Library Progress International *Print version ISSN 0970 1052*
Vol.44 No. 3, Jul-Dec 2024: P.1373-1385 *Online version ISSN 2320 317X*

Original Article *Available online at www.bpasjournals.com*

Comparative Analysis of Propagation of Own Damping Waves in A Viscoelastic Waveguide of a Sectoral Cross Section

I.I. Safarov[1], M.Kh.Teshaev[2,3,4], Z.I. Boltaev[2], T.R. Ruziev[5], R.A.Sobirova[2]

[1]Tashkent Institute of Chemical Technology, Tashkent, Uzbekistan
[2]Bukhara engineering-technological institute, Bukhara, Uzbekistan
[3]Bukhara Branch of Institute of Mathematics AS RUz., Bukhara, Uzbekistan
[4]Bukhara branch of the Tashkent Institute of Irrigation and Mechanization Engineering in Agricultural, Bukhara, Uzbekistan
[5]Pedagogical institute of bukhara state university, Bukhara, Uzbekistan

How to cite this article: I.I. Safarov, M.Kh.Teshaev, Z.I. Boltaev, T.R. Ruziev, R.A.Sobirova (2024) Comparative Analysis of Propagation of Own Damping Waves in A Viscoelastic Waveguide of a Sectoral Cross Section, 44(3), 1373-1385.

ABSTRACT
The main purpose of the article is a comparative study of the existence of natural waves in an infinite viscoelastic waveguide with a sectoral cross section and a plate (based on the hypotheses of Kirchhoff and Timoshenko) with a variable cross section (in the form of a wedge) depending on various parameters of the object (wave number and geometric parameters). Using the Navier equations and the physical relation, a system of six differential equations in partial derivatives for a sector waveguide is obtained. After simple transformations, a system of differential equations with complex coefficients is obtained, which is further solved using the method of straight lines, which will allow using the software tool of the orthogonal sweep method with a combination of the Muller method on complex arithmetic in solving. Also, on the basis of the variational principle, a system of ordinary differential equations with complex coefficients is given. On the basis of numerical calculations, it has been established that the real parts of the phase velocity of propagation of the first mode are less from the Rayleigh wave velocity to 20%, on the segment (the central angle in the wedge-shaped waveguide), and further asymptotically approaches the Rayleigh wave, which propagates in the viscoelastic half-plane. Similar results were obtained for a wedge-shaped plate according to the theory of plates by Kirchhoff and Timoshenko, in the entire wave range. The results on the dynamic theory of elasticity and the plate (based on the hypotheses of Kirchhoff and Timoshenko) differ by no more than 6% for wedge apex angles not exceeding 28°. In calculation results differ up to 20 %. It has been established that at small wedge angles, the simplified theory of Kirchhoff – Lyava and Timoshenko can be used in the entire wave range.

Key words: damped wave, viscoelastic cylinder, sectoral cross section, Navier equation, spectral boundary value problem, orthogonal sweep.

INTRODUCTION

Elastic waves have been studied for more than a hundred years, but work in this direction does not stop, which indicates both the undying interest in this problem and the lack of knowledge of the subject. In [1], a three-dimensional problem of wave propagation in an elastic layer is considered. A characteristic equation is obtained for the phase velocity of symmetric and antisymmetric oscillations. The limiting case is considered: the wavelength is very large and very small compared to the layer thickness.

The article [2] studied the propagation of shear waves in a two-layer medium in an antiplanar setting. We consider the propagation of shear waves in a two-layer medium, when one layer is homogeneous and the other is inhomogeneous with exponential inhomogeneity. The dispersion equation has been studied. In monograph [3], the propagation of natural waves in extended waveguides is studied and dispersion relations are constructed. The change in phase and group wave velocities as a function of wave numbers is analyzed. In [4], at the limiting values of the parameters, the lower modes of the roots of the Pochhammer-Cree dispersion equation were obtained. As a result, they obtained a relationship between the phase velocity and the wave number. And in [5], these results were confirmed. They derive the dispersion equation from the system of differential equations of elasticity theory. The rod equation [6] was used to study the propagation of waves in a loaded reinforcement. In [7], with the help of a correction factor, dispersion curves are given for the phase and group velocities of a wave propagating in round rods. In the article [8], the Pochhammer-Cree dispersion equation was studied in a wide frequency range. The phase velocities of a one-dimensional wave were studied in [9] with and without dispersion. It was concluded in [9] that a one-dimensional problem (although with dispersion taken into account) will not characterize all dynamic processes occurring in a rod in a spatial formulation.

In the article [10], differential equations of longitudinal free vibrations of a rod were obtained based on the variational principle of Hamilton. And also, in the article [11] the results of the above mentioned works were compared with the results obtained on the basis of the Pochhammer - Cree model. They concluded that the Bishop model quite accurately describes the dispersion relation of a rod with a circular cross section, but the dispersion curve asymptotically tends to the transverse wave velocity, while in the Pochhammer-Cree model, the asymptote is the Rayleigh wave velocity. In [12], the propagation of longitudinal waves in a rod in an axisymmetric formulation was studied and a characteristic equation was obtained.

In [13, 14, 15], the solution of the Lame equation was obtained using the Green's expansion, i.e., displacements are constructed through the potentials of longitudinal and transverse waves. It is emphasized that according to the representation completeness theorem (Green's expansion), any wave process in an infinite elastic body can be described as a superposition of wave motions with velocities of longitudinal and transverse waves. In [16], for the first time, for plates, a theory of Rayleigh wave propagation based on Kirchhoff's hypotheses was constructed, an analogue of the Rayleigh wave in the theory of bending vibrations of a plate. The plate was considered under conditions of a plane stress state. Since the problems of plane deformation of a cylindrical body and the plane stress state of a plate are mathematically identical. Then a planar wave of the Rayleigh type must exist in the plate, the velocity of which is determined from the same equation as the velocity of the Rayleigh wave, with the change of the Lamé elastic constant to the corresponding constant for the plate. This wave is called the Rayleigh-type bending wave or the Konenkov wave [17]. In [18], infinitely long plates or strips of variable thickness were studied taking into account the viscoelastic properties of materials.

Investigation of eigenwaves in a cylindrical viscoelastic waveguide with a sector cross section is an urgent task. Therefore, the present work is devoted to the study of the propagation of natural waves in a cylindrical viscoelastic waveguide with a sectoral cross section.

Methods

Statement of the problem and basic relations

In this work, propagation of natural waves in an infinite viscoelastic cylinder with a sector cross section (Fig. 1) is considered. The waveguide has a collinear axis directed along the axis Oz. The problem of analyzing the spectra of normal elastic waves along the considered waveguide is formulated using the relations of a spatial linear mathematical model of the dynamic stress-strain state of deformable bodies, taking into account viscoelastic properties, in a cylindrical coordinate system.

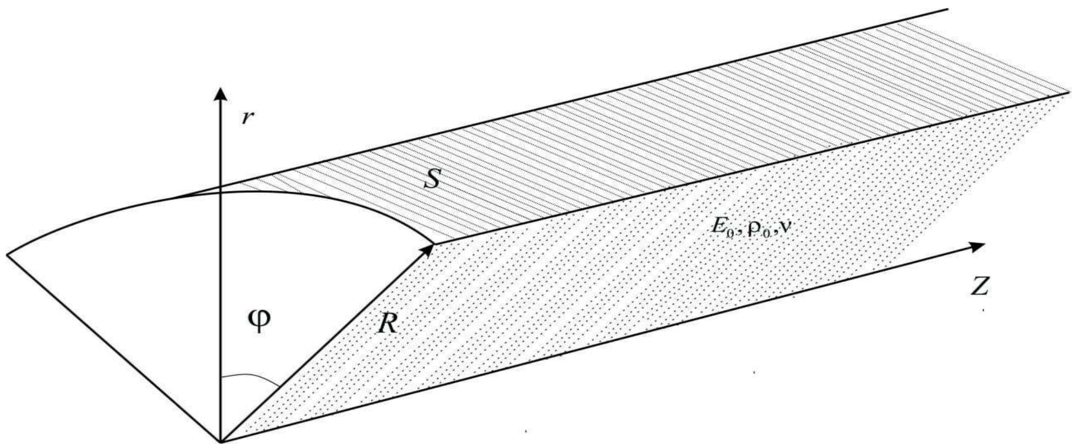

Figure 1. Calculation scheme of an infinite viscoelastic cylinder with sectoral cross section.

.

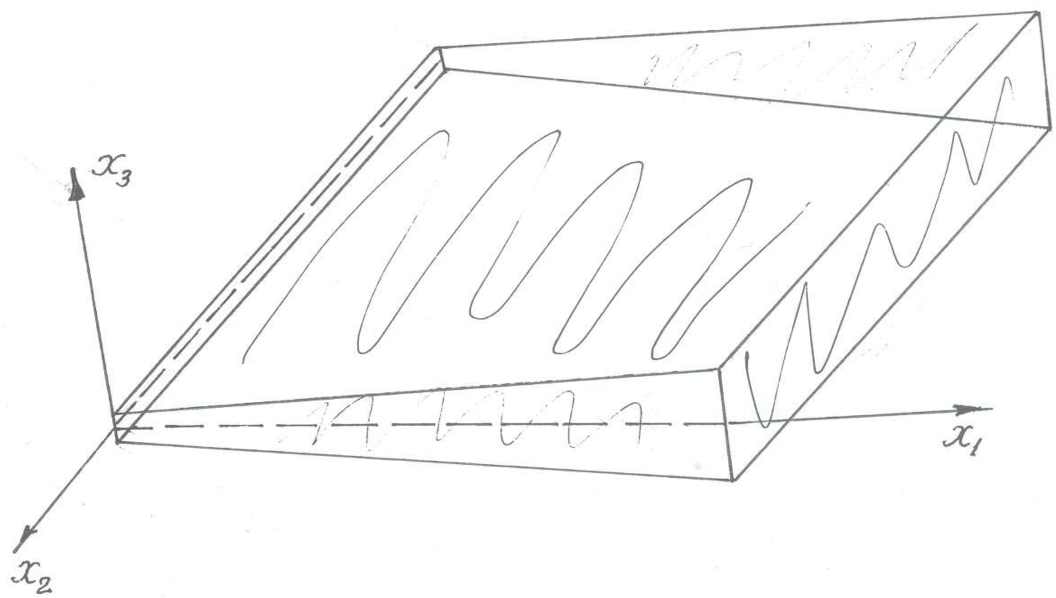

Figure 2. Calculation scheme for a plate of variable thickness.

These relations are formulated for the projections of the dimensionless vector of dynamic elastic wave displacements on the axes of the cylindrical coordinate system $\{u_r, u_\varphi, u_z\}$, as well as for the dimensionless characteristics of the stress-strain state of the object under consideration on the main areas of the cylindrical coordinate system $\{\sigma_{rr}, \sigma_{r\varphi}, \sigma_{\varphi\varphi}, \sigma_{zz}, \sigma_{rz}, \sigma_{\varphi z}\}$. The basic equations of motion of an elastic medium occupying region B are given by three groups of relations [19]:

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad , \tag{1}$$

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), \ \sigma_{ik} = \tilde{\lambda}\,\theta\delta_{ik} + 2\tilde{\mu}\,\varepsilon_{ik} \quad .$$

Here σ_{ik} - stress tensor elements, ε_{ik} - strain tensor elements, θ - bulk deformation, $\tilde{\lambda}$ and $\tilde{\mu}$ - complex quantities:

$$\tilde{\lambda}\left[f(t)\right] = \frac{\nu E_0}{(1+\nu)(1-2\nu)}\left[f(t)-\int_0^t R_\lambda(t-\tau)f(\tau)d\tau\right]; \quad , \quad (2)$$
$$\tilde{\mu}\left[f(t)\right] = \frac{\nu E_0}{2(1+\nu)}\left[f(t)-\int_0^t R_\mu(t-\tau)f(\tau)d\tau\right]$$

where $\nu-$ Poisson's ratio, taken constant; $\varphi(t)$– arbitrary function of time; $R_\lambda(t-\tau), R_\mu(t-\tau)$– relaxation nuclei; E_0– instant modulus of elasticity.

We take the integral terms in (2) small. Then the function $f(t)=\psi(t)e^{-i\omega_R t}$, where $\psi(t)$- slowly changing function of time, ω_R - real constant. Next, applying the freezing procedure [16], we replace relations (2) with approximate ones of the form

$$\bar{E}f = E\left[1-\Gamma^C(\omega_R)-i\Gamma^S(\omega_R)\right]f, \quad (3)$$

where $\Gamma^C(\omega_R)=\int_0^\infty R(\tau)\cos\omega_R\tau\, d\tau$, $\Gamma^S(\omega_R)=\int_0^\infty R(\tau)\sin\omega_R\tau\, d\tau$ - respectively, the cosine and sine Fourier images of the material relaxation kernel. As an example of a viscoelastic material, we take a three-parametric relaxation kernel: $R(t)=Ae^{-\beta t}/t^{1-\alpha}$.

Relations (1), (2), (3) after identical algebraic transformations are reduced to a system of six differential equations resolved with respect to the first derivative with respect to the radial coordinate

$$\begin{cases} \frac{\partial u_r}{\partial r} = \frac{1}{K}\sigma_{rr} - \frac{\lambda}{K}\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}\right); \\ \frac{\partial u_\varphi}{\partial r} = \frac{1}{\mu}\sigma_{r\varphi} - \frac{1}{r}\left(\frac{\partial u_r}{\partial\varphi}-u_\varphi\right); \\ \frac{\partial u_z}{\partial r} = \frac{1}{\mu}\sigma_{rz} - \frac{\partial u_r}{\partial z}; \\ \frac{\partial\sigma_{rr}}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2} - \frac{\tilde{A}}{r} - \frac{1}{r}\frac{\partial\sigma_{r\varphi}}{\partial\varphi} - \frac{\partial\sigma_{rz}}{\partial z}; \\ \frac{\partial\sigma_{r\varphi}}{\partial r} = \rho\frac{\partial^2 u_\varphi}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial\varphi}\left[\sigma_{rr}-\tilde{A}\right] - \frac{2\sigma_{r\varphi}}{r} - \frac{\partial}{\partial z}\tilde{B}; \\ \frac{\partial\sigma_{rz}}{\partial r} = \rho\frac{\partial^2 u_z}{\partial t^2} - \frac{\partial}{\partial z}\left[\sigma_{rr}-2\mu\left(\frac{\partial u_r}{\partial r}-\frac{\partial u_z}{\partial z}\right)\right] - \frac{\sigma_{rz}}{r} - \frac{1}{r}\frac{\partial}{\partial\varphi}\tilde{B}; \end{cases} \quad (4)$$

where the notation

$$\tilde{A} = 2\mu\left[\frac{\partial u_r}{\partial r} - \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial\varphi}+u_r\right)\right]; \quad \tilde{B} = \mu\left(\frac{\partial u_\varphi}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\varphi}\right).$$

Boundary conditions for φ, for an arbitrary sector angle in the case of a free side surface, should be written as:

$$\varphi = -\frac{\varphi_0}{2}, \frac{\varphi_0}{2}; \quad \sigma_{\varphi\varphi} = \sigma_{\varphi r} = \sigma_{\varphi z} = 0, \quad (5)$$

the boundary conditions along the radius are set in the form:

$$r = r_0 \to 0, R: \quad \sigma_{rz} = \sigma_{rr} = \sigma_{r\phi} = 0, \quad (6)$$

where φ_0 - angle at the apex of the wedge. Harmonic waves propagating along the axis z, are the solutions of the boundary value problem (1), (4), (5) ,(6), periodic in z and by time.

The periodicity conditions make it possible to eliminate the dependence of the main unknowns on time

and the axial coordinate z using the following substitution
variables:

$$\sigma_{rr} = \sigma(r,\varphi)e^{i(\kappa z-\omega t)}; u_r = w(r,\varphi)e^{i(\kappa z-\omega t)};$$
$$u_\phi = v(r,\varphi)e^{i(\kappa z-\omega t)}; u_z = u(r,\varphi)e^{i(\kappa z-\omega t)}, \quad (7)$$
$$\sigma_{rz} = \tau_z(r,\varphi)e^{i(\kappa z-\omega t)}, \sigma_{r\varphi} = \tau_\varphi(r,\varphi)e^{i(\kappa z-\omega t)},$$

where $\sigma(r,\varphi), \tau_\varphi(r,\varphi), \tau_z(r,\varphi), w(r,\varphi), v(r,\varphi), u(r,\varphi)$ - complex amplitude functions, ω – circular frequency (complex value), $\omega = 2\pi\nu$, ν – oscillation frequency (complex value); $\lambda=2\pi/\alpha$ - wavelength, *c* – phase velocity of waves (complex value), $\omega = \alpha c$, α- wave number.
Taking into account (7), the system of equations (4) takes the form:

$$\begin{cases} w' = \dfrac{\sigma}{K} - \dfrac{\lambda}{K}\left(ku + \dfrac{1}{r}\left(w + \dfrac{\partial v}{\partial\varphi}\right)\right) \\ v' = \dfrac{\tau_\phi}{\overline{\mu}} + \dfrac{1}{r}\left(v - \dfrac{\partial w}{\partial\varphi}\right) \\ u' = \dfrac{\tau_z}{\overline{\mu}} + kw \\ \sigma' = -\omega^2\rho w + \dfrac{1}{r}\left(A - \dfrac{\partial\tau_\phi}{\partial\varphi}\right) - k\tau_z \\ \tau_\phi' = -\omega^2\rho v - \dfrac{1}{r}\left(\dfrac{\partial(A+\sigma)}{\partial\varphi} + 2\tau_\phi\right) - kB \\ \tau_z' = -\omega^2\rho u - \dfrac{1}{r}\left(\dfrac{\partial B}{\partial\varphi} + \tau_z\right) + k\left(\sigma + 2\overline{\mu}\left(ku - w'\right)\right) \end{cases} \quad (8)$$

where $A = 2\overline{\mu}\left(\dfrac{1}{2}\left(\dfrac{\partial v}{\partial\varphi} + w\right) - w'\right), B = \overline{\mu}\left(\dfrac{1}{r}\dfrac{\partial u}{\partial\varphi} - kv\right).$

The boundary conditions are transformed similarly (6):

$$r = 0, R: \quad \sigma = \tau_\varphi = \tau_z = 0 \ . \quad (9)$$

It is easy to see that the components of the stress tensor $\sigma_{\varphi\varphi}$, $\sigma_{\varphi z}$ and σ_{zz} are expressed in terms of the main unknowns by the formulas:

$$\sigma_{\varphi\varphi} = \sigma_{rr} + 2\overline{\mu}\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} - \frac{\partial u_r}{\partial r}\right), \ \sigma_{\varphi z} = \overline{\mu}\left(\frac{\partial u_z}{\partial\varphi} + \frac{\partial u_\varphi}{\partial z}\right) \quad (10)$$

$$\sigma_{zz} = \sigma_{rr} + 2\overline{\mu}\left(\frac{\partial u_z}{\partial z} - \frac{\partial u_r}{\partial r}\right).$$

Then, taking into account the first equation of system (5), the boundary conditions (9) take the form:

$$\sigma_\varphi = A + \sigma_r = a\sigma_r + b\frac{1}{r}\left(\frac{\partial v}{\partial\varphi} + w\right) + cku = 0$$

$$\varphi = -\frac{\varphi_0}{2}, \frac{\varphi_0}{2}; \qquad \tau_\varphi = 0 \quad (11)$$

$$B = \overline{\mu}\left(\frac{\partial u}{r\partial\varphi} - kw\right) = 0,$$

where

$$a = 1 + \frac{2\overline{\mu}}{k} \qquad b = 2\overline{\mu}\left(1 + \frac{\lambda}{K}\right) \qquad c = 2\mu\frac{\lambda}{k} \qquad .$$

The boundary value problem for the system of partial differential equations (7), (8), (10) can be reduced to the boundary value problem for the system of ordinary differential equations using the method of lines, which will allow using the software of the orthogonal sweep method in the solution.

The method of lines refers to semi-discrete methods. Semi-discrete methods are based on the following idea: to discretize a partial differential equation not completely, not in all independent variables, but only partially. For example, one can discretize only with respect to the variable φ, while leaving r as a continuous variable. The essence of the method of lines is that the rectangular domain of the function of the main unknowns is covered with lines parallel to the axis r and evenly spaced from each other. Numerical values in the given areas are searched only on these lines, and the directional derivative φ, is replaced by approximate finite differences.

Note that the choice of boundary conditions on the slot faces in the form (9) was determined, first of all, by the possibility of separating the variables over the coordinates r and φ, which greatly simplifies the solution of the original problem. According to the method of lines, the rectangular domain of the function of the main unknowns is covered with lines parallel to the axis r and evenly spaced from each other. The solution is sought only on these lines, and the directional derivative φ, is replaced by approximate finite differences. Used approximating formulas of the second order for the first and second derivatives have the form:

$$y_{i,\varphi} \cong \frac{y_{i+1} - y_{i-1}}{2\Delta} \cong \frac{-3y_i + 4y_{i+1} - y_{i+2}}{2\Delta} \cong \frac{3y_i - 4y_{i-1} + y_{i-2}}{2\Delta}$$

$$y''_{i,\varphi} \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta^2}, \tag{12}$$

where i changes from 0 to $N+1\left(i=\overline{0,N+1}\right)$, y_i - projection of an unknown function onto a straight line with number i; Δ – splitting step by coordinate φ.

As a result of discretization, the vector of main unknowns with a total dimension of $6N$ can be written as:

$$Y = \left(\{w_i\},\{v_i\},\{u_i\},\{\sigma_{ri}\},\{\tau_{\varphi i}\},\{\tau_{zi}\}\right)^T \cdot \left(i = \overline{1,N}\right) \tag{13}$$

Central differences (12) are used for internal straight lines (1 <i<N), the left and right differences make it possible to take into account the boundary conditions for φ. In the first case, the derivative φ in the right parts of the system of equations (8) is expressed by the formulas:

1 <i<N:

$$w_{i,\varphi} = \left(w_{i+1} - w_{i-1}\right)/2\Delta\ , u_{i,\varphi} = \left(u_{i+1} - u_{i-1}\right)/2\Delta\ , \tag{14}$$

$$v_{i,\varphi} = \left(v_{i+1} - v_{i-1}\right)/2\Delta\ , \tau_{\varphi_i,\varphi} = \left(\tau_{\varphi(i+1)} - \tau_{\varphi(i-1)}\right)/2\Delta\ ,$$

$$\tau_{\varphi_i,\varphi} = \left(\tau_{\varphi(i+1)} - \tau_{\varphi(i-1)}\right)/2\Delta$$

$$\sigma_{\varphi_i,\varphi} = a\left(\sigma_{i+1} - \sigma_{i-1}\right)/2\Delta + \frac{b}{r}\left[\left(v_{i+1} - 2v_i + v_{i-1}\right)/\Delta^2 + w_{i,\varphi}\right] + cku_{i,\varphi}$$

$$B_i = \left(u_{i+1} - 2u_i + u_{i-1}\right)/\Delta^2/k - kv_{i,\varphi}\ .$$

Boundary conditions at $\varphi = -\frac{\varphi_0}{2}$ is taken into account in the equations corresponding to the lines $i = I$. For the main unknowns not included in the boundary conditions, $\boldsymbol{w}_i$, $\boldsymbol{v}_i$, $\boldsymbol{u}_i$ the right differences of the first relation are used (11):

$$w_{i,\varphi} = \left(-3w_1 + 4w_2 - w_3\right)/2\Delta \,, v_{i,\varphi} = \left(-3v_1 + 4v_2 - v_3\right)/2\Delta \,,$$
$$u_{i,\varphi} = \left(-3u_1 + 4u_2 - u_3\right)/2\Delta \,. \tag{15}$$

For a variable τ_φ conditions (10) are taken into account using the central differences

$$\tau_{\varphi_i,\varphi} \cong \left(\tau_{\varphi_2} - \tau_{\varphi_0}\right)/2\Delta = -\tau_{\varphi_2}/2\Delta \ . \tag{16}$$

The first and third conditions (10) are taken into account when approximating the derivatives of the function B to φ, σ_φ

$$\sigma_{\varphi_1,\varphi} \cong \left(\sigma_{\varphi_2} - \sigma_{\varphi_0}\right)/2\Delta = \sigma_{\varphi_2}\Big/2\Delta = \left(a\sigma_{r_2} + \frac{b}{r}\left[\left(v_3 - v_1\right)/2\Delta + w_2\right] - cku_2\right)\Big/2\Delta$$
$$B_{1,\varphi} \cong \left(B_2 - B_0\right)/2\Delta = B_2/2\Delta = \left[\left(u_3 - u_1\right)/2\Delta/r - kv_2\right]/2\Delta \tag{17}$$

The derivatives for the straight line with the number $i = N$, taking into account the boundary conditions at $\varphi = \frac{\varphi_0}{2}$. The only difference is the replacement of right finite differences by left ones:

$i=N$:

$$w_{i,\varphi} = \left(3w_N - 4w_{N-1} + w_{N-2}\right)/2\Delta$$
$$v_{i,\varphi} = \left(3v_N - \ldots\right)/2\Delta$$
$$U_{i,\varphi} = \left(U_{i+1} - U_{i-1}\right)/2\Delta$$
$$u_{i,\varphi} = \left(3u_N - \ldots\right)/2\Delta \tag{18}$$
$$\tau_{\varphi,\varphi} = -\tau_{\varphi(N-1)}\Big/2\Delta$$

$$\tau_{\varphi_i,\varphi} = \left(\tau_{\varphi(i+1)} - \tau_{\varphi(i-1)}\right)/2\Delta$$
$$\sigma_{i,\varphi} = \left(a\sigma_{N-1} + \frac{b}{r}\left[\left(v_N - v_{N-2}\right)/2\Delta + w_{N-1}\right] + cku_{N-1}\right)\Big/2\Delta = -\frac{\sigma_{N-1}}{2\Delta}$$
$$B_{i,\varphi} = -\left[\left(u_N - u_{N-2}\right)/2\Delta/r - kv_{N-1}\right]\Big/2\Delta = -\frac{B_{N-1}}{2\Delta}$$

The number of straight lines can be halved if we use the conditions of antisymmetry of the transverse vibrations of the plate at $\varphi = 0$:

$$w = u = \sigma_\varphi = 0 \ . \tag{19}$$

The corresponding difference relations taking into account conditions (19) can be written as:

$$i=N: \quad w_{i,\varphi} = -w_{N-1}/2\Delta \qquad u_{i,\varphi} = -u_{N-1}/2\Delta$$
$$v_{i,\varphi} = \left(3v_N - \ldots\right)/2\Delta \tag{20}$$
$$\tau_{\varphi_i,\varphi} = \left(3\tau_{\varphi N} - 4\tau_{\varphi(N-1)} + \tau_{\varphi(N-2)}\right)/2\Delta$$
$$\sigma_{i,\varphi} = -\left(a\sigma_{N-1} + \frac{b}{r}\left[\left(v_N - v_{N-2}\right)/2\Delta + w_{N-1}\right] + cku_{N-1}\right)\Big/2\Delta = -\frac{\sigma_{N-1}}{2\Delta}$$
$$B_{i,\varphi} = -\left(-2u_N + u_{N-1}\right)/\Delta^2/r - kv_{i,\varphi} \ .$$

The resolving system of ordinary differential equations, according to

(19), has the form:

$$w_i' = \sigma_i / k - a\left(ku_i + \left(w_i + v_{i,\varphi}\right)/R\right);$$
$$v_i' = \tau_{\varphi i} + \left(v_i - w_{i,\varphi}\right)/R \qquad (21)$$
$$u_i' = \tau_{zi} + kw_i$$
$$\sigma_i' = -\omega^2 w_i + \left[2\left(\left(w_i + v_{i,\varphi}\right)/R - w_i'\right) - \tau_{\varphi_i,\varphi}\right]/r - k\tau_{zi}$$
$$\tau_{zi}' = -\omega^2 u_i - \left(B_{i,\varphi} + \tau_{zi}\right)/r + k\left(\sigma + 2\left(ku_i - w_i'\right)\right);$$
$$\tau_{\varphi i}' = -\omega^2 v_i + \left(\sigma_{i,\varphi} + 2\tau_{\varphi i}\right)/r - k\left(u_{i,\varphi}/R - kv_i\right).$$

In equations (21), the expressions for the derivatives $w_{i\varphi}$, $v_{i\varphi}$, $u_{i\varphi}$, $\sigma_{i,\varphi}$ $\beta_{i,\varphi}$ $\tau_{i,\varphi}$ are chosen from relations (19) - (21) depending on the boundary conditions along the coordinate φ. The free surface conditions, which are equivalent to conditions (19) and together with Eqs. (21) form a boundary value problem, are obtained in the form

$$B_i = 0, \quad \tau_{\varphi i} = 0, \quad \sigma_\varphi = 0, \quad (i=1,N) \qquad (22)$$

Thus, the original spectral problem (21), (22) with the help of discretization with respect to the coordinate φ, by the method of lines, is reduced to a canonical problem, for the solution of which we apply the previously used method of orthogonal sweep.

The system of ordinary differential equations of the first order for a wedge-shaped plate, resolved with respect to derivatives, has the form [15]:

$$\begin{cases} z_1' = z_2; \\ z_2' = -\dfrac{6(1-v)}{h^3} z_3 + v\kappa^2 z_1; \\ z_3' = z_4 - \dfrac{h^3 \Gamma_k}{3}\kappa^2 z_2; \\ z_4' = v\kappa^2 z_3 + \dfrac{(1+v)h}{6}\kappa^4 z_1 - h\left(\dfrac{\omega}{C_s}\right)^2 \Gamma_k z_1; \end{cases} \qquad (23)$$

The boundary conditions for this system can be written in the following form:

a) free left edge of the record:

$z_3(0) = z_4(0) = 0$ (24)

b) free right edge of the plate:

$z_3(l_1) = z_4(l_1) = 0$ (25)

c) pinched right edge of the record:

$z_1(l_1) = z_2(l_1) = 0$ (26)

Thus, the spectral boundary value problem (23-26) has been formed with respect to the parameter ω, describing the propagation of bending plane edge waves in a Kirchhoff plate.

When the Timoshenko hypothesis is fulfilled, then the system of partial differential equations (23) takes the following form:

$$\begin{cases} z_1' = z_2 + \dfrac{z_n}{\chi h}; \\ z_2' = -\nu\kappa z_3 - \dfrac{6(1-\nu)}{3} z_5; \\ z_3' = \kappa z_2 - \dfrac{12}{h^3} z_6; \\ z_4' = \chi h\kappa z_3 + \kappa^2\left(\chi h - \dfrac{hc^2}{\Gamma_л}\right) z_1; \\ z_5 = -\kappa z_6 + z_4 + \dfrac{h^3}{12\,\Gamma_л}\omega^2 z_2; \\ z_6' = -\chi h\kappa z_1 - \left[\chi h + \dfrac{\kappa^2 h^3}{12\,\Gamma_л}\left(2(1+\nu) - \dfrac{c^2}{\Gamma_л}\right)\right] z_3 + \nu\kappa z_5. \end{cases} \qquad (27)$$

The boundary conditions for this system can be written in the following form:

a) the free left edge of the plate:

$z_4 = z_5 = z_6 = 0, \quad x_1 = 0;$ (28)

b) free right edge of the plate:

$z_4 = z_5 = z_6 = 0, \quad x_1 = l_1;$ (29)

c) pinched right edge of the record:

$z_1 = z_2 = z_3 = 0, \quad x_1 = l_1;$ (30)

Thus, the spectral boundary value problem (27-30) has been formulated with respect to the parameter ω, describing the propagation of bending plane edge waves in the Timoshenko plate.

Results and Analysis

As an example of a viscoelastic material, we take a three-parametric relaxation kernel: $R_\lambda(t) = R_\mu(t) = Ae^{-\beta t}/t^{1-\alpha}$. The dimensionless quantities are chosen so that the shear rate C_s, density ρ and outer radius R have unit values, Poisson's ratio $\upsilon = 0{,}25$, and kernel parameters $A = 0{,}048; \quad \beta = 0{,}05; \quad \alpha = 0{,}1$ The calculations were carried out in dimensionless parameters. The change in the complex phase velocity $C_0 = C_{0R} + iC_{0I}$ from corners $m^{\square} = m\varphi$ at different values of the instantaneous modulus of elasticity. The calculation results are shown in Fig.3 and Fig.4.

Figure 3 shows the change in the real part of the complex phase velocity C_0 to $m^{\square}$ for different values of the instantaneous modulus of elasticity: ($C_{0R} = C_{0R}^{\square}/C_R$, C_R- rayleigh ox speed) 1. E=0.01;2. E=0.05; 3.E=0.1; 4.E=0.8, where $E = E_0/E_p$, E_p- modulus of elasticity of steel). From Fig. 3 it can be seen that with an increase $m^{\square}$ the phase velocities increase and asymptotically approach the Rayleigh wave. Figure 4 shows the change in the imaginary part of the complex phase velocity from $m^{\square}$ at different values of the instantaneous modulus of elasticity

($C_{0R} = C_{0R}^{\square}/C_R$, C_R- rayleigh ox speed): 1. E=0.01;2. E=0.05; 3.E=0.1; 4.E=0.8, where $E = E_0/E_p$, E_p- modulus of elasticity of steel).

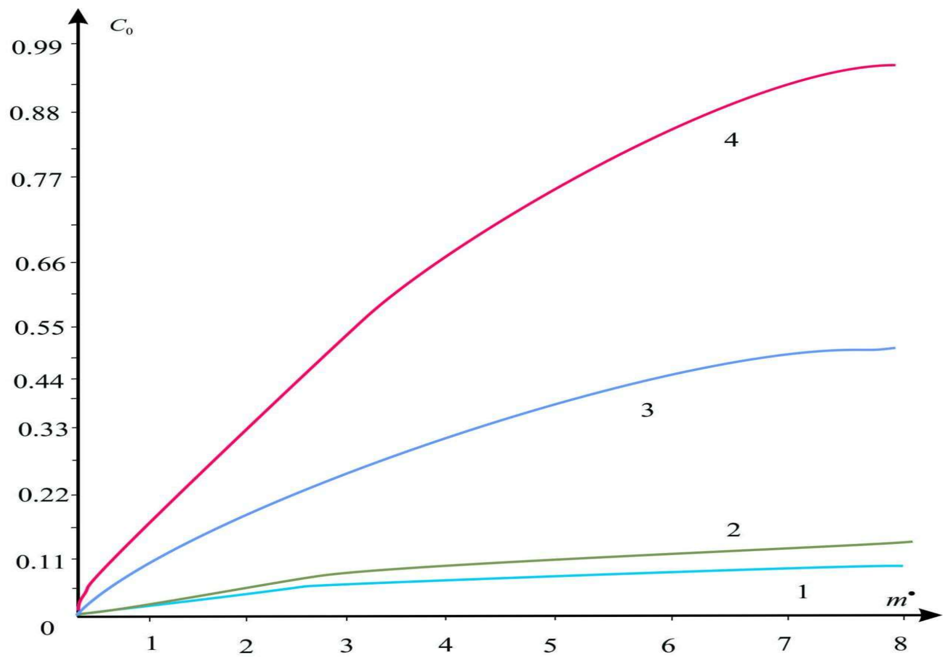

Fig. 3. Dependence of the real part of the complex phase velocity on at different values of the instantaneous modulus of elasticity.

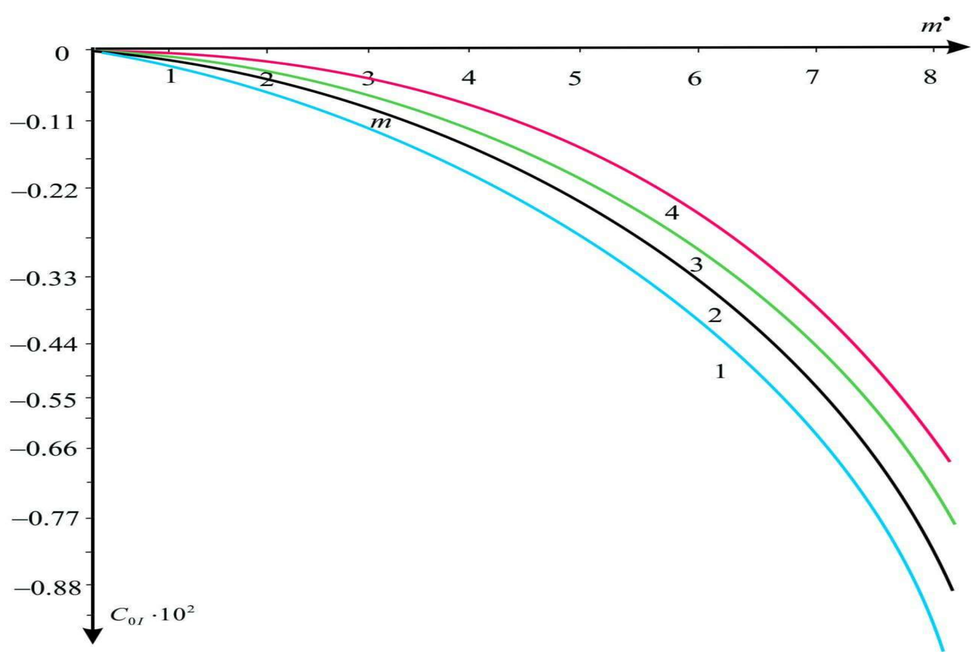

Fig. 4. Dependence of the imaginary part of the complex phase velocity on at different values of the instantaneous modulus of elasticity.

For a Kirchhoff plate of variable thickness, we studied the first five modes with the lowest complex phase propagation velocities $C = C_R + iC_I$, where C_R - phase velocities propagate waves; C_I - damping speed. Figure 5 shows the dispersion curves of the first mode depending on the thickness, which varies linearly. It is assumed here that both edges of the plate are free. Straight line I corresponds to a constant thickness $h_1=h_2=0,1$. In this case, the plate oscillates like a rod. Curve II - option $h_1=h_2/2=0,05$; curve III - option $h_1=h_2/100=0,001,$

curve IV- $h_1 = h_2 / 1000 = 0{,}0001$ and $E_{\min} = 6{,}9 \cdot 10^6\ \kappa / \text{м}^2$, $E_{\max s} = 6{,}9 \cdot 10^8\ \kappa / \text{м}^2$, $\beta = 10^{-4}$. It is seen that at $\kappa > 9$ damping speed increases depending on κ. Для пластинки постоянной толщины C_Γ for a segment $10^{-4} < C < 70$ decreases in a straight line. It can be seen that the dependence of the damping coefficient on the wave number starts from the wave number 3.6. With the entrainment of the wave number, the damping coefficient tends to the side of decrease.

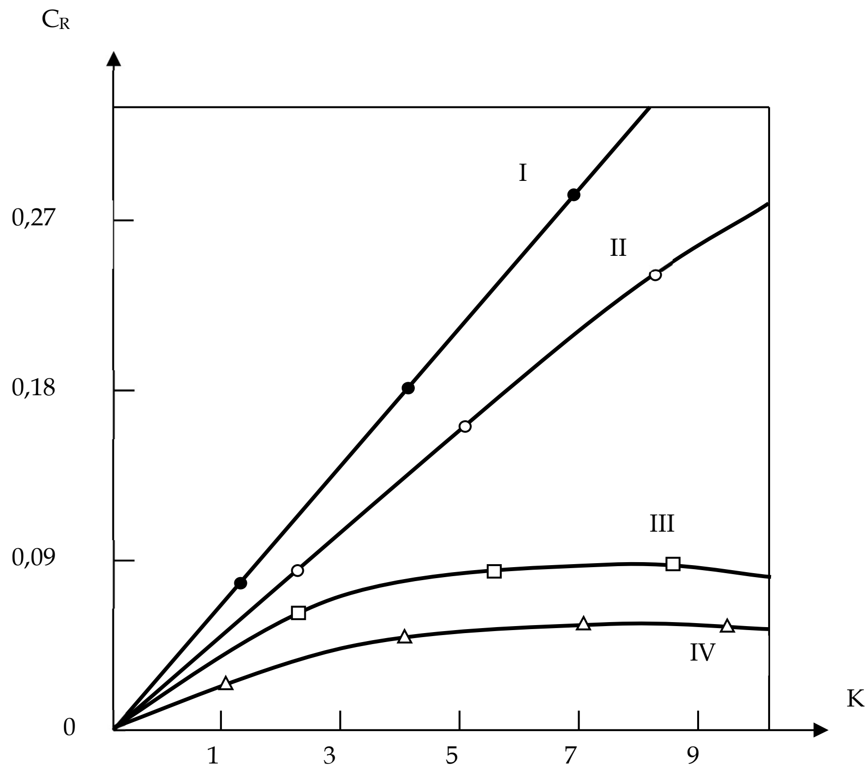

Figure 5. Dispersion curves of the first mode

I. $h_1=h_2=0{,}1$; II. $h_1=h_{2/2}=0{,}05$; III. $h_{2/100} = 0{,}001$; IV. $h_1=h_{2/1000}=0{,}001$

The calculation results obtained by equations (8)-(11) are compared with approximate equations obtained on the basis of the Kirchhoff and Timoshenko hypotheses (23)-(30). Based on the numerical results, the following relationship was obtained

$$C_0 = C_R \sin(m\varphi), \quad (31)$$

in $m\varphi < 90°$ (m = 1, 2, …,), which is given in [16,18]. The waves that appeared in the acute angle of the wedge were called wedge waves or Troyanovsky-Safarov waves. The damping decrements of these waves turned out to be small numbers, but they change rapidly, i.e. turned out to be nonmonotonic functions of the parameters of harmonic waves.

When taking into account the viscosity C_0 the velocities of these waves are reduced to 15%. As the opening angle decreases, the corresponding damping coefficients also decrease exponentially. For the damping factor, based on the numerical results, the following empirical relationship is proposed

$$C_{0I} = -C_R e^{-3.2m} \sin(m\varphi). \quad (32)$$

Table 1 lists the limiting values of the real part of the phase velocity of the first edge mode as a function of the wedge angle. The phase velocities found within the framework of the above method for calculating a three-dimensional wedge are given in columns 5–6 for various boundary conditions. Column 5 corresponds to the calculation variant with three internal straight lines (N = 3) and boundary conditions (11), column 6 corresponds to the boundary conditions:

$$\varphi = -\frac{\theta_0}{2}: \quad \sigma_{\varphi\varphi} = \sigma_{\varphi r} = \sigma_{\varphi z} = 0; \qquad (32)$$
$$\varphi = 0: \quad u_r = u_z = \sigma_{\varphi\varphi} = 0.$$

Column 6 shows the results of calculations obtained by formula (31).

Also, in columns 3 and 4, respectively, the limiting values of the real part of the phase velocity of the first edge mode, obtained in [20, 21] according to the theory of Kirchhoff and Timoshenko plates, are given.

In accordance with the numerical results and the results given in Table 1, it can be concluded that the calculation options according to the Kirchhoff, Timoshenko and three-dimensional theory methods are consistent with each other within 7% for wedge angles with a thickness at the base h_2, *not exceeding 0.5 (wedge angle* $\varphi_0 = 28°$). Note that for the angle $\varphi = 90°$ the limiting phase velocity was also calculated in the works [15], where the value is given for it 0,901 (ν = 0,25). Thus, in contrast to waveguides with a rectangular cross section, in wedge-shaped waveguides with a sufficiently small wedge angle, when analyzing the dispersion dependences of the first mode, it is permissible to use the Kirchhoff–Lyava plate theory.

Table 1. Limiting values of the phase velocity of the first edge mode depending on the wedge angle

R	θ_0	By Krichhoff hypothesis	By Timoshenko's hypothesis	By three-dimensional theory, boundary conditions (11)	By 3D theory, boundary conditions	According to the formula (3.36) $R_\lambda = R_\mu = 0$
0,2	10^0	0,2260	0,1862	0,1761	0,2542	0,1827
0,3	15^0	0,3802	0,2964	0,2982	0,3082	0,2763
0,5	28^0	0,5385	0,4329	0,4626	0,4755	0,4335
0,7	40^0	0,7014	0,5535	0,5923	0,6059	0,5746
1	60^0	1,0144	0,6718	0,7294	0,7414	0,7368
2	90^0	2,8637	0,8441	0,8792	0,8949	0,9212

It turns out that for sufficiently small wedge angles, when analyzing the dispersion dependences of the first mode, it is permissible to use the theory of Kirchhoff plates, and the oscillation modes near an acute wedge angle are also satisfactorily described by the plate theory [15]. This phenomenon should be considered as a characteristic feature of the dynamic behavior of a variable thickness waveguide. In the case of a cylinder with a sectorial cross section, the first mode has a cut-off frequency, and the phase velocity tends to infinity. At large wave numbers, the limiting phase velocity of this mode also coincides with the velocity of the Rayleigh wave. On locking, the axial displacements are equal to zero and the oscillations of an infinite viscoelastic cylinder of a sectorial cross section occur in a flat deformed state. In the second mode at the cut-off frequency, only real and opinion parts of the axial displacement are observed, the circular and radial displacements are equal to zero. Unlike edge waves in a sharp wedge, viscoelastic waves in an infinite cylinder of a sector cross section do not have a limiting solution as the outer radius tends to infinity.

Conclusion

Based on the results of the research, the following conclusions were made:

1. The results of calculations of the limiting velocity of propagation of the first mode in a wedge-shaped waveguide according to the theory of Kirchhoff plates and according to the dynamic theory of elasticity differ by no more than 6% , for angles whose wedge vertices do not exceed *28°*. In $28^0 < \varphi < 90^0$ results differ by up to 20%. At small wedge angles, the simplified theory of Kirchhoff and Timoshenko can be used in the entire wave range.
2. Asymptotic dependences are obtained that describe the phase velocities of wedge-shaped waves (31) and (32), the so-called Troyanovsky-Safarov formulas.

3. Damping decrements turned out to be small numbers, but they change quickly, those. turned out to be nonmonotonic functions of the waveguide parameters. When taking into account the viscosity C_{0R} drops to 15%. As the opening angle decreases, the corresponding damping coefficients also decrease exponentially.

References

1. Melkonyan A.V. Three-dimensional problem of wave propagation in an elastic layer // Proceedings of the National Academy of Sciences of Armenia. 2011. - 66, - No. 1, - P.11-16.
2. Pogosyan N.D., Sanoyan Yu.G., Terzyan S.A. Propagation of shear waves in a two-layer medium in an anti-planar setting. Izvestiya NAS Armenii. 2013. - 64, - No. 4, - P. 12-16.
3. Gavrilyuk, S. L., Makarenko, N. I., Sukhinin, S. V. Waves in Continuous Media.
Lecture Notes in Geosystems Mathematics and Computing, Springer International Publishing AG, 2017. - 140 rubles.
4. Bancroft, D. The velocity of longitudinal waves in cylindrical bars // Phys. Rev.- 1941. - v. 59. - P. 588-593.
5. Wang L., Rokhlin S.I. Stable reformulation of transfer matrix method for wave propagation in layered anisotropic media, Ultrasonics, 39(2001). -2002. -pp. 413–424.
6. Kessler D., Kosloff, D. Elastic wave propagation using cylindrical coordinates // Geophysics, - 1991. - Vol. XI. - No. I. - pp. 2080-2089
7. Mindlin R. D., Herrmann G. A. One-Dimensional Theory of Compressional
Waves in an Elastic Rod // Proceedings of the First U.S. National Congress of Applied Mechanics, 1951. - pp. 187-191
8. Onoe M., McNiven, H.D., Mindlin, R.D. Dispersion of axially symmetric waves in elastic rods // J. Appl. Mech., - 1962. - 29.- pp. 729-734.
9. V. I. Erofeev, Waves in rods. Dispersion. Dissipation. Nonlinearity / V.I. Erofeev, V.V. Kazhaev, N.P. Semerikov. - M. : FIZMATLIT, 2002. - 208 p.
11. Sarkar A., Venkata R. Sonti. Simplified dispersion curves for circular cylindrical shells using shallow shell theory // Journal of Sound and Vibration. 322.–2009. -pp. 1–7.
12. Zaitsev B.D., Kuznetsova I.E., Borodina I.A. Characteristics of acoustic plate waves in potassium neonate (KNbO3) single crystal, Ultrasonics, 39(2001). -2001. -pp. 51–55.
13. Ewing W. M., Jardetzku W. S., Press F. Elastic Waves in Layered Media, McGraw-Hill, New York, 1962, 357 p.
14. I. I. Safarov, Z. F. Dzhumaev, Z. I. Boltaev. Harmonic waves in an infinite cylinder with a radial crack, taking into account the damping ability of the material. Problems of Mechanics of Uzbekistan. 2011, No. 4, pp. 20-25.
15. Safarov II, Boltaev ZI Propagation of harmonic waves in a plate of variable thickness. News of higher educational institutions. Volga region. Series: physics-mathematical sciences, No. 4, 2011, p. 31-39.
16. Safarov I.I., Teshaev M.Kh., Boltaev Z.I. Mathematical modeling of the wave process in a mechanical waveguide with allowance for internal friction. Germany. LAP. 2013. 243p.
17. Yu. K. Konenkov, "On a Rayleigh-type bending wave," Acoust. magazine - 1960. - V. 6, issue. 1. - pp. 124–126.
18.Safarov I.I, Akhmedov M. Sh., Boltaev Z.I. Ducting in Extended Plates of Variable Thickness. Global Journal of Science Frontier Research: F Mathematics & Decision Sciences. 2016. Volume 16, Issue 2 (Ver.1.0). P.33-66.
19. Koltunov M.A. Creep and Relaxation. - M .: Higher School, 1976.- 276s.
20. S.K. Godunov. On numerical solutions of boundary value problems for systems of linear ordinary differential equations. – Uspekhi matematicheskikh nauk, 1961, vol. 16, no. 3, pp. 171-174
21. V.I. Myachenkov, I.V. Grigoriev. Calculation of composite shell structures on a computer: Spavochnik. - Moscow: Mashinostroenie, 1981.-216p.