Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics

Website : https : //www.bpasjournals.com/

Bull. Pure Appl. Sci. Sect. E Math. Stat.
38E(2), 586–591 (2019)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI: 10.5958/2320-3226.2019.00059.6

$1-$ Uniform ideals in $N-$ groups *

S. Bhavanari[1], S.P. Kuncham[2], V.R. Paruchuri[3] and M. Bhavanari[4]

1. Satyanarayana Bhavanari, Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar-522510, India.
2. Syam Prasad Kuncham, Department of Mathematics, Manipal Institute of Technology, MAHE Manipal-576104, India.
3. Venugopala Rao Paruchuri, Department of Mathematics, Andhra Loyola College (Autonomous), Vijayawada-520008, India.
4. Mallikarjuna Bhavanari, Institute of Energy Engineering, Department of Mechanical Engineering, National Central University Jhongli, Taoyuan, Taiwan-32001, Republic of China.

1. E-mail: bhavanari2002@yahoo.co.in , 2. E-mail: kunchamsyamprasad@gmail.com
3. E-mail: venugopalparuchuri@gmail.com , 4. E-mail: bhavanarim@yahoo.co.in

Abstract In this paper, we consider $N-$groups and explore the notions $H-$ essential and strictly essential ideals of an $N-$group G. We prove the elementary properties of essential ideals and strictly essential ideals which are closed under finite intersections and transitive closures. Further, we study the notions $i-$uniform $(i = 0, 1)$ ideals of an $N-$ -group G. We provide necessary examples of each of these notions, and examined the cases wherein these two concepts coincide.

Key words $H-$ essential ideal, strictly essential ideal, $0-$uniform ideal, $1-$uniform ideal.

2010 Mathematics Subject Classification 16Y30.

1 Introduction

Nearrings are generalized rings which are crucial in the nonlinear theory of group mappings. Nearrings are defined in a natural way. For a group $(G, +)$ (not necessarily abelian), the set $M(G) = \{f : G \to G\}$ together with component-wise addition and composition of mappings forms a nearring but not a ring. The module over a nearring is a generalization of the module over an associative ring. For the comprehensive survey on module theory, we refer the reader to Anderson and Fuller [1]. The authors Camillo and Zelmanowitz [2, 3], and Fleury [5] studied the dimension concepts in modules over rings. Bhavanari [12,13] studied the notions of tertiary decomposition and primary decomposition in Noetherian $N-$ Groups. In Satyanarayana [14, 15, 17, 20], Satyanarayana and Rao [21], Satyanarayana, Syam Prasad and Nagaraju [27], the notions of essential ideals, uniform ideals and corresponding Goldie dimension and its dualization concept, namely, finite spanning dimension in $N-$ groups were studied and these

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received February 14, 2019 / Revised May 28, 2019 / Accepted June 24, 2019. Online First Published on December 24, 2019 at https://www.bpasjournals.com/.
Corresponding author Satyanarayana Bhavanari, E-mail: bhavanari2002@yahoo.co.in

authors made significant contributions to these topics. The concepts like the direct and inverse systems on modules over rings were introduced and the characterizations for E-direct systems and related Goldie dimension aspects were studied in Satyanarayana [16, 18, 19], Reddy and Satyanarayana [10], whereas the modules over nearrings were studied by Oswald [7], Satyanarayana and Syam Prasad [22–25] and Syam Prasad [31]. Regarding the further details of additional concepts and for a comprehensive study on nearrings and related notions, we refer the reader to Pilz [8], Ferrero and Ferrero [4], and Meldrum [6], Satyanarayana and Syam Prasad [26, 30], Syam Prasad et al. [32].

A right nearring is a set N together with two binary operations '+' and '·' such that
(i) $(N, +)$ is a group (not necessarily abelian),
(ii) Multiplicatively associative,
(iii) Right distributive property of multiplication over addition.
In view of (iii), N satisfies the right distributive law, and so it is called a right nearring. It is obvious that $0 \cdot x = 0$ for all $x \in N$. However, $n \cdot 0$ need not be equal to 0, in general. We denote by $N_0 = \{n \in N : n \cdot 0 = 0\}$ the zero-symmetric part of the right nearring. We call N is zero-symmetric if N coincides with N_0.
Let G be a group written additively. Then G is said to an $N-$group if there exists a mapping $N \times G \to G$ (an element (a, g) in $N \times G$ is denoted by ag), satisfying:
(i) $(a + b)\, g = ag + bg$ and
(ii) $(ab)\, g = a\,(bg)$ for all $g \in G$ and $a, b \in N$.
We denote $N-$group by ${}_NG$ or simply by G. For introductory definitions and results we refer to Pilz [8], and Syam Prasad and Satyanarayana [29]. Let $(H, +)$be a normal subgroup of G. Then H is called an ideal of G (denoted by $H \leq_N G$) if $n\,(x + a) - nx \in H$ for all $n \in N, x \in G$ and $a \in H$. Further, $I \trianglelefteq_N G$ is said to be essential (strictly essential, resp.) in an ideal J of G if it fulfills the condition: K is an ideal ($N-$subgroup of G, resp.) of G, $I \cap K = (0)\,, K \subseteq J$ imply $K = (0)$.

2 The concepts essential and strictly essential ideals in $N-$groups

In this section, we study the definitions of $H-$ essential and strictly essential ideals of G, to ascertain the elementary properties of essential ideals and strictly essential ideals, such as their closure under finite intersections and transitive closures.

Definition 2.1. Let G be an $N-$ group and H be an ideal of G . An ideal (or $N-$ subgroup) K of H is said to be
(i) $H-$ essential in H if $K \cap B = (0)$ and B is an ideal of H, then $B = (0)$. We denote this by $K \leq_e H$.
(ii) Strictly essential in H if $K \cap B = (0)$, B is an $N-$subgroup in H, then $B = (0)$. We denote this by $K \leq_{se} H$.

Proposition 2.2. *(i) Intersection of finite number of strictly essential $N-$ subgroups of G is strictly essential.*
(ii) If N satisfies the condition that every ideal is $N-$ subgroup, then intersection of finite number of strictly essential ideals is strictly essential.

Proof. (i) Let H_1 and H_2 be two strictly essential $N-$ subgroups of G . To show that $H_1 \cap H_2$ is strictly essential, we take an $N-$ subgroup K of G with $(H_1 \cap H_2) \cap K = (0)$. Now $H_1 \cap (H_2 \cap K) = (0)$, $H_2 \cap K$ is $N-$ subgroup of G and H_1 is strictly essential, we have, $H_2 \cap K = (0)$. Since H_2 is strictly essential, we get that $K = (0)$. This shows that $H_1 \cap H_2$ is a strictly essential $N-$ subgroup.
By using Mathematical Induction, we conclude that any finite intersection of strictly essential $N-$ subgroups is strictly essential.
(ii) Let A, B be strictly essential ideals in G. To verify that $A \cap B$ is also a strictly essential ideal in G. Let K be an $N-$ subgroup of G with $(A \cap B) \cap K = (0)$. Since $A \cap (B \cap K) = (0)$, $B \cap K$ is $N-$ subgroup (by the hypothesis) and $A-$ is strictly essential, we have, $B \cap K = (0)$. Since B is strictly essential, it follows that $K = (0)$.

□

Proposition 2.3. *For any $N-$ subgroups I, J, K of G , we have $I \leq_{se} J$ and $J \leq_{se} K$ then $I \leq_{se} K$.*

Proof . Let L be any $N-$ subgroup of G such that $L \subseteq K$ and $I \cap L = (0)$. Now $I \cap (L \cap J) = (0)$ (since, $L \cap J \subseteq L \subseteq K$) $\Rightarrow L \cap J = (0)$ (since, $I \leq_{se} J$) $\Rightarrow L = (0)$ (since, $J \leq_{se} K$). Therefore, $I \leq_{se} K$.

□

Corollary 2.4. *Let $N = N_0$. If I, J, K are ideals of G such that $I \leq_{se} J$ and $J \leq_{se} K$ then $I \leq_{se} K$.*

Proof. Since each ideal of G is an $N-$ subgroup, the proof follows from the above Proposition 2.3. □

Proposition 2.5. *Suppose I, J are $N-$ subgroups. Then $I \leq_{se} J$ if and only if $I \cap K \leq_{se} J \cap K$, for every $N-$ subgroup K of G.*

Proof . Let K be any $N-$ subgroup of G. To show $I \cap K \leq_{se} J \cap K$. Let L be any $N-$ subgroup of G with $L \subseteq J \cap K$, and $(I \cap K) \cap L = (0)$. Then $I \cap (K \cap L) = (0)$. Now since $(K \cap L) \subseteq L \subseteq J$ and $I \leq_{se} J$, we have that $K \cap L = (0)$. Further since $L \subseteq K$, we get $L = (0)$.

□

Corollary 2.6. *Suppose each ideal of G be an $N-$ subgroup. Let I, J be two ideals of G.*

Proof. To show $I \cap K \leq_{se} J \cap K$. Take L an $N-$ subgroup of G such that $L \subseteq J \cap K$ and $(I \cap K) \cap L = (0)$. This implies that $I \cap (K \cap L) = (0)$ (by hypothesis each ideal is an $N-$ subgroup, $I \cap K$ is an $N-$ subgroup). This means $K \cap L = (0)$ (since, $I \leq_{se} J$). Hence, $L = (0)$. □

Proposition 2.7. *Let A, B, C be either ideals or $N-$ subgroups of G. If $A \subseteq B \subseteq C$ then $A \leq_{se} C$ if and only if $A \leq_{se} B$ and $B \leq_{se} C$.*

Proof . Suppose $A \subseteq B \subseteq C$ and $A \leq_{se} C$. Let L be an $N-$ subgroup of G with $L \subseteq B$, and $A \cap L = (0)$. Since $L \subseteq B \subseteq C$ and $A \leq_{se} C$, we get $L = (0)$. Therefore, $A \leq_{se} B$.
Next we show that $B \leq_{se} C$. Let H be an $N-$ subgroup of G such that $H \subseteq C$ and $B \cap H = (0)$.
Now, $A \cap H \subseteq B \cap H = (0)$. This implies that $A \cap H = (0)$. Now since $A \leq_{se} C$ and $H \subseteq C$, we get $H = (0)$. Hence, $B \leq_{se} C$. □

The conditions (i) and (iii) of the following Proposition 2.8 are known for nearrings (see, Satyanarayana, Bh. and Syam Prasad, K. [30, Proposition 2.2.11 and Theorem 2.3.9]). Though the proofs for ideals of $N-$ groups are similar, we include them here for the purpose of completeness.

Proposition 2.8. *(i) Let G be an $N-$ group and let I, J be the ideals of G with $G = I \oplus J$. Then $a + b = b + a$ for all $a \in I$ and $b \in J$.*
(ii) If $N = N_0, n \in N, a \in I, b \in J$ and the sum $I + J$ is direct in G then $n(a + b) = na + nb$.
(iii) Let $N = N_0$, I an ideal of G, is a direct summand. Then each ideal of I is an ideal of G .

Proof. (i). Since $a + b - a - b \in I \cap J = (0)$ (as, $a + b - a \in I$), we get $a + b - a - b = 0$, and so, $a + b = b + a$.
(ii) Take $n \in N, a \in I, b \in J$. Now, $n(a + b) - nb - na \in I$ (since, N is zero symmetric and $n(a + b) - nb \in I$). Also, $n(a + b) - na + (na - nb - na) \in J$. Therefore, $n(a + b) - nb - na \in I \cap J = (0)$, and hence $n(a + b) = na + nb$.
(iii) Suppose I is an ideal of G which is a direct summand. Then $G = I \oplus J$, where J is the complement of I . Let K be an ideal of I . Then K is a normal subgroup of I. To show that K is a normal subgroup in G, we take $x \in K, g \in G$. Since $g \in G$, we have $g = i + j$ for some $i \in I$ and $j \in J$. By (i), we get, $g = i + j = j + i$. Now $g + x - g = (j + i) + x - (j + i) = j + (i + x - i) - j = j + (-j) + (i + x - i) = 0 + i + x - i \in K$ (since K is an ideal of I).Therefore, K is normal in G.
Next take $n \in N, g \in G$ and $k \in K$. Let $g = a + b$ for some $a \in I$ and $b \in J$. Now $n(g + k) - ng = n(a + b + k) - n(a + b) = n(a + k) + nb - (na + nb)$ (by (ii))
$= n(a + k) + nb - nb - na = n(a + k) - na \in K$ (since K is an ideal of I). Therefore, K is an ideal of G. □

3 The concepts $0-$ uniform and $1-$ uniform ideals in $N-$ groups

In this section, we study the notions: $0-$ uniform and $1-$ uniform ideals of an $N-$ group G . We provide necessary examples of each of these notions and examine the cases wherein these two concepts coincide.

Definition 3.1. Let H be an ideal (or $N-$ subgroup) of G . Then H is said to be
(i) $0-$ uniform if every non-zero ideal of G contained in H is essential in H.
(ii) $1-$ uniform if every non-zero ideal of G contained in H is strictly essential in H .

Note 3.2. Let N be a zero-symmetric nearring and G be an $N-$ group. Let H be an ideal (or an $N-$ subgroup).
(i) If $G = H$, then the concepts 'essential' and '$H-$ essential' coincide.
(ii) If $G = H$, then the concept 'uniform' (defined in Reddy and Satyanarayana [11]) coincides with $0-$ uniform.

Example 3.3. Suppose that G is a simple group. Then G contains no non-trivial normal subgroups. Consider G as an $N-$ group, where $N = \mathbb{Z}$, the ring of integers. Then G contains no non-trivial ideals. Therefore, G is a $0-$ uniform ideal (or $0-$ uniform $N-$ subgroup).

Example 3.4. Take $N = \mathbb{Z}_6$, the ring of integers modulo 6. Define $a \cdot b = 0$ for all $a, b \in N$. Then N is a zero-symmetric nearring. The only non-trivial ideals of N are $\{0, 3\}$ and $\{0, 2, 4\}$. Also $\{0, 3\} \cap \{0, 2, 4\} = (0)$. Hence N is not a $0-$ uniform ideal (or $N-$ subgroup). Hence N is not $1-$ uniform.

Example 3.5. Take $N = S_3$, the symmetric group (written additively) on $\{1, 2, 3\}$. Write $0 = (1), a = (1\ \ 2), b = (1\ \ 2\ \ 3)$. Then N is a nearring with trivial multiplication $*$ defined by $x * y = 0$ for all $x, y \in N$. Now consider N as an $N-$ group.
Then ${}_NN$ has only four non-trivial subgroups $\{0, b, 2b\}, \{0, a + b\}, \{0, b + a\}, \{0, a\}$ in which $\{0, b, 2b\}$ is the only non-trivial normal subgroup. Therefore $\{0, b, 2b\}$ is an ideal whereas $\{0, a + b\}, \{0, b + a\}, \{0, a\}$ are not ideals. Therefore,
(i) $\{0, b, 2b\}$ is an essential ideal of ${}_NN$ but not strictly essential.
(ii) ${}_NN$ is $0-$ uniform.
(iii) Since $\{0, b, 2b\} \cap \{0, a\} = (0)$ but $\{0, a\} \neq (0)$, we see that ${}_NN$ is not $1-$ uniform.

Lemma 3.6. *If N is zero symmetric, then every $1-$ uniform ideal of G is $0-$ uniform.*

Proof . Suppose I is $1-$ uniform. We have to show I is $0-$ uniform. Let K be any non-zero ideal of G contained in I . Let A be an ideal of G such that $K \cap A = (0)$. Since $N = N_0$, we have A is an $N-$ subgroup of G and also, since I is $1-$ uniform we get $A = (0)$. Hence I is $0-$ uniform. □

Remark 3.7. If $N = N_0$ is not true then the above Lemma 3.6 may not be true. So, in general, $1-$ uniform may not imply $0-$uniform. The following example justifies this remark by exhibiting the existence of an ideal of an $N-$ group which is not an $N-$subgroup.

Example 3.8. Take the $(\mathbb{Z}_6, +)$ group. Define $a * b = a$, for all $a, b \in \mathbb{Z}_6$. Then $N = (\mathbb{Z}_6, +, *)$ is a near ring which is not a ring. We further note that N is not a zero symmetric near ring. Here, $I_1 = \{0, 2, 4\}, I_2 = \{0, 3\}$ are non-trivial ideals of N. Take $G = \mathbb{Z}_6$ and $N = \mathbb{Z}_6$. Now I_1 and I_2 are ideals of G which are not $N-$ subgroups of G. For instance, $3 \in N, 2 \in I_1$ but $3 * 2 = 3 \notin I_1$. Therefore, I_1 is not an $N-$ subgroup. Also, $2 \in N, 3 \in I_2$ but $2 * 3 = 2 \notin I_2$. Therefore, I_2 is not an $N-$ subgroup. Since the only $N-$ subgroup of G is G itself, G is $1-$ uniform. Since $I_1 \cap I_2 = (0)$ but $I_1 \neq (0)$ and $I_2 \neq (0)$, we conclude that G is not $0-$ uniform. Hence G is $1-$ uniform but not $0-$ uniform $N-$ group (or ideal).
Further, if we consider I_1 as $N-$ group, then $I_1 = \{0, 2, 4\}$ is $1-$ uniform and also $0-$ uniform (in its own rights).

Proposition 3.9. *Each ideal I contained in a $1-$ uniform ideal of G is $1-$uniform.*

Proof . Let U be the $1-$ uniform ideal of G and A be an ideal of G such that $A \subseteq U$.
Now to show that A is $1-$ uniform, we take an ideal $(0) \neq H$ of G such that $H \subseteq A$. Let K be an $N-$ subgroup of G with $K \subseteq A$ and $H \cap K = (0)$. Now $H, K \subseteq A \subseteq U$, and U is $1-$uniform, we obtain that $K = (0)$. $\square$

Acknowledgements The first author acknowledges Acharya Nagarjuna University, the second author acknowledges Manipal Institute of Technology, MAHE (Deemed to be University), the third author acknowledges Andhra Loyola College (Autonomous) for their kind support and encouragement. The authors thank the referees for their comments and suggestions.

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