1},u_{n(i)-1},u_{m(i)}\right)\\ &\quad -\rho\left(S\left(u_{m(i)-1},u_{m(i)-1},u_{n(i)}\right),S\left(u_{n(i)-1},u_{n(i)-1},u_{m(i)}\right),S\left(u_{m(i)-1},u_{m(i)-1}u_{n(i)-1}\right)\right)\Big]\end{aligned}$$

$$\begin{aligned}\varepsilon \leq \frac{1}{3}[&S\left(u_{m(i)-1},u_{m(i)-1},u_{n(i)}\right)+S\left(u_{n(i)-1},u_{n(i)-1},u_{m(i)}\right)\\ &-\rho(S\left(u_{m(i)-1},u_{m(i)-1},u_{n(i)}\right),S\left(u_{n(i)-1},u_{n(i)-1},u_{m(i)}\right),\\ &S\left(u_{m(i)-1},u_{m(i)-1}u_{n(i)-1}\right))]\end{aligned} \tag{3.3.1}$$

Also,

$$\begin{aligned}\varepsilon \leq S\left(u_{m(i)},u_{m(i)},u_{n(i)}\right) &= S\left(u_{n(i)},u_{n(i)},u_{m(i)}\right)\\ &\leq 2S\left(u_{n(i)},u_{n(i)},u_{n(i)-1}\right)+S\left(u_{n(i)-1},u_{n(i)-1},u_{m(i)}\right) && \text{[by Lemma 2.8]}\\ &= 2S\left(u_{n(i)-1},u_{n(i)-1},u_{n(i)}\right)+S\left(u_{m(i)},u_{m(i)},u_{n(i)-1}\right) && \text{[by Lemma 2.7]}\\ &< 2S\left(u_{n(i)-1},u_{n(i)-1},u_{n(i)}\right)+\varepsilon\end{aligned}$$

Making $i\to\infty$, we get $S\left(u_{n(i)-1},u_{n(i)-1},u_{n(i)}\right)=0$

The above inequality becomes,

$$\lim_{i\to\infty} S\left(u_{m(i)},u_{m(i)},u_{n(i)}\right)=\varepsilon$$

Also,

$$\lim_{i\to\infty} S\left(u_{m(i)},u_{m(i)},u_{n(i)-1}\right)=\varepsilon$$

Again, `

$$\begin{aligned}S\left(u_{m(i)},u_{m(i)},u_{n(i)-1}\right) &\leq 2S\left(u_{m(i)},u_{m(i)},u_{m(i)-1}\right)+S\left(u_{m(i)-1},u_{m(i)-1},u_{n(i)-1}\right) && \text{[by Lemma 2.8]}\\ &= 2S\left(u_{m(i)},u_{m(i)},u_{m(i)-1}\right)+S\left(u_{n(i)-1},u_{n(i)-1},u_{m(i)-1}\right)\end{aligned}$$

[by Lemma 2.7]

$$= 2S(u_{m(i)}, u_{m(i)}, u_{m(i)-1}) + 2S(u_{n(i)-1}, u_{n(i)-1}, u_{n(i)}) + S(u_{m(i)-1}, u_{m(i)-1}, u_{n(i)})$$

$$\therefore S(u_{m(i)}, u_{m(i)}, u_{n(i)-1}) \leq 2S(u_{m(i)}, u_{m(i)}, u_{m(i)-1}) + 2S(u_{n(i)-1}, u_{n(i)-1}, u_{n(i)}) + S(u_{m(i)-1}, u_{m(i)-1}, u_{n(i)}) \tag{3.3.2}$$

And,

$$S(u_{m(i)-1}, u_{m(i)-1}, u_{n(i)}) \leq 2S(u_{m(i)-1}, u_{m(i)-1}, u_{m(i)}) + S(u_{m(i)}, u_{m(i)}, u_{n(i)}) \tag{3.3.3}$$

Making $i \to \infty$ in the inequality (3.3.2) and (3.3.3), we get,

$$\varepsilon \leq \lim_{i\to\infty} S(u_{m(i)-1}, u_{m(i)-1}, u_{n(i)}) \text{ and}$$

$$\lim_{i\to\infty} S(u_{m(i)-1}, u_{m(i)-1}, u_{n(i)}) \leq \varepsilon$$

Combining the above inequalities, we get,

$$\lim_{i\to\infty} S(u_{m(i)-1}, u_{m(i)-1}, u_{n(i)}) = \varepsilon$$

Inequality (3.3.1) can be rewritten as,

$$\begin{aligned}
\varepsilon \leq \frac{1}{3}&[S(u_{n(i)}, u_{n(i)}, u_{m(i)-1}) + S(u_{n(i)-1}, u_{n(i)-1}, u_{m(i)}) \\
&-\rho\left(S(u_{n(i)}, u_{n(i)}, u_{m(i)-1}), S(u_{n(i)-1}, u_{n(i)-1}, u_{m(i)}), S(u_{m(i)-1}, u_{m(i)-1}, u_{n(i)-1})\right)] \\
= \frac{1}{3}&[2S(u_{n(i)}, u_{n(i)}, u_{m(i)}) + S(u_{m(i)}, u_{m(i)}, u_{m(i)-1}) + S(u_{n(i)-1}, u_{n(i)-1}, u_{m(i)}) \\
&-\rho(2S(u_{n(i)}, u_{n(i)}, u_{m(i)}) + S(u_{m(i)}, u_{m(i)}, u_{m(i)-1}), \\
&S(u_{n(i)-1}, u_{n(i)-1}, u_{m(i)}), S(u_{m(i)-1}, u_{m(i)-1}, u_{n(i)}))] \\
= \frac{1}{3}&[2S(u_{m(i)}, u_{m(i)}, u_{n(i)}) + S(u_{m(i)-1}, u_{m(i)-1}, u_{m(i)}) + S(u_{m(i)}, u_{m(i)}, u_{n(i)-1}) \\
&-\rho(2S(u_{m(i)}, u_{m(i)}, u_{n(i)}) + S(u_{m(i)-1}, u_{m(i)-1}, u_{m(i)}), \\
&S(u_{m(i)}, u_{m(i)}, u_{n(i)-1}), S(u_{m(i)-1}, u_{m(i)-1}, u_{n(i)-1}))]
\end{aligned}$$

Making $i \to \infty$, we get,

$$\varepsilon \leq [2\varepsilon + \varepsilon - \rho(2\varepsilon, \varepsilon, \varepsilon)]$$

$$\varepsilon \leq \varepsilon - \frac{\rho}{3}(2\varepsilon, \varepsilon, \varepsilon)$$

$$\Rightarrow 0 \leq -\frac{\rho}{3}(2\varepsilon, \varepsilon, \varepsilon)$$

$$\Rightarrow \rho(2\varepsilon, \varepsilon, \varepsilon) \leq 0$$

which is a contradiction.

Since $\rho(u, v, w) > 0$ if $u \neq v \neq w \neq 0$ and $\rho(u, v, w) = 0$ only if one of u, v, w is zero.

Here, $\varepsilon > 0$. Which implies $2\varepsilon > 0$ and hence $\rho(2\varepsilon, \varepsilon, \varepsilon) > 0$.

Thus we get, $\{u_n\}$ is cauchy.

Also, our space (X, S) is complet S-metric space, every Cauchy Sequence is necessarily convergent in X.

This leads us to $\{u_n\}$ is converge to a point in X.

Let $\{u_n\} \to z$.

Then

$$\begin{aligned}
S(z, z, Cz) &\leq 2S(z, z, u_{n+1}) + S(u_{n+1}, u_{n+1}, Cz) \qquad \text{[by Lemma 2.8]} \\
&= 2S(z, z, u_{n+1}) + S(Cu_n, Cu_n, Cz) \\
&\leq 2S(z, z, u_{n+1}) + \frac{1}{3}[S(u_n, u_n, Cz) + S(z, z, Cu_n) \\
&\quad -\rho(S(u_n, u_n, Cz), S(z, z, Cu_n), S(u_n, u_n, z))] \\
&= 2S(z, z, u_{n+1}) + \frac{1}{3}[S(u_n, u_n, Cz) + S(z, z, u_{n+1}) \\
&\quad -\rho(S(u_n, u_n, z), S(z, z, u_{n+1}), S(u_n, u_n, z))]
\end{aligned}$$

Taking $n \to \infty$

$$S(z,z,Cz) \leq 2S(z,z,z) + \frac{1}{3}[S(z,z,Cz) + S(z,z,z)$$
$$-\rho\big(S(z,z,z), S(z,z,z), S(z,z,z)\big)$$
$$S(z,z,Cz) \leq \frac{1}{3}S(z,z,Cz)$$

This is possible only if $S(z,z,Cz) = 0$

$$\Rightarrow z = Cz \qquad [\because \text{by } (S2)]$$

(i.e) C has a fixed point z.

To prove the uniqueness

Suppose C has two fixed points say z_1 and z_2. (i.e) $Cz_1 = z_1$ and $Cz_2 = z_2$.

Then,

$$\begin{aligned} S(z_1,z_1,z_2) &= S(Cz_1,Cz_1,Cz_2) \\ &\leq \frac{1}{3}[S(z_1,z_1,Cz_2) + S(z_2,z_2,Cz_1) \\ &\quad -\rho\big(S(z_1,z_1,Cz_2), S(z_2,z_2,Cz_1), S(z_1,z_1,z_2)\big)] \\ &\leq \frac{1}{3}[S(z_1,z_1,z_2) + S(z_2,z_2,z_1) \\ &\quad -\rho\big(S(z_1,z_1,z_2), S(z_2,z_2,z_1), S(z_1,z_1,z_2)\big)] \\ &\leq \frac{1}{3}[2S(z_1,z_1,z_2) - \rho\big(S(z_1,z_1,z_2), S(z_2,z_2,z_1), S(z_1,z_1,z_2)\big)] \end{aligned}$$

$$\Rightarrow \frac{1}{3}S(z_1,z_1,z_2) \leq -\frac{\rho}{3}\big(S(z_1,z_1,z_2), S(z_2,z_2,z_1), S(z_1,z_1,z_2)\big)$$

Suppose $z_1 \neq z_2$.

Since $\frac{1}{3}S(z_1,z_1,z_2) > 0$ and $\rho\big(S(z_1,z_1,z_2), S(z_2,z_2,z_1), S(z_1,z_1,z_2)\big) > 0$

$$\Rightarrow 0 < -\rho\big(S(z_1,z_1,z_2), S(z_2,z_2,z_1), S(z_1,z_1,z_2)\big)$$
$$\Rightarrow \rho\big(S(z_1,z_1,z_2), S(z_2,z_2,z_1), S(z_1,z_1,z_2)\big) < 0$$

which is a contradiction.

$$\therefore z_1 = z_2.$$

Hence proved.

Example 3.4

The self mapping C defined in example (3.2) is weakly C_S'-contraction in complete S-metric Space.

Hence by Theorem (3.3), the function C has unique fixed point and the fixed point is $'0'$.

Example 3.5

Define $Cu = \begin{cases} \frac{1}{5}, u \in \left[0, \frac{1}{2}\right] \\ \frac{3}{4}, u \in \left(\frac{1}{2}, 1\right] \end{cases}$

the metric S and the mapping ρ as in Example (3.2).

Let $u = \frac{1}{3} \in \left[0, \frac{1}{2}\right], v = \frac{5}{6} \in \left(\frac{1}{2}, 1\right]$, then,$Cu = \frac{1}{5}, Cv = \frac{3}{4}$

$$S(Cu,Cu,Cv) = S\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{4}\right) = 2\left|\frac{1}{5} - \frac{3}{4}\right| = \frac{2 \times 11}{20} = \frac{11}{10}$$
$$S(u,u,Cv) = S\left(\frac{1}{3}, \frac{1}{3}, \frac{3}{4}\right) = 2\left|\frac{1}{3} - \frac{3}{4}\right| = 2 \times \frac{5}{12} = \frac{5}{6}$$
$$S(v,v,Cu) = S\left(\frac{5}{6}, \frac{5}{6}, \frac{1}{5}\right) = 2\left|\frac{5}{6} - \frac{1}{5}\right| = 2 \times \frac{19}{30} = \frac{19}{15}$$
$$S(u,u,v) = S\left(\frac{1}{3}, \frac{1}{3}, \frac{5}{6}\right) = 2\left|\frac{1}{3} - \frac{5}{6}\right| = 2 \times \frac{9}{18} = 1$$

$$\frac{1}{3}[S(u,u,Cv) + S(v,v,Cu) - \rho\big(S(u,u,Cv), S(v,v,Cu), S(u,u,v)\big)]$$

$$= \frac{1}{3}\left[\frac{5}{6} + \frac{19}{15} - \rho\left(\frac{5}{6}, \frac{19}{15}, 1\right)\right]$$
$$= \frac{1}{3}\left[\frac{5}{6} + \frac{19}{15} - \frac{5}{6}\right]$$
$$= \frac{1}{3}\left[\frac{19}{15}\right]$$
$$= \frac{19}{45}$$
$$< \frac{11}{10}$$
$$S(Cu, Cu, Cv) > \frac{1}{3}\left[S(u, u, Cv) + S(v, v, Cu) - \rho\big(S(u, u, Cv), S(v, v, Cu), S(u, u, v)\big)\right]$$

Hence, by Theorem (3.3), it has no unique fixed point.

Here, the function C has 2 fixed points $\frac{1}{5}$ and $\frac{3}{4}$.

Example 3.6

Define $Cu = \begin{cases} u + \frac{1}{2}, u \in \left[0, \frac{1}{2}\right] \\ 0, \text{otherwise} \end{cases}$

the metric S and the mapping ρ as in Example (3.2).

Let $u = \frac{1}{3}, v = \frac{2}{3}$, then $Cu = \frac{5}{6}, Cv = 0$

$$S(Cu, Cu, Cv) = S\left(\frac{5}{6}, \frac{5}{6}, 0\right) = 2\left|\frac{5}{6} - 0\right| = \frac{5}{3}$$
$$S(u, u, Cv) = S\left(\frac{1}{3}, \frac{1}{3}, 0\right) = 2\left|\frac{1}{3}\right| = \frac{2}{3}$$
$$S(v, v, Cu) = S\left(\frac{2}{3}, \frac{2}{3}, \frac{5}{6}\right) = 2\left|\frac{5}{6}\right| = \frac{5}{3}$$
$$S(u, u, v) = S\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) = 2\left|\frac{1}{3} - \frac{2}{3}\right| = \frac{2}{3}$$
$$\frac{1}{3}\left[S(u, u, Cv) + S(v, v, Cu) - \rho\big(S(u, u, Cv), S(v, v, Cu), S(u, u, v)\big)\right]$$
$$= \frac{1}{3}\left[\frac{2}{3} + \frac{1}{3} - \rho\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)\right]$$
$$= \frac{1}{3}\left[1 - \frac{1}{3}\right]$$
$$= \frac{1}{3}\left[\frac{2}{3}\right]$$
$$= \frac{2}{9}$$
$$< \frac{5}{3}$$
$$S(Cu, Cu, Cv) > \frac{1}{3}[S(u, u, Cv) + S(v, v, Cu) - \rho(S(u, u, Cv), S(v, v, Cu), S(u, u, v))]$$

Hence, by Theorem (3.3), C has no unique fixed point.

Here, the function C h as no fixed point.

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