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Internally Consistent Fuzzy Linear Operator in a Fuzzy Hilbert Space

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ABSTRACT

This work-study is concerned with the adjoint fuzzy linear operator and the self-adjoint fuzzy linear operator operating on a fuzzy Hilbert space (FH-Space). Several definitions, several basic statements about positive fuzzy operators, and a multitude of theorems are covered in depth, along with the characteristics of the adjoint and self-adjoint fuzzy operators in FH-adjoint fuzzy operators in an FH-space.

Key Words: Adjoint Fuzzy Operators, Self-Adjoint Fuzzy operator, FH-space, FIP-Space.
AMS subject classifications: 54H25, 47S40, 03E72.

1. Introduction

In 1965, Zadeh [7] introduced the concept of a fuzzy set. Later, Heilpern [3] extended this idea by defining fuzzy mappings as functions from an arbitrary set to a subset of fuzzy sets in a metric linear space, also proving a fixed point theorem for these fuzzy mappings. Subsequent work by various authors [5], [6] has expanded upon Heilpern's findings. In this study, we establish fixed point theorems for fuzzy mappings as initially proposed by Heilpern and adapted for use in Hilbert spaces [3]. The notion of a fuzzy inner product space (FIP-space), which generalizes the traditional inner product space, was further explored by Felbin [1], Gani, and Manikandan [4]. Research into fuzzy Hilbert spaces has been conducted by Goudarzi, MandVaezpour, and S. M.

The following rules apply to the paper: There are various early findings in Section 2. The concept of adjoint fuzzy linear operators, self-adjoint fuzzy linear operators, many theorems, and a discussion of some of these fuzzy operators' features are introduced in section three.

2 Preliminaries

In the following discussions, we mainly follow the definitions and notations due to Heilpern. Let H represent a Hilbert space, with F(H) indicating the entire collection of fuzzy sets contained in H. Let $\mathbf{P_\sigma} \in \mathbf{F(H)}$ and $\boldsymbol{\sigma} \in [\mathbf{0}, \mathbf{1}]$. The $\boldsymbol{\sigma}$-level set of $\mathbf{P}$, denoted by $\mathbf{P_\sigma}$ is defined as

$$P_\sigma = \{y: P(y) \geq \sigma\} \text{if } \alpha \in (0,1]$$

$$P_0 = \{y: P(y) > 0\},$$

Where $\overline{\mathbf{B}}$ stands for the closure of a set B.

2.1 Definition

A fuzzy subset A of F(H) is termed an approximate quantity if and only if its α-level set is a non fuzzy, compact, convex subset of F(H) for each $\alpha \in [0,1]$ and $\sup_{x \in F(H)} A(x) = 1$. The sub collection of all approximate quantities within F(H) is denoted by W(H).

2.2 Definition

Let A in $F(H)$ and α in [0,1] such that ||A||=α or $n(P_A) = \alpha$, then the pair (P_A, α) is called a fuzzy point in $F(H)$ and it is denoted by P_A^α. The dual fuzzy point P_A^α is the point with norm (1-α) denoted by $P^* = P^{1-\alpha}$.

2.3 Definition

The set of all fuzzy points in $F(H)$ is gven by $P^*(F(H)) = \{P_A^\alpha | A \epsilon F(H), \ \alpha \epsilon [0,1]\}$. In F we follow the usual ≤ order relation correspondingly we define an order relation in $P^*(F(H))$.

2.4 Definition

We define $P_A^\alpha < P_B^\beta$ iff $\alpha < \beta$ and $P_A^\alpha = P_B^\beta \text{ iff } A = B$ (then automatically $\alpha = \beta$).

2.5 Definition

A fuzzy Hilbert space $(F(H))$is a vector space over $[0,1]$with a mapping $P^*(F(H)) \times P^*(F(H)) \to [0,1]$ referred to as the scalar product and denoted by (P_x, P_y) which satisfies the following

(i) $(P_x, P_y) = (P_y, P_x)$

(ii) $(P_{x_1} + P_{x_2}, P_y) = (P_{x_1}, P_y) + (P_{x_2}, P_y)(P_{x_1}, P_{x_2}, P_y \in P^*(F(H)))$

(iii) $(\alpha P_x, P_y) = \alpha(P_x, P_y)(P_x, P_y \in P^*(F(H)), \alpha \in [0,1])$

(iv) $(P_x, P_y) > 0 \; for x \neq 0; (P_x, P_x) = 0$ $for P_x = 0 (P_x \in P^*(F(H)))$

(v) $P^*(F(H))$ is a Banach space with the norm $n(P_x) = (P_x, P_x)^{\frac{1}{2}}$.

Let S be the class of all F-bounded linear mappings of T into itself and F-bounded symmetric operators in F(H), where F(H) is a real or complex Hilbert space.

$$(TP_x, P_y) = (P_x, TP_y)(P_x, P_y \in P^*(F(H)))$$

In the case where S=S∗, the class bounded self adjoint operators. When introducing a relation ≤ into S, write A ≤ B or B ≥ A to indicate that

$$(P_A P_x, P_x) \leq (P_B P_x, P_x)\left(P_x \in P^*(F(H))\right).$$

Positive operators are those operators T belonging to S such that T ≥ 0.

2.6 Definition

Let $(E, F, *)$ be a probabilistic inner product space.

1. A sequence $\{P_{x_n}\} \in E$is called F–converges to $P_x \in E$, iff or any$\epsilon > 0$and $\lambda > 0, \exists N \in Z+, N = N(\epsilon, \lambda)$ Such that $Fxn - x, xn - x(\epsilon) > 1 - \lambda$ whenever $n > N$.
2. A linear functional$f(P_x)$defined on E is called F−continuous, if $P_{x_n} \to P_x$ implies $f(P_{x_n}) \to f(P_x)$ for any $\{P_{x_n}\}, P_x \in E$.

2.7 Definition

Let $(E, G, *)$ be a F(H) – space with $(u, v) = \sup\{x \in \mathbb{R}: G(u, v, x) < 1\}, \forall u, v \in E$ and let $S \in FB(E)$, then S is self–adjoint Fuzzy operator, if $S = S^*$ where S^* is adjoint Fuzzy operator of S.

3. Some Elementary Propositions on

Positive Fuzzy Operators.

Proposition. 1

According to the generalized Schwartz inequality, for any positive operator T,$|(\mathbf{TP_x}, \mathbf{P_y})|^2 \leq (\mathbf{TP_x}, \mathbf{P_x})(\mathbf{TP_y}, \mathbf{P_y})$

Proof.

If $\mathbf{B}(\mathbf{P_x}, \mathbf{P_y}) = (\mathbf{TP_x}, \mathbf{P_y})$ is a bi-linear form that is positive semi-definite symmetric, and hence the generalized Schwarz inequality for this form.

Proposition. 2

If the operator T is positive, then $\mathbf{n}(\mathbf{T}) = \mathbf{sup}\{(\mathbf{TP_x}, \mathbf{P_x}) : \mathbf{n}(\mathbf{P_x}) \leq \mathbf{1}\}$

Proof.

If the operator T is positive, and let $\mathbf{M} = \mathbf{sup}\{(\mathbf{TP_x}, \mathbf{P_x}) : \mathbf{n}(\mathbf{P_x}) \leq \mathbf{1}\}$

By Schwarz inequality

$|(\mathbf{TP_x}, \mathbf{P_x})| \leq \mathbf{n}(\mathbf{TP_x})\mathbf{n}(\mathbf{P_x})$,

$\mathbf{M} \leq \mathbf{n}(\mathbf{T})$ …………………………(1)

putting $\mathbf{P_y}$=T$\mathbf{P_x}$ in the generalized Schwarz inequality, we have

$$\mathbf{n}(\mathbf{TP_x})^4 = (\mathbf{TP_x}, \mathbf{TP_x})^2 = |\mathbf{TP_x}, \mathbf{P_y}|^2$$
$$\leq (\mathbf{TP_x}, \mathbf{P_x})(\mathbf{TP_y}, \mathbf{P_y})$$
$$\leq \mathbf{M}^2\mathbf{n}(\mathbf{P_x})^2\mathbf{n}(\mathbf{TP_x})^2,$$
$$\mathbf{n}(\mathbf{T}) \leq \mathbf{M} \quad \text{……………………..} \quad (2)$$

From (1) and (2)

$\mathbf{n}(\mathbf{T}) = \mathbf{sup}\{(\mathbf{TP_x}, \mathbf{P_x}) : \mathbf{n}(\mathbf{P_x}) \leq \mathbf{1}\}$.

Proposition. 3

For each element of S, let $(\mathbf{T_n})$ be a fuzzy bounded rising sequence, $\mathbf{T_n} \leq \mathbf{T_{n+1}} \leq \mathbf{M.I}$. Then, to an element T of S, $(\mathbf{T_n})$ F-converges strongly.

$$\lim_{n\to\infty} \mathbf{T_n P_x} = \mathbf{TP_x}(\mathbf{P_x} \in \mathbf{P}^*(\mathbf{F}(\mathbf{H})))$$

Proof.

For m < n,

Let $\mathbf{P_{A_{m,n}}}$ = Tn − Tm.

By the generalized Schwartz inequality with $\mathbf{T} = \mathbf{P_{A_{m,n}}}$ and $\mathbf{y} = \mathbf{P_{A_{m,n}}P_x}$,

We have $\mathbf{n}(\mathbf{P_{A_{m,n}}P_x})^4 = |(\mathbf{P_{A_{m,n}}P_x}, \mathbf{P_{A_{m,n}}P_x})|^2$

$= |\mathbf{P_{A_{m,n}}P_x}, \mathbf{P_y}|^2$

$\leq (\mathbf{P_{A_{m,n}}P_x}, \mathbf{P_x})(\mathbf{P_{A_{m,n}}P_y}, \mathbf{P_y})$.

Since$\mathbf{0} \leq \mathbf{P_{A_{m,n}}} \leq \mathbf{MI}$,wehave$(\mathbf{P_{A_{m,n}}P_y}, \mathbf{P_y}) \leq \mathbf{M}^3\mathbf{n}(\mathbf{P_x})^2$

Hence $\mathbf{n}(\mathbf{T_nP_x} - \mathbf{T_mP_x})^4 \leq \mathbf{M}^3\mathbf{n}(\mathbf{P_x})^2\,\{\mathbf{T_nP_x}, \mathbf{P_x} - \mathbf{T_mP_x}, \mathbf{P_x}\}$.

Since the F-sequence (Tn$\mathbf{P_x}$, $\mathbf{P_x}$) is a F-bounded rising sequence of real numbers, it follows $(\mathbf{T_nP_x})$ is a F-Cauchy sequence which F-converges to $\mathbf{TP_x} \in \mathbf{P}^*(\mathbf{F}(\mathbf{H}))$.

Proposition. 4

If $\mathbf{T} \geq \mathbf{0}, \mathbf{I} + \mathbf{T}$ is invertible, $(\mathbf{I} + \mathbf{T} - \mathbf{1}) \geq \mathbf{0}$, and $(\mathbf{I} + \mathbf{T}) - \mathbf{1} \in (\mathbf{T})$.

Proof.

Now, we have

$$\mathbf{I} \leq \mathbf{I} + \mathbf{T} \leq (\mathbf{1} + \mathbf{M})\mathbf{I},$$
$$\frac{\mathbf{1}}{\mathbf{1} + \mathbf{M}} \leq \mathbf{P_A} \leq \mathbf{I}$$

Where $\mathbf{P_A} = \frac{\mathbf{1}}{\mathbf{1+M}}(\mathbf{I} + \mathbf{T})$

Therefore

$$\mathbf{n}(\mathbf{I} - \mathbf{P_A}) \leq \mathbf{n}\left(\mathbf{1} - \left(\frac{\mathbf{1}}{\mathbf{1+M}}\right)\mathbf{I}\right) = \frac{\mathbf{M}}{\mathbf{1+M}} < \mathbf{1}.$$

Proposition. 5

If $\mathbf{P_A} \geq \mathbf{0}, \mathbf{P_B} \geq \mathbf{0}, \mathbf{P_AP_B} = \mathbf{P_BP_A}$ **then** $\mathbf{P_AP_B} \geq \mathbf{0}$.

Proof.

Since $\mathbf{P_A} \in (\mathbf{P_B})$, we have$(\mathbf{P_B})^{\frac{1}{2}} \in (\mathbf{P_A})$ and so

$$\mathbf{P_AP_B} = \mathbf{P_A}(\mathbf{P_B})^{\frac{1}{2}}(\mathbf{P_B})^{\frac{1}{2}}$$

$$\mathbf{P_A P_B = (P_B)^{\frac{1}{2}} P_A (P_B)^{\frac{1}{2}}}$$

Therefore $\left((\mathbf{P_B})^{\frac{1}{2}}\mathbf{P_A}(\mathbf{P_B})^{\frac{1}{2}}\mathbf{x}, \mathbf{x}\right) = (\mathbf{P_A}(\mathbf{P_B})^{\frac{1}{2}}\mathbf{x}, (\mathbf{P_B})^{\frac{1}{2}}\mathbf{x}) \geq \mathbf{0}$.

Theorem.1:

Let $\mathbf{P_A} \geq \mathbf{0}$, and let $\mathbf{P_B} = \mathbf{2}(\mathbf{P_A})^2(\mathbf{I} + (\mathbf{P_A})^2)^{-1}$. Then

1. $\mathbf{P_B} \in (\mathbf{P_A})$,
2. $\mathbf{0} \leq \mathbf{P_B} \leq \mathbf{P_A}$,
3. $\mathbf{I} - \mathbf{P_B} = (\mathbf{I} - \mathbf{P_A})(\mathbf{I} + \mathbf{P_A})(\mathbf{I} + (\mathbf{P_A})^2)^{-1}$,
4. if $\mathbf{P}_\wp$ is a projective permutable with $\mathbf{P_A}$ and $\mathbf{P}_\wp \leq \mathbf{P_A}$, then $\mathbf{P}_\wp \leq \mathbf{P_B}$, for some $\mathbf{P_A}, \mathbf{P_B}, \mathbf{P}_\wp \in \mathbf{P}^*(\mathbf{F(H)})$

Proof.

Proposition (5) ⇒ (1).

That $\mathbf{P_B} \geq \mathbf{0}$. since $(\mathbf{P_A})^2$ and $(\mathbf{I} + (\mathbf{P_A})^2)^{-1}$ are permutable.

Also $(\mathbf{I} + (\mathbf{P_A})^2)(\mathbf{P_A} - \mathbf{P_B}) = \mathbf{P_A} + (\mathbf{P_A})^3 - \mathbf{2}(\mathbf{P_A})^2 = \mathbf{P_A}(\mathbf{I} - \mathbf{P_A})^2 \geq \mathbf{0}$,

$$\mathbf{P_A} - \mathbf{P_B} = (\mathbf{I} + (\mathbf{P_A})^2) - \mathbf{1}(\mathbf{I} + (\mathbf{P_A})^2)(\mathbf{P_A} - \mathbf{P_B}) \geq \mathbf{0}$$

This proves (2), and (iii) is obvious.

Let $\mathbf{P}_\wp$ be a projection such that $\mathbf{P}_\wp \in \mathbf{P_A}'$ and $\mathbf{P}_\wp \leq \mathbf{P_A}$. We have

$$\mathbf{P}_\wp = (\mathbf{P}_\wp)^2 \leq \mathbf{P}_\wp \mathbf{P_A} \leq (\mathbf{P_A})^2$$

$$\mathbf{P}_\wp = (\mathbf{P}_\wp)^2 \leq (\mathbf{P_A})^2 \mathbf{P}_\wp.$$

$$(\mathbf{I} + (\mathbf{P_A})^2)(\mathbf{P_B} - \mathbf{P}_\wp) = \mathbf{2}(\mathbf{P_A})^2 - (\mathbf{I} + (\mathbf{P_A})^2)\mathbf{P}$$

$$\geq \mathbf{2}(\mathbf{P_A})^2 - \mathbf{2}(\mathbf{P_A})^2\mathbf{P}$$

$$= \mathbf{2}(\mathbf{P_A})^2(\mathbf{I} - \mathbf{P}_\wp) \geq \mathbf{0}$$

Since all the fuzzy operators involved in $(\mathbf{I} + (\mathbf{P_A})^2)^{-1}$ are permutable,

for some $\mathbf{P_A}, \mathbf{P_B}, \mathbf{P}_\wp \in \mathbf{P}^*(\mathbf{F(H)})$

$$\mathbf{P_B} - \mathbf{P}_\wp \geq (\mathbf{I} + (\mathbf{P_A})^2)^{-1}\mathbf{2}(\mathbf{P_A})^2(\mathbf{I} - \mathbf{P}_\wp) \geq \mathbf{0}.$$

Theorem.2:

Assume that $\mathbf{P_A} \in \mathbf{P}^*(\mathbf{F(H)})$ is a positive operator. Then, define the sequence $(\mathbf{P_{A_m}})$ inductively by $\mathbf{P_{A_1}} = \mathbf{P_A}, \mathbf{P_{A_{m+1}}} = \mathbf{2}(\mathbf{P_{A_m}})^2(\mathbf{I} + (\mathbf{P_{A_m}})^2)^{-1}(\mathbf{m} = \mathbf{1}, \mathbf{2}, \ldots)$

Then

1. $\mathbf{0} \leq \mathbf{P_{A_{m+1}}} \leq \mathbf{P_{A_m}}(\mathbf{m} = \mathbf{1}, \mathbf{2}, \ldots)$,
2. The sequence $(\mathbf{P_{A_m}})$ converges strongly to a projection $\mathbf{Q}$ belonging to $(\mathbf{P_A})$.
3. $\mathbf{Q} \leq \mathbf{P_A}$,
4. $(\mathbf{I} - \mathbf{P_A})(\mathbf{I} - \mathbf{Q}) \geq \mathbf{0}$,
5. If $\mathbf{P}_\wp$ is a projection that is permutable with $\mathbf{P_A}$, then Q is maximal in that sense. Furthermore, if $\mathbf{P}_\wp \leq \mathbf{P_A}$, then $\mathbf{P}_\wp \leq \mathbf{Q}$.

Proof.

(1) This follows from the previous theorem. (2) and (3) follows from (1) and Proposition (3) that $(\mathbf{P_{A_m}})$ strongly converges to a positive operator $\mathbf{Q}$ with $\mathbf{Q} \leq \mathbf{P_A}$, and that $\mathbf{Q} \in (\mathbf{P_A})$. It remains that $\mathbf{Q}$ is a projection.

Since $\mathbf{0} \leq \mathbf{P_{A_m}} \leq \mathbf{P_A}$, we have

$$\mathbf{n}(\mathbf{P_{A_m}}) \leq \mathbf{n}(\mathbf{P_A})(\mathbf{m} = \mathbf{1}, \mathbf{2}, \ldots);$$

Therefore $\lim\limits_{\mathbf{n}\to\infty} \mathbf{P_{A_m}} \mathbf{P_x} = \mathbf{Q}\mathbf{P_x}$ for some $\mathbf{P_A}, \mathbf{P_x} \in \mathbf{P}^*(\mathbf{F(H)})$

$\lim\limits_{\mathbf{n}\to\infty} (\mathbf{P_{A_m}})^2\mathbf{P_x} = \mathbf{Q}^2\mathbf{P_x}$ for some $\mathbf{P_A}, \mathbf{P_x} \in \mathbf{P}^*(\mathbf{F(H)})$

$\lim\limits_{\mathbf{n}\to\infty} \mathbf{P_{A_{m+1}}} \{(\mathbf{I} + \mathbf{Q}^2)\mathbf{P_x} - (\mathbf{I} + (\mathbf{P_{A_m}})^2)\mathbf{P_x}\} = \mathbf{0}$, for some $\mathbf{P_x} \in \mathbf{P}^*(\mathbf{F(H)})$

Therefore $(\mathbf{Q} - \mathbf{Q}^2)^2 = \mathbf{0}$, But $\mathbf{Q} - \mathbf{Q}^2$ is symmetric, this gives $(\mathbf{Q} - \mathbf{Q})^2 = \mathbf{0}$, hence Q is a projection.

(4) $(\mathbf{I} - \mathbf{P_{A_m}}) = (\mathbf{I} - \mathbf{P_{A_{m-1}}})(\mathbf{I} + \mathbf{P_{A_{m-1}}})(\mathbf{I} + (\mathbf{P_{A_{m-1}}})^{-1})$

$$(\mathbf{I} - \mathbf{P_A})(\mathbf{I} - \mathbf{P_{A_m}}) \geq \mathbf{0}$$

$$(\mathbf{I} - \mathbf{P_A})(\mathbf{I} - \mathbf{Q}) \geq \mathbf{0}$$

(5) Assume that $\mathbf{P_A}$ and $\mathbf{P}_\wp$ are

permutable projections such that $\mathbf{P}_{\wp} \leq \mathbf{P}_{\mathbf{A}}$. When (1) and (4) are applied repeatedly, $\mathbf{P}_{\wp}$ becomes permutable with $\mathbf{P}_{\mathbf{A}_\mathbf{m}}$ and $\mathbf{P}_{\wp} \leq \mathbf{P}_{\mathbf{A}_\mathbf{m}}$. We have $\mathbf{P}_{\wp} \leq \mathbf{Q}$ in the limit.

4. Conclusion

The adjacent in this study, definitions of fuzzy linear operators are inserting able into a fuzzy Hilbert space. Several basic theorems and elementary assertions were illustrated using adjoint and self-adjoint fuzzy operators in FH-space.

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