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CONSTRUCTION OF FINITE FIELDS

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Abstract: The theory of Finite fields plays a significant role in the theory of Galois extensions. Finite fields have many applications in Coding theory, Computing and Statistics. Therefore, this paper makes an attempt to study some finite fields and their properties. Especially it concentrates on the construction of finite fields. It is known that the characteristic of a finite field is a prime number. Here, first we construct the finite fields containing 4, 8, and 16 elements of characteristic 2, then the fields with 9 and 27 elements of characteristic 3, and finally the fields of 25 elements with characteristic 5. It is well known that the multiplicative group of a finite field is a cyclic group. Therefore, the generators of all the cyclic groups of the finite fields are also found.

Keywords: finite fields, characteristic of a finite field, Galois extension.

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1. INTRODUCTION

A non-empty set F together with the operations addition (+) and multiplication (.) is known as a field if $(F,+)$ and $(F_0,\cdot)$ are abelian groups and two distributive laws hold in F, where F_0 is the set of all non-zero elements of F. For example, $(F_2,+_2,\times_2)$ is a finite field containing two elements. If the number of elements in F is finite, then it is called a finite field and the number is defined to be the order of the field F. The least positive integer n, if it exists, such that $na = a + a + \cdots + a\ (n-terms) = 0$ for all $a \in F$, then n is called the characteristic of the field F, and the field F is said to be of finite characteristic. It is well known that the characteristic of a finite field is a prime number. The set $F[x]$ of all polynomials in x over a field F forms an Integral domain which is called the Polynomial domain. The Polynomial domain $F[x]$ is a Principal ideal domain. That is, every ideal in the Polynomial domain $F[x]$ is generated by a single element. A polynomial $f(x)$ is said to divide a polynomial $g(x)$ in $F[x]$ if there exists a polynomial $r(x)$ in $F[x]$ such that $g(x) = f(x)r(x)$. If $\deg(g(x)) > \deg(f(x)) > 0$, then $f(x)$ is called a proper divisor of $g(x)$. If $\deg(g(x)) \geq 1$ and it has no proper divisors, then it is called an irreducible polynomial in $F[x]$. A polynomial is called monic if its leading coefficient is 1. An Ideal I of a Commutative ring R with identity is called a maximal ideal of R if there does not exist any other ideal J of R such that $I \subset J \subset R$. The Mobius $\mu-$function is defined as $\mu(n)=\begin{cases}1 & if\ n=1\\ 0 & if\ n\ has\ a\ square\ factor\\ (-1)^t & if\ n\ has\ t\ distinct\ prime\ factors\end{cases}$.

For example, $\mu(4)=0,\ \mu(2)=(-1)^1=-1$.

If F is a subfield of a field E, then E is called an extension field of F. It is well known that the multiplicative group of a finite field is cyclic. Also, a finite cyclic group of order n has $\phi(n)$ generators, where ϕ is Euler's $\phi-$function. Hence, F_n is a finite field of order n, then the order of its multiplicative group F_n^* is $n-1$. Since the multiplicative group is cyclic, the order of its generators is also $n-1$ as the order of a cyclic group is equal to the order of its generators. Throughout this paper, F denotes a field with respect to the operations addition (+) and multiplication ($\cdot$), and F_n denotes a finite field, containing n elements, with respect to the operations addition and multiplication modulo n. R stands for a Commutative ring with unity. The ideal generated by an element x of a Commutative ring with unity is denoted by $\langle x\rangle$.

2. MAIN RESULTS

Lemma 1: An element $f(x)$ in a Polynomial domain $F[x]$ is irreducible if and only if the ideal $\langle f(x)\rangle$ generated by $f(x)$ is a maximal ideal.

Proof: Since $F[x]$ is a polynomial integral domain (PID), $g(x)$ divides $f(x)$ implies $\langle f(x)\rangle \subseteq \langle g(x)\rangle \subset F[x]$. The maximality of the ideal $\langle f(x)\rangle$ implies $\langle f(x)\rangle = \langle g(x)\rangle$ and hence $f(x)$ has no non-trivial divisors which implies the irreducibility of $f(x)$. If $f(x)$ is irreducible, then it has no non-trivial divisors and hence the ideal generated by $f(x)$ is maximal. •

Lemma 2: The ideal I of a Commutative ring R with unity is maximal if and only if the quotient ring R/I is a field.

Proof: See [1]. •

By combining Lemmas 1 & 2, we get

Theorem 3: The Quotient ring $F(x)/\langle f(x)\rangle$ is a field if and only if $f(x)$ is irreducible.

Proof: Follows immediately from the Lemmas 1 & 2. •

Theorem 4: If $f(x)$ is a polynomial of degree n in the Polynomial domain $F[x]$, then the Quotient ring $F[x]/\langle f(x)\rangle = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} : a_i \in F\}$ forms an $n-$dimensional vector space over the field F with basis $\{1, \alpha, \alpha^2, \cdots, \alpha^{n-1}\}$, where $\alpha = x + \langle f(x)\rangle$. Also, α is a root of the polynomial $f(x)$.

Proof: Follows from Lemma 2 and Theorem 3. •

The following theorem gives the existence of finite fields.

Theorem 5: For any prime number p and any positive integer n, there exists a finite field F with characteristic p containing p^n elements.

Proof: See Dummit and Foote [1]. •

Towards the context of irreducible polynomials, the following theorem is crucial.

Theorem 6: For any positive integer n, there exists an irreducible polynomial of degree n in $F_p[x]$, where F_p is a finite field containing p elements and p is a prime.

Proof: See Bhattacharya, Jain and Nagpaul [2]. •

Since the construction of irreducible polynomials over finite fields is a separate process, we are not discussing it here. But, the number of monic irreducible polynomials over a finite field is given by the following Lemma.

Lemma 7: If q is a prime number and F_q is a finite field containing q elements, then the number of monic irreducible polynomials of degree n over F_q is $\frac{1}{n}\sum_{d\backslash n}\mu(d)q^{n/d}$, where $\mu(d)$ is the Mobius function and d is a divisor of n.

Proof: See Dummit and Foote [1], page 568. •

Example 1: If $n=2,\ q=2,$ then the number of monic irreducible polynomials of degree 2 over the field F_2 is given by $\frac{1}{2}\left(\mu(1)2^2+\mu(2)2^1\right)=\frac{1}{2}(4-2)=1.$

Example 2: If $n=3,\ q=2,$ then the number of monic irreducible polynomials of degree 3 over the field F_2 is given by $\frac{1}{3}\left(\mu(1)2^3+\mu(3)2^1\right)=\frac{1}{3}(8-2)=2.$

Example 3: If $n=4, q=2,$ then the number of monic irreducible polynomials of degree 4 over the field F_2 is given by $\frac{1}{4}\left(\mu(1)2^4+\mu(2)2^2+\mu(4)2^1\right)=\frac{1}{4}(16-4+0)=3.$

Example 4: If $n=2,\ q=3,$ then the number of monic irreducible polynomials of degree 2 over the field F_3 is given by $\frac{1}{2}\left(\mu(1)3^2+\mu(2)3^1\right)=\frac{1}{2}(9-3)=3.$

Example 5: If $n=2,\ q=5,$ then the number of monic irreducible polynomials of degree 2 over the field F_5 is given by $\frac{1}{2}\left(\mu(1)5^2+\mu(2)5^1\right)=\frac{1}{2}(25-5)=10.$

Example 6 : If $n=3,\ q=3$, then the number of monic irreducible polynomial of degree 3 over the field F_3 is given by $\frac{1}{3}\left(\mu(1)3^3+\mu(3)3^1\right)=\frac{1}{3}(27-3)=8$.

Before we proceed to construct the finite fields of orders 4, 8, 16, 9, 27 and 25, it is required to find at least one monic irreducible polynomial of the required degree over each of the finite fields F_2, F_3, F_5.

Field containing 4 elements:

To construct the field F_{2^2} containing 4 elements, it is required to find a monic irreducible polynomial of degree 2. From Example 1, it is clear that there exists only one monic irreducible polynomial of degree 2 in the Polynomial domain $F_2[x]$. By trial and inspection, it is clear that the polynomial $f(x)=x^2+x+1$ is a monic irreducible polynomial of degree 2 in the Polynomial domain $F_2[x]$ over the field F_2. Therefore, the Quotient ring $\frac{F_2[x]}{\langle f(x)\rangle}$ is a field by the Theorem 3, and its elements are in the form of $a+b\alpha$ by the Theorem 4, where $\alpha\notin F_2$ is a root of the polynomial $f(x)$. Therefore, $F_2\subset F_2(\alpha)$, where $F_2(\alpha)$ is an extension filed of F_2, generated by α. Hence the elements of the field F_{2^2} are $\{0, 1, \alpha, \alpha+1\}$.The composition tables of F_{2^2} with respect to addition and multiplication modulo 2 under the condition

$$\alpha^2+\alpha+1=0 \qquad \ldots(A)$$

are the following:

Table 1:

$+_2$	0	1	α	$\alpha+1$
0	0	1	α	$\alpha+1$
1	1	0	$\alpha+1$	α
α	α	$\alpha+1$	0	1
$\alpha+1$	$\alpha+1$	α	1	0

Table 2:

$\times_2$	0	1	α	$\alpha+1$
0	0	0	0	0
1	0	1	α	$\alpha+1$
α	0	α	$\alpha+1$	1
$\alpha+1$	0	$\alpha+1$	1	α

From the composition tables, it is clear that set F_{2^2} satisfies all the axioms of a field and hence it forms a field containing $2^2 = 4$ elements. Since $\left(F_4^*, \times_2\right)$ is a cyclic group of order 3, it has $\phi(3)=2$ generators. We have $o(1)=1$. Also, $(\alpha)^1=\alpha;\ (\alpha)^2=\alpha+1;\ (\alpha)^3=\alpha.\alpha^2=\alpha(\alpha+1)=\alpha^2+\alpha=1. \Rightarrow o(\alpha)=3$ and $(\alpha+1)^1=\alpha+1;\ (\alpha+1)^2=\alpha;\ (\alpha+1)^3=\alpha(\alpha+1)=\alpha^2+\alpha=1; \Rightarrow o(\alpha+1)=3$ (by condition (A)).

Since the orders of α and $\alpha+1$ are equals, the order of the multiplicative group $\left(F_4^*, \times_2\right)$, $\alpha, \alpha+1$ are the generators.

Field containing 8 elements:

To construct the field F_{2^3} containing 8 elements, it is required to find a monic irreducible polynomial of degree 3. By Example 2, it is clear that there exist two monic irreducible polynomials of degree 3 in the Polynomial domain $F_{2^3}[x]$. By trial and inspection, it is clear that the polynomial $f(x)=x^3+x+1$ is a monic irreducible polynomial of degree 3 in the Polynomial domain $F_2[x]$ over the field F_2. Therefore, the Quotient ring $\dfrac{F_2[x]}{\langle f(x)\rangle}$ is a field by Theorem 3 and its elements are in the form of $a+b\alpha+c\alpha^2$ by Theorem 4, where $\alpha \notin F_2$ is a root of the polynomial $f(x)$. Therefore, $F_2 \subset F_2(\alpha)$, where $F_2(\alpha)$ is an extension field of F_2, generated by α. Hence the elements of the field F_{2^3} are $\{0, 1, \alpha, \alpha+1, \alpha^2, \alpha^2+1, \alpha^2+\alpha, \alpha^2+\alpha+1\}$.

The composition tables of F_{2^3} with respect to addition and multiplication modulo 2 under the condition

$$\alpha^3+\alpha+1=0 \qquad \text{...(B)}$$

are the following:

Table 3:

$+_2$	0	1	α	$\alpha+1$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
0	0	1	α	$\alpha+1$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
1	1	0	$\alpha+1$	α	α^2+1	α^2	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$
α	α	$\alpha+1$	0	1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2	α^2+1
$\alpha+1$	$\alpha+1$	α	1	0	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2+1	α^2
α^2	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	0	1	α	$\alpha+1$
α^2+1	α^2+1	α^2	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	1	0	$\alpha+1$	α
$\alpha^2+\alpha$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2	α^2+1	α	$\alpha+1$	0	1
$\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2+1	α^2	$\alpha+1$	α	1	0

Table 4:

$\times_2$	0	1	α	$\alpha+1$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
0	0	0	0	0	0	0	0	0
1	0	1	α	$\alpha+1$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
α	0	α	α^2	$\alpha^2+\alpha$	$\alpha+1$	1	$\alpha^2+\alpha+1$	α^2+1
$\alpha+1$	0	$\alpha+1$	$\alpha^2+\alpha$	α^2+1	$\alpha^2+\alpha+1$	α^2	1	α
α^2	0	α^2	$\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α	α^2+1	1
α^2+1	0	α^2+1	1	α^2	α	$\alpha^2+\alpha+1$	$\alpha+1$	$\alpha^2+\alpha$
$\alpha^2+\alpha$	0	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	1	α^2+1	$\alpha+1$	α	α^2
$\alpha^2+\alpha+1$	0	$\alpha^2+\alpha+1$	α^2+1	α	1	$\alpha^2+\alpha$	α^2	$\alpha+1$

From the composition tables, it is clear that set F_{2^3} satisfies all the axioms of a field and hence it forms a field containing 2^3 = 8 elements. Since $(F_8^*, \times_2)$ is a cyclic group of order 7, it has $\phi(7)=6$ generators. We have $o(1)=1$. Therefore, all other non-zero elements $\alpha,\ \alpha+1,\ \alpha^2,\ \alpha^2+1,\ \alpha^2+\alpha,\ \alpha^2+\alpha+1$ are generators of this cyclic group as their orders also equal to 7.

Field containing 9 elements:

To construct the field F_{3^2} containing 9 elements, it is required to find a monic irreducible polynomial of degree 2. By Example 4, it is clear that there exist three monic irreducible polynomials of degree 2 in the Polynomial domain $F_3[x]$. By trial and inspection, it is clear that the polynomial $f(x)=x^2+x+2$ is a monic irreducible polynomial of degree 2 in the Polynomial domain $F_3[x]$ over the field F_3. Therefore, the Quotient ring $\frac{F_3[x]}{\langle f(x)\rangle}$ is a field by Theorem 3 and its elements are in the form of $a+b\alpha$ by the Theorem 4, where $\alpha \notin F_3$ is a root of the polynomial $f(x)$. Therefore, $F_3 \subset F_3(\alpha)$, where $F_3(\alpha)$ is an extension filed of F_3, generated by α. Hence the elements of the field F_{3^2} are $F_{2^4} = \{0,1,2,\alpha,\alpha+1,\alpha+2, 2\alpha, 2\alpha+1, 2\alpha+2\}$. The composition tables of F_{3^2} with respect to addition and multiplication modulo 3 under the condition

$$\alpha^2+\alpha+2=0 \qquad \ldots(C)$$

are the following:

Table 5:

$+_3$	0	1	2	α	$\alpha+1$	$\alpha+2$	2α	$2\alpha+1$	$2\alpha+2$
0	0	1	2	α	$\alpha+1$	$\alpha+2$	2α	$2\alpha+1$	$2\alpha+2$
1	1	2	0	$\alpha+1$	$\alpha+2$	α	$2\alpha+1$	$2\alpha+2$	2α
2	2	0	1	$\alpha+2$	α	$\alpha+1$	$2\alpha+2$	2α	$2\alpha+1$
α	α	$\alpha+1$	$\alpha+2$	2α	$2\alpha+1$	$2\alpha+2$	0	1	2
$\alpha+1$	$\alpha+1$	$\alpha+2$	α	$2\alpha+1$	$2\alpha+2$	2α	1	2	0
$\alpha+2$	$\alpha+2$	α	$\alpha+1$	$2\alpha+2$	2α	$2\alpha+1$	2	0	1
2α	2α	$2\alpha+1$	$2\alpha+2$	0	1	2	α	$\alpha+1$	$\alpha+2$
$2\alpha+1$	$2\alpha+1$	$2\alpha+2$	2α	1	2	0	$\alpha+1$	$\alpha+2$	α
$2\alpha+2$	$2\alpha+2$	2α	$2\alpha+1$	2	0	1	$\alpha+2$	α	$\alpha+1$

Table 6:

$\times_3$	0	1	2	α	$\alpha+1$	$\alpha+2$	2α	$2\alpha+1$	$2\alpha+2$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	α	$\alpha+1$	$\alpha+2$	2α	$2\alpha+1$	$2\alpha+2$
2	0	2	1	2α	$2\alpha+2$	$2\alpha+1$	α	$\alpha+2$	$\alpha+1$
α	0	α	2α	$2\alpha+1$	1	$\alpha+1$	$\alpha+2$	$2\alpha+2$	2
$\alpha+1$	0	$\alpha+1$	$2\alpha+2$	1	$\alpha+2$	2α	2	α	$2\alpha+1$
$\alpha+2$	0	$\alpha+2$	$2\alpha+1$	$\alpha+1$	2α	2	$2\alpha+2$	1	α
2α	0	2α	α	$\alpha+2$	2	$2\alpha+2$	$2\alpha+1$	$\alpha+1$	1
$2\alpha+1$	0	$2\alpha+1$	$\alpha+2$	$2\alpha+2$	α	1	$\alpha+1$	2	2α
$2\alpha+2$	0	$2\alpha+2$	$\alpha+1$	2	$2\alpha+1$	α	1	2α	$\alpha+2$

From the composition tables, it is clear that set F_{3^2} satisfies all the axioms of a field and hence it forms a field containing $3^2 = 9$ elements. Since $(F_9^*, \times_3)$ is a cyclic group of order 8, it has $\phi(8)=4$ generators. Since the order of an element of group divides the order of the group, we check the powers 1, 2, 4 and 8 of all elements of the cyclic group as they are only the divisors of 8. Using the condition (C), we get

$o(1)=1$; $o(2)=2$; $(\alpha)^1=\alpha$; $(\alpha)^2=2\alpha+1$; $(\alpha)^4=2$; $(\alpha)^8=1 \Rightarrow o(\alpha)=8$;

$(\alpha+1)^1=\alpha+1$; $(\alpha+1)^2=\alpha+2$; $(\alpha+1)^4=2$; $(\alpha+1)^8=1 \Rightarrow o(\alpha+1)=8$;

$(\alpha+2)^1=\alpha+2$; $(\alpha+2)^2=2$; $(\alpha+2)^4=1 \Rightarrow o(\alpha+2)=4$;

$(2\alpha)^1=2\alpha$; $(2\alpha)^2=2\alpha+1$; $(2\alpha)^4=2$; $(2\alpha)^8=1 \Rightarrow o(2\alpha)=8$;

$(2\alpha+1)^1 = 2\alpha+1;\ (2\alpha+1)^2 = 2;\ (2\alpha+1)^4 = 1 \Rightarrow o(2\alpha+1) = 4;$

$(2\alpha+2)^1 = 2\alpha+2; (2\alpha+2)^2 = \alpha+2; (2\alpha+2)^4 = 2; (2\alpha+2)^8 = 1 \Rightarrow o(2\alpha+2) = 8;$

Therefore, the elements $\alpha,\ 2\alpha,\ \alpha+1,\ 2\alpha+2$ are the generators of this group.

Field containing 16 elements:

To construct the field F_{2^4} containing 16 elements, it is required to find a monic irreducible polynomial of degree 4. By Example 3, it is clear that there exist three monic irreducible polynomials of degree 4 in the Polynomial domain $F_2[x]$. By Trial and Inspection, it is clear that the polynomial $f(x) = x^4 + x + 1$ is a monic irreducible polynomial of degree 4 in the Polynomial domain $F_2[x]$ over the field F_2. Therefore, the Quotient ring $\frac{F_2[x]}{\langle f(x) \rangle}$ is a field by Theorem 3 and its elements are in the form of $a + b\alpha + c\alpha^2 + d\alpha^3$ by Theorem 4, where $\alpha \notin F_2$ is a root of the polynomial $f(x)$. Therefore, $F_2 \subset F_2(\alpha)$, where $F_2(\alpha)$ is an extension filed of F_2 generated by α. Hence the elements of the field F_{2^4} are

$$F_{2^4} = \{0, 1, \alpha, \alpha+1, \alpha^2, \alpha^2+1, \alpha^2+\alpha, \alpha^2+\alpha+1, \alpha^3, \alpha^3+1, \alpha^3+\alpha, \alpha^3+\alpha+1, \alpha^3+\alpha^2, \alpha^3+\alpha^2+1, \alpha^3+\alpha^2+\alpha, \alpha^3+\alpha^2+\alpha+1\}$$

The composition tables of F_{2^3} with respect to addition and multiplication modulo 2 under the condition

$$\alpha^4 + \alpha + 1 = 0 \qquad \ldots(D)$$

are given in the Tables 7(a) and 7(b) and the Tables 8(a) and 8(b) respectively.

Field containing 25 elements:

To construct the field F_{5^2} containing 25 elements, it is required to find a monic irreducible polynomial of degree 2 in the Polynomial domain $F_5[x]$. By Example 5, it is clear that there exist 10 monic irreducible polynomials of degree ten in the Polynomial domain $F_5[x]$. By trial and inspection, it is clear that the polynomial $f(x) = x^2 + x + 1$ is a monic irreducible polynomial of degree 2 in the Polynomial domain $F_5[x]$ over the field F_2. Therefore, the Quotient ring $\frac{F_5[x]}{\langle f(x) \rangle}$ is a field by Theorem 3 and its elements are in the form of $a + b\alpha$ by Theorem 4, where $\alpha \notin F_5$ is a root of the polynomial $f(x)$. Therefore, $F_2 \subset F_5(\alpha)$, where $F_5(\alpha)$ is an extension filed of F_5 generated by α. Hence the elements of the field F_{5^2} are

$$F_{5^2} = \left\{ \begin{matrix} 0,1,2,3,4,\alpha,2\alpha,3\alpha,4\alpha,\alpha+1,\alpha+2,\alpha+3,\alpha+4,2\alpha+1,2\alpha+2,2\alpha+3,2\alpha+4, \\ 3\alpha+1,3\alpha+2,3\alpha+3,3\alpha+4,4\alpha+1,4\alpha+2,4\alpha+3,4\alpha+4 \end{matrix} \right\}$$

The composition tables of F_{5^2} with respect to addition and multiplication modulo 5 under the condition

$$\alpha^2 + \alpha + 1 = 0 \qquad \ldots(E)$$

are given in the Tables 9(a) and 9(b) and the Tables 10(a) and 10(b) respectively.

Table 7(a):

$+_2$	0	1	α	α+1	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
0	0	1	α	α+1	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
1	1	0	α+1	α	α^2+1	α^2	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$
α	α	α+1	0	1	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2	α^2+1
α+1	α+1	A	1	0	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2+1	α^2
α^2	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	0	1	α	α+1
α^2+1	α^2+1	α^2	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	1	0	α+1	α
$\alpha^2+\alpha$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2	α^2+1	α	α+1	0	1

$\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2+1	α^2	$\alpha+1$	α	1	0
α^3	α^3	α^3+1	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$
α^3+1	α^3+1	α^3	$\alpha^3+\alpha+1$	$\alpha^3+\alpha$	$\alpha^3+\alpha^2+1$	α^3+1	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$
$\alpha^3+\alpha$	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$	α^3	α^3+1	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+1$
$\alpha^3+\alpha+1$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha$	α^3+1	α^3	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2$
$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$	α^3	α^3++1	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$
$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2++\alpha+1$	$\alpha^3+\alpha^2+\alpha$	α^3+1	α^3	$\alpha^3+\alpha+1$	$\alpha^3+\alpha$
$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$	α^3	α^3+1
$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha$	α^3+1	α^3

Table 7(b):

$+_2$	α^3	α^3+1	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$
0	α^3	α^3+1	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$
1	α^3+1	α^3	$\alpha^3+\alpha+1$	$\alpha^3+\alpha$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$
α	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$	α^3	α^3+1	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+1$
$\alpha+1$	$\alpha3+\alpha+1$	$\alpha^3+\alpha$	α^3+1	α^3	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2$
α^2	$\alpha3+\alpha^2$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$	α^3	α^3+1	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$
α^2+1	$\alpha^{3+}\alpha^2+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$	α^3+1	α^3	$\alpha^3+\alpha+1$	$\alpha^3+\alpha$
$\alpha^2+\alpha$	α^3 $+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2$	α^3+1	$A^3+\alpha$	$\alpha^3+\alpha+1$	α^3	α^3+1
$\alpha^2+\alpha+1$	α^3 $+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha$	α^3+1	α^3
α^3	0	1	α	$\alpha+1$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
α^3+1	1	0	$\alpha+1$	α	α^2+1	α^2	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$
$\alpha^3+\alpha$	α	$\alpha+1$	0	1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2	α^2+1
$\alpha^3+\alpha+1$	$\alpha+1$	A	1	0	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2+1	α^2
$\alpha^3+\alpha^2$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	0	1	α	$\alpha+1$
$\alpha^3+\alpha^2+1$	α^2+1	α^2	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	1	0	$\alpha+1$	α
$\alpha^3+\alpha^2+\alpha$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2	α^2+1	α	$\alpha+1$	0	1
$\alpha^3+\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2+1	α^2	$\alpha+1$	α	1	0

Table 8(a):

$\times_2$	0	1	α	$\alpha+1$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
0	0	0	0	0	0	0	0	0
1	0	1	α	$\alpha+1$	α^2	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
α	0	A	α^2	$\alpha^2+\alpha$	α^3	$\alpha^3+\alpha$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+\alpha$
$\alpha+1$	0	$\alpha+1$	$\alpha^2+\alpha$	α^2+1	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha$	α^3+1
α^2	0	α^2	α^3	$\alpha^3+\alpha^2$	$1+\alpha$	$\alpha^2+\alpha+1$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha^2+\alpha+1$
α^2+1	0	α^2+1	$\alpha^3+\alpha$	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^2+1+\alpha$	α	$\alpha^3+\alpha^2+1$	α^3
$\alpha^2+\alpha$	0	$\alpha^2+\alpha$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha^2+1$	$\alpha^2+\alpha+1$	1
$\alpha^2+\alpha+1$	0	$\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$	α^3+1	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$	1	$\alpha^2+\alpha$
α^3	0	α^3	$\alpha+1$	$\alpha^3+\alpha+1$	$\alpha^2+\alpha$	$\alpha^3+\alpha+1$	α^2+1	$\alpha^3+\alpha^2+1$
α^3+1	0	α^3+1	1	α^3	$\alpha^2+\alpha+1$	α^2	$1+\alpha$	$\alpha^3+\alpha$
$\alpha^3+\alpha$	0	$\alpha^3+\alpha$	α^3+1	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha$		α^3+1	$\alpha+1$
$\alpha^3+\alpha+1$	0	$\alpha^3+\alpha+1$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha$	α^3+1	$\alpha^3+\alpha^2+\alpha+1$	α^2
$\alpha^2+\alpha^2$	0	$\alpha^3+\alpha^2$	$1+\alpha^2+\alpha$	$1+\alpha+\alpha^2$	α^2+1	α^3	$\alpha^3+\alpha^2+\alpha$	α
$\alpha^3+\alpha^2+1$	0	$\alpha^3+\alpha^2+1$	α^2+1	α^2	1	$\alpha^3+\alpha^2$	α^3	α^2+1
$\alpha^3+\alpha^2+\alpha$	0	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+1$	1	$\alpha^3+\alpha^2+1$	$\alpha+1$	α	$\alpha^3+\alpha^2$
$\alpha^3+\alpha^2+\alpha+1$	0	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+1$	α	α^3+1	$\alpha^2+\alpha$	α^2	$\alpha^3+\alpha+1$

Table 8(b):

$\times_2$	α^3	α^3+1	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$
0	0	0	0	0	0	0	0	0
1	α^3	α^3+1	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$	$\alpha^3+\alpha^2$	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$
α	$1+\alpha$	1	$\alpha^2+\alpha+1$	α^2+1	$\alpha^3+\alpha+1$	α^3+1	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+1$
$\alpha+1$	$\alpha^3+\alpha+1$	α^3	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2	1	α
α^2	$\alpha^2+\alpha$	A	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha$	α^2+1	1	$\alpha^3+\alpha^2+1$	α^3+1
α^2+1	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha+1$	α^2	1	α^3+1	$\alpha^3+\alpha^2$	$\alpha+1$	$\alpha+\alpha^2$
$\alpha^2+\alpha$	α^2+1	$1+\alpha$	α^3+1	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$	α^3	α	α^2
$\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^3+\alpha$	$\alpha+1$	α^2	α	α^2+1	$\alpha^3+\alpha^2$	$\alpha^3+\alpha+1$
α^3	$\alpha^2+\alpha^3$	α^2	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^3+\alpha$	α	α^3+1	1
α^3+1	α^2	$\alpha^3+\alpha^2+1$	α^2+1	$\alpha^3+\alpha^2$	$\alpha^2+\alpha$	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^3+\alpha^2+\alpha$
$\alpha^3+\alpha$	$\alpha^3+\alpha^2+\alpha+1$	α^2+1	α^3	α	1	$\alpha^3+\alpha+1$	$\alpha^2+\alpha$	$\alpha^2+\alpha^3$
$\alpha^3+\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^3+\alpha^2$	α	$\alpha^3+\alpha+1$	$\alpha^3+\alpha^2+1$	α^3+1	α^3	$\alpha+1$
$\alpha^2+\alpha^2$	$\alpha^3+\alpha$	$\alpha^2+\alpha$	1	$\alpha^3+\alpha^2+1$	$\alpha^3+\alpha^2+\alpha+1$	$1+\alpha$	α^2	α^3
$\alpha^3+\alpha^2+1$	α	$\alpha^3+\alpha^2+\alpha+1$	$\alpha^3+\alpha+1$	$\alpha^2+\alpha$	$1+\alpha$	$\alpha^3+\alpha^2+\alpha$	$\alpha^3+\alpha$	$\alpha^2+\alpha+1$
$\alpha^3+\alpha^2+\alpha$	α^3+1	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^3	α^2	$\alpha^3+\alpha$	$\alpha^3+\alpha+1$	α^2+1
$\alpha^3+\alpha^2+\alpha+1$	1	$\alpha^3+\alpha^2+\alpha$	$\alpha^2+\alpha^3$	$\alpha+1$	α^3	α^2+1	α^2+1	$\alpha^3+\alpha$

From the composition tables, it is clear that set F_{2^4} satisfies all the axioms of a field and hence it forms a field containing 2^4 = 16 elements. Since $\left(F_{15}^*, \times_2\right)$ is a cyclic group of order 15, it has $\phi(15)=8$ generators. Since the order of an element of a group divides the order of the group, we check powers 3 and 5 of all elements of the cyclic group as they are the only divisors of 15. Using the condition (D), we get

$$o(1)=1;\ (\alpha)^5=\alpha^2+\alpha \Rightarrow (\alpha)^{15}=1 \Rightarrow o(\alpha)=15;$$

$$(\alpha+1)^3=\alpha^3+\alpha^2+\alpha+1; (\alpha+1)^5=\alpha^2+\alpha \Rightarrow (\alpha+1)^{15}=1 \Rightarrow o(\alpha+1)=15;$$

$$(\alpha^2)^3=\alpha^3+\alpha^2; (\alpha^2)^5=\alpha^2+\alpha+1 \Rightarrow (\alpha^2)^{15}=1 \Rightarrow o(\alpha^2)=15;$$

$$(\alpha^2+1)^3=\alpha^3+1;\ (\alpha^2+1)^5=\alpha^2+\alpha+1 \Rightarrow (\alpha^2+1)^{15}=1 \Rightarrow o(\alpha^2+1)=15;\ (\alpha^2+\alpha)^3=1 \Rightarrow o(\alpha^2+\alpha)=3;$$

$$(\alpha^3)^5=1 \Rightarrow o(\alpha^3)=5; (\alpha^2+\alpha+1)^3=1 \Rightarrow o(\alpha^2+\alpha+1)=3;$$

$$(\alpha^3+1)^3=\alpha^3+\alpha^2+\alpha+1;\ (\alpha^3+1)^5=\alpha^2+\alpha+1 \Rightarrow (\alpha^3+1)^{15}=1 \Rightarrow o(\alpha^3+1)=15;$$

$$(\alpha^3+\alpha)^5=1 \Rightarrow o(\alpha^3+\alpha)=5;$$

$$(\alpha^3+\alpha+1)^3=\alpha^3+\alpha^2;\ (\alpha^3+\alpha+1)^5=\alpha^2+\alpha \Rightarrow (\alpha^3+\alpha+1)^{15}=1 \Rightarrow o(\alpha^3+\alpha+1)=15;$$

$$(\alpha^3+\alpha^2)^5=1 \Rightarrow o(\alpha^3+\alpha^2)=5;$$

$$\left(\alpha^3+\alpha^2+1\right)^3=\alpha^3+\alpha;\left(\alpha^3+\alpha^2+1\right)^5=\alpha^2+\alpha \Rightarrow \left(\alpha^3+\alpha^2+1\right)^{15}=1 \Rightarrow o\left(\alpha^3+\alpha^2+1\right)=15;$$

$$\left(\alpha^3+\alpha^2+1\right)^3=\alpha^3;\left(\alpha^3+\alpha^2+\alpha\right)^5=\alpha^2+\alpha+1 \Rightarrow \left(\alpha^3+\alpha^2+\alpha\right)^{15}=1 \Rightarrow o\left(\alpha^3+\alpha^2+\alpha\right)=15;$$

$$\left(\alpha^3+\alpha^2+\alpha+1\right)^5=1 \Rightarrow o\left(\alpha^3+\alpha^2+\alpha+1\right)=5;$$

There are 8 elements of order 15 and their orders are equal to the order of the group. Therefore, the generators are $\alpha, \alpha+1, \alpha^2, \alpha^2+1, \alpha^3+1, \alpha^3+\alpha+1, \alpha^3+\alpha^2+1, \alpha^3+\alpha^2+\alpha;$

Table 9(a):

$+_5$	0	1	2	3	4	α	2α	3α	4α	α+1	α+2	α+3	α+4
0	0	1	2	3	4	α	2α	3α	4α	α+1	α+2	α+3	α+4
1	1	2	3	4	0	α+1	2α+1	3α+1	4α+1	α+2	α+3	α+4	α
2	2	3	4	0	1	α+2	2α+2	3α+2	4α+2	α+3	α+4	α	α+1
3	3	4	0	1	2	α+3	2α+3	3α+3	4α+3	α+4	α	α+1	α+2
4	4	0	α+1	2	3	α+4	2α+4	3α+4	4α+4	α	α+1	α+2	α+3
α	α	α+1	2α+1	α+3	α+4	2α	3α	4α	0	2α+1	2α+2	2α+3	2α+4
2α	2α	2α+1	3α+1	2α+3	2α+4	3α	4α	0	α	3α+1	3α+2	3α+3	3α+4
3α	3α	3α+1	4α+1	3α+3	3α+4	4α	0	α	2α	4α+1	4α+2	4α+3	4α+4
4α	4α	4α+1	α+2	4α+3	4α+4	0	α	2α	3α	1	2	3	4
α+1	α+1	α+2	α+3	α+4	α	2α+1	3α+1	4α+1	1	2α+2	2α+3	2α+4	2α
α+2	α+2	α+3	α+4	A	α+1	2α+2	3α+2	4α+2	2	2α+3	2α+4	2α	2α+1
α+3	α+3	α+4	α	α+1	α+2	2α+3	3α+3	4α+3	3	2α+4	2α	2α+1	2α+2
α+4	α+4	α	2α+2	α+2	α+3	2α+4	3α+4	4α+4	4	2α	2α+1	2α+2	2α+3
2α+1	2α+1	2α+2	2α+3	2α+4	2α	3α+1	4α+1	1	α+1	3α+2	3α+3	3α+4	3α
2α+2	2α+2	2α+3	2α+4	2α	2α+1	3α+2	4α+2	2	α+2	3α+3	3α+4	3α	3α+1
2α+3	2α+3	2α+4	2α	2α+1	2α+2	3α+3	4α+3	3	α+3	3α+4	3α	3α+1	3α+2
2α+4	2α+4	2α	3α+2	2α+2	2α+3	3α+4	4α+4	4	α+4	3α	3α+1	3α+2	3α+3
3α+1	3α+1	3α+2	3α+3	3α+4	3α	4α+1	1	α+1	2α+1	4α+2	4α+3	4α+4	4α
3α+2	3α+2	3α+3	3α+4	3α	3α+1	4α+2	2	α+2	2α+2	4α+3	4α+4	4α	4α+1
3α+3	3α+3	3α+4	3α	3α+1	3α+2	4α+3	3	α+3	2α+3	4α+4	4α	4α+1	4α+2
3α+4	3α+4	3α	4α+2	3α+2	3α+3	4α+4	4	α+4	2α+4	4α	4α+1	4α+2	4α+3
4α+1	4α+1	4α+2	4α+3	4α+4	4α	1	α+1	2α+1	3α+1	2	3	4	0
4α+2	4α+2	4α+3	4α+4	4α	4α+1	2	α+2	2α+2	3α+2	3	4	0	1

4α+3	4α+3	4α+4	4α	4α+1	4α+2	3	α+3	2α+3	3α+3	4	0	1	2
4α+4	4α+4	4α	4α+1	4α+2	4α+3	4	α+4	2α+4	3α+4	0	1	2	3

Table 9(b):

$+_5$	2α+1	2α+2	2α+3	2α+4	3α+1	3α+2	3α+3	3α+4	4α+1	4α+2	4α+3	4α+4
0	2α+1	2α+2	2α+3	2α+4	3α+1	3α+2	3α+3	3α+4	4α+1	4α+2	4α+3	4α+4
1	2α+2	2α+3	2α+4	2α	3α+2	3α+3	3α+4	3α	4α+2	4α+3	4α+4	4α
2	2α+3	2α+4	2α	2α+1	3α+3	3α+4	3α	3α+1	4α+3	4α+4	4α	4α+1
3	2α+4	2α	2α+1	2α+2	3α+4	3α	3α+1	3α+2	4α+4	4α	4α+1	4α+2
4	2α	2α+1	2α+2	2α+3	3α	3α+1	3α+2	3α+3	4α	4α+1	4α+2	4α+3
α	3α+1	3α+2	3α+3	3α+4	4α+1	4α+2	4α+3	4α+4	1	2	3	4
2α	4α+1	4α+2	4α+3	4α+4	1	2	3	4	α+1	α+2	α+3	α+4
3α	1	2	3	4	α+1	α+2	α+3	α+4	2α+1	2α+2	2α+3	2α+4
4α	α+1	α+2	α+3	α+4	2α+1	2α+2	2α+3	2α+4	3α+1	3α+2	3α+3	3α+4
α+1	3α+2	3α+3	3α+4	3α	4α+2	4α+3	4α+4	4α	2	3	4	0
α+2	3α+3	3α+4	3α	3α+1	4α+3	4α+4	4α	4α+1	3	4	0	1
α+3	3α+4	3α	3α+1	3α+2	4α+4	4α	4α+1	4α+2	4	0	1	2
α+4	3α	3α+1	3α+2	3α+3	4α	4α+1	4α+2	4α+3	0	1	2	3
2α+1	4α+2	4α+3	4α+4	4α	2	3	4	0	α+2	α+3	α+4	α
2α+2	4α+3	4α+4	4α	4α+1	3	4	0	1	α+3	α+4	α	α+1
2α+3	4α+4	4α	4α+1	4α+2	4	0	1	2	α+4	α	α+1	α+2
2α+4	4α	4α+1	4α+2	4α +3	0	1	2	3	α	α+1	α+2	α+3
3α+1	2	3	4	0	α+2	α+3	α+4	α	2α+2	2α+3	2α+4	2α
3α+2	3	4	0	1	α+3	α+4	α	α+1	2α+3	2α+4	2α	2α+1
3α+3	4	0	1	2	α+4	α	α+1	α+2	2α+4	2α	2α+1	2α+2
3α+4	0	1	2	3	α	α+1	α+2	α+3	2α	2α+1	2α+2	2α+3
4α+1	α+2	α+3	α+4	A	2α+2	2α+3	2α+4	2α	3α+2	3α+3	3α+4	3α
4α+2	α+3	α+4	α	α+1	2α+3	2α+4	2α	2α+1	3α+3	3α+4	3α	3α+1
4α+3	α+4	α	α+1	α+2	2α+4	2α	2α+1	2α+2	3α+4	3α	3α+1	3α+2
4α+4	α	α+1	α+2	α+3	2α	2α+1	2α+2	2α+3	3α	3α+1	3α+2	3α+3

Table 10(a):

$\times_5$	0	1	2	3	4	α	2α	3α	4α	α+1	α+2	α+3	α+4
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	α	2α	3α	4α	α+1	α+2	α+3	α+4
2	0	2	4	1	3	2α	4α	α	3α	2α+2	2α+4	2α+1	2α+3
3	0	3	1	4	2	3α	α	4α	2α	3α+3	3α+1	3α+4	3α+2
4	0	4	3	2	1	4α	3α	2α	α	4α+4	4α+3	4α+2	4α+1
α	0	α	2α	3α	4α	4α+4	3α+3	2α+2	α+1	4	α+4	2α+4	3α+4
2α	0	2α	4α	A	3α	3α+3	α+1	4α+4	2α+2	3	2α+3	4α+3	α+3
3α	0	3α	α	4α	2α	2α+2	4α+4	α+1	3α+3	2	3α+2	α+2	4α+2
4α	0	4α	3α	2α	α	α+1	2α+2	3α+3	4α+4	1	4α+1	3α+1	2α+1
α+1	0	α+1	2α+2	3α+3	4α+4	4	3	2	1	α	2α+1	3α+2	4α+3
α+2	0	α+2	2α+4	3α+1	4α+3	α+4	2α+3	3α+2	4α+1	2α+1	3α+3	4α	2
α+3	0	α+3	2α+1	3α+4	4α+2	2α+4	4α+3	α+2	3α+1	3α+2	4α	3	α+1
α+4	0	α+4	2α+3	3α+2	4α+1	3α+4	α+3	4α+2	2α+1	4α+3	2	α+1	2α
2α+1	0	2α+1	4α+2	α+3	3α+4	4α+3	3α+1	2α+4	α+2	α+4	3α	1	2α+2
2α+2	0	2α+2	4α+4	α+1	3α+3	3	1	4	2	2α	4α+2	α+4	3α+1
2α+3	0	2α+3	4α+1	α+4	3α+2	α+3	2α+1	3α+4	4α+2	3α+1	4	2α+2	4α
2α+4	0	2α+4	4α+3	α+2	3α+1	2α+3	4α+1	α+4	3α+2	4α+2	α+1	3α	4
3α+1	0	3α+1	α+2	4α+3	2α+4	3α+2	α+4	4α+1	2α+3	α+3	4α+4	2α	1
3α+2	0	3α+2	α+4	4α+1	2α+3	4α+2	3α+4	2α+1	α+3	2α+4	1	3α+3	α

3α+3	0	3α+3	α+1	4α+4	2α+2	2	4	1	3	3α	α+3	4α+1	2α+4
3α+4	0	3α+4	α+3	4α+2	2α+1	α+2	2α+4	3α+1	4α+3	4α+1	2α	4	3α+3
4α+1	0	4α+1	3α+2	2α+3	α+4	2α+1	4α+2	α+3	3α+4	α+2	3	4α+4	3α
4α+2	0	4α+2	3α+4	2α+1	α+3	3α+1	α+2	4α+3	2α+4	2α+3	α	2	4α+4
4α+3	0	4α+3	3α+1	2α+4	α+2	4α+1	3α+2	2α+3	α+4	3α+4	2α+2	α	3
4α+4	0	4α+4	3α+3	2α+2	α+1	1	2	3	4	4α	3α+4	2α+3	α+2

Table 10(b):

$\times_5$	2α+1	2α+2	2α+3	2α+4	3α+1	3α+2	3α+3	3α+4	4α+1	4α+2	4α+3	4α+4
0	0	0	0	0	0	0	0	0	0	0	0	
1	2α+1	2α+2	2α+3	2α+4	3α+1	3α+2	3α+3	3α+4	4α+1	4α+2	4α+3	4α+4
2	4α+2	4α+4	4α+1	4α+3	α+2	α+4	α+1	α+3	3α+2	3α+4	3α+1	3α+3
3	α+3	α+1	α+4	α+2	4α+3	4α+1	4α+4	4α+2	2α+3	2α+1	2α+4	2α+2
4	3α+4	3α+3	3α+2	3α+1	2α+4	2α+3	2α+2	2α+1	α+4	α+3	α+2	α+1
α	4α+3	3	α+3	2α+3	3α+2	4α+2	2	α+2	2α+1	3α+1	4α+1	1
2α	3α+1	1	2α+1	4α+1	α+4	3α+4	4	2α+4	4α+2	α+2	3α+2	2
3α	2α+4	4	3α+4	α+4	4α+1	2α+1	1	3α+1	α+3	4α+3	2α+3	3
4α	α+2	2	4α+2	3α+2	2α+3	α+3	3	4α+3	3α+4	2α+4	α+4	4
α+1	α+4	2α	3α+1	4α+2	α+3	2α+4	3α	4α+1	α+2	2α+3	3α+4	4α
α+2	3α	4α+2	4	α+1	4α+4	1	α+3	2α	3	α	2α+2	3α+4
α+3	1	α+4	2α+2	3α	2α	3α+3	4α+1	4	4α+4	2	α	2α+3
α+4	2α+2	3α+1	4α	4	1	α	2α+4	3α+3	3α	4α+4	3	α+2
2α+1	2	2α+3	4α+4	α	4α	α+1	3α+2	3	3α+3	4	2α	4α+1
2α+2	2α+3	4α	α+2	3α+4	2α+1	4α+3	α	3α+2	2α+4	4α+1	α+3	3α
2α+3	4α+4	α+2	3α	3	2	2α	4α+3	α+1	α	3α+3	1	2α+4
2α+4	α	3α+4	3	2α+2	3α+3	2	2α+1	4α	1	2α	4α+4	α+3
3α+1	4α	2α+1	2	3α+3	2α+2	3	3α+4	α	4	3α	α+1	4α+2
3α+2	α+1	4α+3	2α	2	3	3α	α+2	4α+4	4α	2α+2	4	3α+1
3α+3	3α+2	α	4α+3	2α+1	3α+4	α+2	4α	2α+3	3α+1	α+4	4α+2	2α
3α+4	3	3α+2	α+1	4α	α	4α+4	2α+3	2	2α+2	1	3α	α+4
4α+1	3α+3	2α+4	α	1	4	4α	3α+1	2α+2	2α	α+1	2	4α+3
4α+2	4	4α+1	3α+3	2α	3α	2α+2	α+4	1	α+1	3	4α	3α+2
4α+3	2α	α+3	1	4α+4	α+1	4	4α+2	3α	2	4α	3α+3	2α+1
4α+4	4α+1	3α	2α+4	α+3	4α+2	3α+1	2α	α+4	4α+3	3α+2	2α+1	α

From the composition tables, it is clear that set F_{5^2} satisfies all the axioms of a field and hence it forms a field containing $5^2 = 25$ elements. Since $\left(F_{24}^{*},\times_2\right)$ is a cyclic group of order 24, it has $\phi(24)=8$ generators. Since the order of an element of a group divides the order of the group, we check powers 2, 3, 4, 6, 8 and 12 of all elements of the cyclic group as they are only the divisors of 24. Using the condition (E), we get $o(1)=1$;

$o(2)=4; o(3)=4; o(4)=2; o(\alpha)=3; o(2\alpha)=12; o(3\alpha)=12; o(4\alpha)=12;$

$o(\alpha+1)=6; o(\alpha+2)=24; o(\alpha+3)=8; o(\alpha+4)=24; o(2\alpha+1)=8; o(2\alpha+2)=12; o(2\alpha+3)=24;$

$o(2\alpha+4)=24; o(3\alpha+1)=24; o(3\alpha+2)=24; o(3\alpha+3)=12; o(3\alpha+4)=8;$

$o(4\alpha+1)=24;\ o(4\alpha+2)=8;\ o(4\alpha+3)=24;\ o(4\alpha+4)=3;$

Therefore, the elements $\alpha+2,\ \alpha+4,\ 2\alpha+3,\ 2\alpha+4, 3\alpha+1,\ 3\alpha+2,\ 4\alpha+1, 4\alpha+3$ are the generators of this group.

Field containing 27 elements:

To construct the field F_{3^3} containing 27 elements, it is required to find a monic irreducible polynomial of degree 3. By Example 6, it is clear that there exist eight monic irreducible polynomials of degree 3 in the Polynomial domain $F_3[x]$. By trial and inspection, it is clear that the polynomial $f(x)=x^3+2x+1$ is a monic irreducible polynomial of degree 3 in the Polynomial domain $F_3[x]$ over the field F_3. Therefore, the Quotient ring $\frac{F_3[x]}{\langle f(x)\rangle}$ is a field by Theorem 3 and its elements are in the form of $a+b\alpha+c\alpha^2$ by Theorem 4, where $\alpha \notin F_3$ is a root of the polynomial $f(x)$. Therefore, $F_3 \subset F_3(\alpha)$, where $F_3(\alpha)$ is an extension filed of F_3, generated by α. Hence the elements of the field F_{3^3} are

$$F_{3^3} = \left\{ \begin{array}{l} 0,1,2,\alpha,2\alpha,\alpha^2,2\alpha^2,\alpha+1,\alpha+2,2\alpha+1,2\alpha+2,\alpha^2+1,\alpha^2+2,2\alpha^2+1,2\alpha^2+2,\alpha^2+\alpha,\alpha^2+2\alpha, \\ \alpha^2+\alpha+1,\alpha^2+\alpha+2,\alpha^2+2\alpha+1,\alpha^2+2\alpha+2,\ 2\alpha^2+\alpha,2\alpha^2+\alpha+1,2\alpha^2+\alpha+2, \\ 2\alpha^2+2\alpha,2\alpha^2+2\alpha+1,2\alpha^2+2\alpha+2 \end{array} \right\}$$

The composition tables of F_{3^3} with respect to addition and multiplication modulo 3 under the condition

$$\alpha^3+2\alpha+1=0 \qquad \text{...(F)}$$

are the following:

Table 11(a):

$+_3$	0	1	2	α	2α	α^2	$2\alpha^2$	$\alpha+1$	$\alpha+2$
0	0	1	2	α	2α	α^2	$2\alpha^2$	$\alpha+1$	$\alpha+2$
1	1	2	0	$\alpha+1$	$2\alpha+1$	α^2+1	$2\alpha^2+1$	$\alpha+2$	α
2	2	0	1	$\alpha+2$	$2\alpha+2$	α^2+2	$2\alpha^2+2$	α	$\alpha+1$
α	α	$\alpha+1$	$\alpha+2$	2α	0	$\alpha^2+\alpha$	$2\alpha^2+\alpha$	$2\alpha+1$	$2\alpha+2$
2α	2α	$2\alpha+1$	$2\alpha+2$	0	α	$\alpha^2+2\alpha$	$2\alpha^2+2\alpha$	1	2
α^2	α^2	α^2+1	α^2+2	$\alpha^2+\alpha$	$\alpha^2+2\alpha$	$2\alpha^2$	0	$\alpha^2+\alpha+1$	$\alpha^2+\alpha+2$
$2\alpha^2$	$2\alpha^2$	$2\alpha^2+1$	$2\alpha^2+2$	$2\alpha^2+\alpha$	$2\alpha^2+2\alpha$	0	α^2	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha+2$
$\alpha+1$	$\alpha+1$	$\alpha+2$	α	$2\alpha+\alpha$	1	$\alpha^2+\alpha+1$	$2\alpha^2+\alpha+1$	$2\alpha+2$	2α
$\alpha+2$	$\alpha+2$	α	$\alpha+1$	$2\alpha+2$	2	$\alpha^2+\alpha+2$	$2\alpha^2+\alpha+2$	2α	$2\alpha+1$
$2\alpha+1$	$2\alpha+1$	$2\alpha+2$	2α	1	$\alpha+1$	$\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$	2	0
$2\alpha+2$	$2\alpha+2$	2α	$\alpha+1$	2	$\alpha+2$	$\alpha^2+2\alpha+2$	1	0	1
α^2+1	α^2+1	α^2+2	α^2	$\alpha^2+\alpha+1$	$\alpha^2+2\alpha+1$	$2\alpha^2+1$	2	$\alpha^2+\alpha+2$	$\alpha^2+\alpha$
α^2+2	α^2+2	α^2	$2\alpha+1$	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+2$	$2\alpha^2+2$	α^2+1	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$
$2\alpha^2+1$	$2\alpha^2+1$	$2\alpha^2+2$	α^2	$2\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+1$	1	α^2+2	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha$
$2\alpha^2+2$	$2\alpha^2+2$	$2\alpha^2$	α^2+1	$2\alpha^2+\alpha+2$	$2\alpha^2+2\alpha+2$	2	α	$2\alpha^2+\alpha$	$2\alpha^2+\alpha+1$
$\alpha^2+\alpha$	α^2+1	$\alpha^2+\alpha+1$	$2\alpha^2$	$\alpha^2+2\alpha$	α^2	$2\alpha^2+\alpha$	2α	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+2$
$\alpha^2+2\alpha$	$\alpha^2+2\alpha$	$\alpha^2+2\alpha+1$	$2\alpha^2+1$	α^2	$\alpha^2+\alpha$	$2\alpha^2+2\alpha$	$\alpha+1$	α^2+1	α^2+2
$\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha+2$	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+1$	α^2+1	$2\alpha^2+\alpha+1$	$\alpha+2$	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha$
$\alpha^2+\alpha+2$	$\alpha^2+\alpha+2$	$\alpha^2+\alpha$	$\alpha^2+\alpha$	$\alpha^2+2\alpha+2$	α^2+2	$2\alpha^2+\alpha+2$	$2\alpha+1$	$\alpha^2+2\alpha$	$\alpha^2+2\alpha+1$
$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+2$	α^2+1	$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+1$	$2\alpha+2$	α^2+2	α^2
$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha$	$\alpha^2+2\alpha$	α^2+2	$\alpha^2+\alpha+2$	$2\alpha^2+2\alpha+2$	$\alpha+1$	α^2	α^2+1
$2\alpha^2+\alpha$	$2\alpha^2+\alpha$	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha+1$	$2\alpha^2+2\alpha$	$2\alpha^2$	$\alpha+2$	$\alpha^2+\alpha$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$
$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha+2$	$2\alpha^2+2\alpha+1$	$2\alpha^2+1$	$\alpha+1$	$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha$
$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha$	$2\alpha^2+2$	$2\alpha^2+2$	$\alpha+2$	0	$2\alpha^2$	$2\alpha^2+1$	$2\alpha^2+2\alpha+1$
$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	2α	$2\alpha^2+\alpha$	2α	α^2	$2\alpha^2+1$	$2\alpha^2+2$	$2\alpha+2$
$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$	$2\alpha^2+1$	$2\alpha^2+1$	$2\alpha+1$	$2\alpha^2+1$	$2\alpha^2+2$	α^2	$2\alpha^2+\alpha+2$
$2\alpha^2+2\alpha+$	$2\alpha^2+2\alpha+$	$2\alpha^2+2\alpha$	$2\alpha+1$	$2\alpha^2$	$2\alpha+2$	$2\alpha^2+2$	α^2	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha$

2	2								

Table 11(b):

$+_3$	$2\alpha+1$	$2\alpha+2$	α^2+1	α^2+2	$2\alpha^2+1$	$2\alpha^2+2$	$\alpha^2+\alpha$	$\alpha^2+2\alpha$	$\alpha^2+\alpha+1$
0	$2\alpha+1$	$2\alpha+2$	α^2+1	α^2+2	$2\alpha^2+1$	$2\alpha^2+2$	$\alpha^2+\alpha$	$\alpha^2+2\alpha$	$\alpha^2+\alpha+1$
1	$2\alpha+2$	$2\alpha+2$	α^2+2	α^2	$2\alpha^2+2$	$2\alpha^2$	$\alpha^2+\alpha+1$	$\alpha^2+2\alpha+1$	$\alpha^2+\alpha+2$
2	2α	$2\alpha+1$	α^2	$\alpha^2+2\alpha+2$	$2\alpha^2$	$2\alpha^2+1$	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+2$	$\alpha^2+\alpha$
α	$2\alpha+\alpha+1$	2	$\alpha^2+\alpha+1$	$\alpha^2+2\alpha+2$	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha+2$	$\alpha^2+2\alpha$	α^2	$2\alpha^2+\alpha+1$
2α	$\alpha+1$	$\alpha+2$	$\alpha^2+2\alpha+1$	$2\alpha^2+2$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$	α^2	$\alpha^2+\alpha$	$\alpha+1$
α^2	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+2$	$2\alpha^2+1$	2	1	2	$2\alpha^2+\alpha$	$2\alpha^2+2\alpha$	$\alpha^2+2\alpha+2$
$2\alpha^2$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$	1	$\alpha^2+\alpha$	α^2+1	α^2+2	α	$2\alpha^2$	$\alpha^2+2\alpha$
$\alpha+1$	2	0	$\alpha^2+\alpha+2$	$\alpha^2+\alpha+1$	$2\alpha^2+\alpha+2$	$2\alpha^2$	$\alpha^2+2\alpha+1$	α^2+1	α^2+2
$\alpha+2$	0	1	$\alpha^2+\alpha$	$\alpha^2+2\alpha$	$2\alpha^2+2$	$2\alpha^2+\alpha+1$	$\alpha^2+2\alpha+2$	α^2+2	α^2
$2\alpha+1$	$\alpha+2$	A	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha$	α^2+1	2	$2\alpha^2+\alpha+2$
$2\alpha+2$	α	$\alpha+1$	$\alpha^2+2\alpha$	$2\alpha^2$	$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	α^2+2	$\alpha^2+\alpha+2$	$2\alpha^2+\alpha$
α^2+1	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha$	$2\alpha^2+2$	$2\alpha^2+1$	2	0	$2\alpha^2+\alpha+2$	$2\alpha^2+2\alpha+1$	α^2+1
α^2+2	$\alpha^2+2\alpha$	$\alpha^2+2\alpha+1$	$2\alpha^2$	0	α^2+1	1	2	$2\alpha^2+2\alpha+2$	α
$2\alpha^2+1$	$2\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha$	2	1	α^2+2	α^2	0	$2\alpha+1$	$2\alpha^2+2\alpha+1$
$2\alpha^2+2$	$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	0	$2\alpha^2+\alpha+2$	α^2	α^2+1	$\alpha+1$	$2\alpha+2$	$2\alpha^2+1$
$\alpha^2+\alpha$	α^2+1	α^2+2	$2\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+2$	$\alpha+1$	$\alpha+2$	$2\alpha^2+2$	$2\alpha^2$	$2\alpha^2+2\alpha+2$
$\alpha^2+2\alpha$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha+2$	$2\alpha^2+2\alpha+1$	$2\alpha^2+\alpha$	$2\alpha+1$	$2\alpha+2$	$2\alpha^2$	$2\alpha^2+\alpha$	$2\alpha^2+2$
$\alpha^2+\alpha+1$	α^2+2	α^2	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha+1$	$\alpha+2$	α	$2\alpha^2+2\alpha+1$	2α	$2\alpha^2+2\alpha+2$
$\alpha^2+\alpha+2$	α^2	α^2+1	$2\alpha^2+\alpha$	$2\alpha^2+2\alpha$	α	$\alpha+1$	$2\alpha^2+2\alpha+2$	$2\alpha^2+1$	$2\alpha^2+2\alpha$
$\alpha^2+2\alpha+1$	$\alpha^2+\alpha+2$	$\alpha^2+\alpha$	$2\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha+1$	$2\alpha+2$	2α	$2\alpha^2+1$	$2\alpha^2+2$	$2\alpha^2+2$
$\alpha^2+2\alpha+2$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha$	$\alpha+2$	2α	$2\alpha+1$	$2\alpha^2+2$	$2\alpha^2+2\alpha+2$	$2\alpha^2$
$2\alpha^2+\alpha$	$2\alpha^2+1$	$2\alpha^2+2$	$\alpha+1$	α	$\alpha^2+\alpha+1$	$\alpha^2+\alpha+2$	2α	0	$2\alpha+1$
$2\alpha^2+\alpha+1$	$2\alpha^2+2$	$2\alpha^2$	$\alpha+2$	$\alpha+1$	$\alpha^2+\alpha+2$	$\alpha^2+\alpha$	$2\alpha+1$	1	$2\alpha+2$
$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha$	α	α^2	$\alpha+1$	α^2+1	1	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$
$2\alpha^2+2\alpha$	$2\alpha^2+1$	$2\alpha^2+2$	$2\alpha+1$	$2\alpha+2$	$\alpha^2+\alpha$	$2\alpha+2$	α	2	1
$2\alpha^2+2\alpha+1$	$2\alpha^2+2$	$2\alpha^2+1$	2α	$\alpha+2$	2α	$2\alpha+1$	$\alpha+2$	$\alpha^2+\alpha$	0
$2\alpha^2+2\alpha+2$	$2\alpha^2$	α^2	$2\alpha+1$	2α	$2\alpha+1$	$\alpha^2+\alpha$	$2\alpha^2+1$	2α	2

Table 11(c):

$+_3$	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+2$	$2\alpha^2+\alpha$	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha+2$	$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$
0	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+2$	$2\alpha^2+\alpha$	$2\alpha^2+2\alpha+1$	$2\alpha^2+\alpha+2$	$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$
1	$\alpha^2+\alpha$	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha$	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha$	$2\alpha^2+2\alpha+$	$2\alpha^2+2\alpha+$	$2\alpha^2+2\alpha$

							1	2	
2	$\alpha^2+\alpha+1$	$\alpha^2+2\alpha$	$\alpha^2+2\alpha+1$	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha$	$2\alpha^2+\alpha+1$	α^2	$2\alpha^2+1$	$2\alpha+1$
A	$\alpha^2+2\alpha+2$	α^2+1	α^2+2	$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2$	$2\alpha^2+\alpha$	$2\alpha+1$	$2\alpha^2$
2α	α^2+2	$\alpha^2+\alpha+1$	$\alpha^2+\alpha+2$	$2\alpha^2$	$2\alpha^2+1$	α^2+2	2α	$2\alpha^2+1$	$2\alpha+2$
α^2	$2\alpha^2+\alpha+2$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$	α	$\alpha+1$	$\alpha+2$	α^2	$2\alpha^2+2$	α^2
$2\alpha^2$	$\alpha+2$	$2\alpha+1$	$2\alpha+2$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	0	$2\alpha^2+1$	α^2	$2\alpha^2+\alpha+1$
$\alpha+1$	$\alpha^2+2\alpha$	α^2+2	α^2	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+1$	$2\alpha^2$	$2\alpha^2+2$	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha$
$\alpha+2$	$\alpha^2+2\alpha+1$	α^2	α^2+1	$2\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha$	$2\alpha^2+1$	$2\alpha+2$	$2\alpha^2+2$	$2\alpha^2$
$2\alpha+1$	α^2	$\alpha^2+\alpha+2$	$\alpha^2+\alpha$	$2\alpha^2+1$	$2\alpha^2+2$	$2\alpha^2+2\alpha+1$	$2\alpha^2+1$	2α	α^2
$2\alpha+2$	α^2+1	α^2+2	$\alpha^2+\alpha+1$	$2\alpha^2+2$	$2\alpha^2$	$2\alpha^2+\alpha+1$	$2\alpha^2+2$	$\alpha+2$	$2\alpha+1$
α^2+1	$2\alpha^2+\alpha$	$2\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha$	$\alpha+1$	$\alpha+2$	$2\alpha^2+\alpha$	$2\alpha+1$	$2\alpha^2$	$\alpha^2+\alpha$
α^2+2	$2\alpha^2+\alpha+1$	$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	$\alpha+2$	α	α	$2\alpha+2$	$2\alpha+1$	$2\alpha^2+1$
$2\alpha^2+1$	α	$2\alpha+2$	2α	0	α^2	α^2	$\alpha^2+\alpha$	$\alpha+2$	2α
$2\alpha^2+2$	$\alpha+1$	2α	$2\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	$\alpha+1$	$2\alpha+2$	$\alpha^2+\alpha$	2
$\alpha^2+\alpha$	$2\alpha^2+2\alpha+2$	$2\alpha^2+1$	$2\alpha+2$	$\alpha^2+\alpha+2$	$2\alpha+1$	α^2+1	α	0	$\alpha^2+\alpha+1$
$\alpha^2+2\alpha$	$2\alpha^2+2$	$2\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+2$	2α	1	1	2	2	$2\alpha^2+2\alpha+1$
$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha$	$2\alpha^2$	$2\alpha^2$	1	$2\alpha+2$	$\alpha^2+\alpha+1$	1	$2\alpha^2+2$	α^2
$\alpha^2+\alpha+2$	$2\alpha^2+2\alpha+1$	$2\alpha^2+\alpha+2$	$2\alpha^2+2$	$2\alpha+1$	2α	$\alpha^2+\alpha+1$	0	$2\alpha^2+2\alpha$	$2\alpha+1$
$\alpha^2+2\alpha+1$	$2\alpha^2$	$2\alpha^2$	$2\alpha^2+\alpha$	$2\alpha+2$	2	$\alpha^2+\alpha$	$\alpha^2+2\alpha$	α^2+1	$\alpha^2+\alpha+2$
$\alpha^2+2\alpha+2$	$2\alpha^2+1$	0	$2\alpha^2+\alpha+1$	2	0	$\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$	$\alpha+2$	$\alpha^2+2\alpha$
$2\alpha^2+\alpha$	$\underline{2\alpha^2+2}$	1	2	$\alpha^2+2\alpha$	$\alpha^2+2\alpha+1$	α	$2\alpha^2+2\alpha+1$	$\alpha^2+2\alpha$	$2\alpha^2+1$
$2\alpha^2+\alpha+1$	2α	2	0	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+2$	$2\alpha^2$	$\alpha^2+\alpha$	$2\alpha^2$	$\alpha^2+\alpha+1$
$2\alpha^2+\alpha+2$	$\alpha^2+\alpha$	A	$2\alpha^2$	$2\alpha^2+1$	$\alpha^2+2\alpha$	$2\alpha^2+1$	2α	α	α^2
$2\alpha^2+2\alpha$	0	$\alpha^2+\alpha$	1	$2\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha+1$	$\alpha^2+2\alpha$	α^2+1	$\alpha+1$	$2\alpha^2$
$2\alpha^2+2\alpha+1$	2	$2\alpha^2+2$	$2\alpha^2+2\alpha$	α^2+1	$\alpha+2$	α^2	α^2	1	α
$2\alpha^2+2\alpha+2$	$\alpha^2+\alpha+1$	$2\alpha^2+\alpha$	$2\alpha^2+2\alpha+1$	α^2	$2\alpha+1$	2	2	$\alpha+2$	0

Table 12(a):

$\times_3$	0	1	2	α	2α	α^2	$2\alpha^2$	$\alpha+1$	$\alpha+2$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	α	2α	α^2	$2\alpha^2$	$\alpha+1$	$\alpha+2$
2	0	2	1	2α	α	$2\alpha^2$	α^2	$2\alpha+2$	$2\alpha+1$
α	0	A	2α	α^2	$2\alpha^2$	$\alpha+2$	$2\alpha+1$	$\alpha^2+\alpha$	$\alpha^2+2\alpha$
2α	0	2α	4α	$\alpha+2$	α^2	$2\alpha+1$	$\alpha+2$	$2\alpha^2+2\alpha$	$2\alpha^2+\alpha$
α^2	0	α^2	α^2	$2\alpha+1$	$2\alpha+1$	$\alpha^2+2\alpha$	$2\alpha^2+2\alpha$	$\alpha^2+\alpha+2$	2
$2\alpha^2$	0	$2\alpha^2$	$2\alpha^2$	$\alpha^2+\alpha$	$\alpha+2$	$2\alpha^2+2\alpha$	$\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	$2\alpha^2+\alpha+2$
$\alpha+1$	0	$\alpha+1$	$\alpha+1$	$\alpha^2+2\alpha$	$2\alpha^2+2\alpha$	$2\alpha^2+\alpha$	$2\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+1$
$\alpha+2$	0	$\alpha+2$	$\alpha+2$	$2\alpha^2+\alpha$	$2\alpha^2+\alpha$	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+1$	α^2+2	α^2+2
$2\alpha+1$	0	$2\alpha+1$	$2\alpha+1$	$2\alpha^2+2\alpha$	$\alpha^2+2\alpha$	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha+2$	$2\alpha^2+1$	$\alpha^2+2\alpha+1$
$2\alpha+2$	0	$2\alpha+2$	$2\alpha+2$	$2\alpha+2$	$\alpha^2+\alpha$	$\alpha^2+2\alpha+1$	$\alpha^2+\alpha+2$	$2\alpha^2+\alpha+2$	α^2+2
α^2+1	0	α^2+1	α^2+1	2	$\alpha+1$	$2\alpha^2+2\alpha+1$	$\alpha^2+\alpha$	$\alpha^2+2\alpha$	$2\alpha^2+1$

α^2+2	0	α^2+2	α^2+2	1	2	$2\alpha^2+2\alpha$	α	α^2+1	$2\alpha^2+2\alpha+1$
$2\alpha^2+1$	0	$2\alpha^2+1$	$2\alpha^2+1$	$\alpha+1$	1	2α	2α	$2\alpha^2+2$	$2\alpha^2$
$2\alpha^2+2$	0	$2\alpha^2+2$	$2\alpha^2+2$	$\alpha^2+\alpha+2$	$2\alpha+2$	α	$2\alpha^2+2\alpha$	$2\alpha^2+\alpha$	$\alpha^2+2\alpha$
$\alpha^2+\alpha$	0	$\alpha^2+\alpha$	$\alpha^2+\alpha$	$2\alpha^2+\alpha+2$	$2\alpha^2+2\alpha+1$	$\alpha^2+\alpha+2$	$2\alpha^2+1$	$2\alpha^2+2\alpha+2$	$\alpha^2+\alpha+2$
$\alpha^2+2\alpha$	0	$\alpha^2+2\alpha$	$\alpha^2+2\alpha$	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+1$	α^2+2	$2\alpha^2+2\alpha+2$	2	1
$\alpha^2+\alpha+1$	0	$\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	α^2+2	$2\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	α^2+1	$2\alpha^2$	$\alpha^2+2\alpha+2$
$\alpha^2+\alpha+2$	0	$\alpha^2+\alpha+2$	$\alpha^2+\alpha+2$	$2\alpha^2+2\alpha+2$	$2\alpha^2+1$	$2\alpha^2+2$	1	$2\alpha^2+2\alpha+1$	$\alpha^2+\alpha$
$\alpha^2+2\alpha+1$	0	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+1$	$2\alpha^2+2$	$\alpha^2+\alpha+1$	2	$\alpha^2+\alpha+2$	α	$2\alpha+2$
$\alpha^2+2\alpha+2$	0	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+1$	α^2+1	$2\alpha^2+\alpha+1$	$2\alpha+2$	$2\alpha+1$	$\alpha+2$
$2\alpha^2+\alpha$	0	$2\alpha^2+\alpha$	$\alpha^2+2\alpha$	α^2+1	$2\alpha^2+\alpha+2$	$\alpha+1$	$\alpha^2+\alpha+1$	1	α^2
$2\alpha^2+\alpha+1$	0	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha$	$\alpha^2+\alpha+1$	$2\alpha^2+2$	$2\alpha^2+2\alpha+2$	$\alpha+1$	$\alpha+2$	$\alpha^2+\alpha+2$
$2\alpha^2+\alpha+2$	0	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$	$2\alpha+2$	$2\alpha^2+\alpha+1$	2α	$\alpha^2+2\alpha+2$
$2\alpha^2+2\alpha$	0	$2\alpha^2+2\alpha$	$2\alpha^2+\alpha+2$	$2\alpha^2+1$	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+2$	α^2+2	$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+1$
$2\alpha^2+2\alpha+1$	0	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha$	$2\alpha^2+2$	α^2+2	$2\alpha^2+1$	2	$\alpha^2+2\alpha+2$	$\alpha+2$
$2\alpha^2+2\alpha+2$	0	$2\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha+1$	$2\alpha^2+\alpha+1$	$\alpha^2+2\alpha+2$	1	$2\alpha^2+2$	α^2	2α

Table 12(b):

$\times_3$	$2\alpha+1$	$2\alpha+2$	α^2+1	α^2+2	$2\alpha^2+1$	$2\alpha^2+2$	$\alpha^2+\alpha$	$\alpha^2+2\alpha$	$\alpha^2+\alpha+1$
0	$2\alpha+1$	0	0	0	0	0	0	0	0
1	$\alpha+2$	$2\alpha+2$	α^2+1	α^2+2	$2\alpha^2+1$	$2\alpha^2+2$	$\alpha^2+\alpha$	$\alpha^2+2\alpha$	$\alpha^2+\alpha+1$
2	$2\alpha^2+\alpha$	$\alpha+1$	$2\alpha^2+2$	$2\alpha^2+1$	α^2+2	α^2+1	$2\alpha^2+2\alpha$	$2\alpha^2+\alpha$	$2\alpha^2+2\alpha+2$
α	$\alpha^2+2\alpha$	$2\alpha^2+2\alpha$	$2\alpha+2$	2	1	$\alpha+1$	$\alpha^2+\alpha+2$	$2\alpha^2+\alpha+2$	$\alpha^2+2\alpha+2$
2α	$\alpha^2+2\alpha+1$	$\alpha^2+\alpha$	$\alpha+1$	1	2	$2\alpha+2$	$2\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+1$	$2\alpha^2+2$
α^2	$2\alpha^2+\alpha+2$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha$	2α	α	α^2	α^2+2	$\alpha^2+\alpha+1$	α^2+1
$2\alpha^2$	$2\alpha^2+1$	$\alpha^2+\alpha+2$	α^2+2	α	2α	$2\alpha^2+2\alpha$	$2\alpha^2+1$	$2\alpha^2+2\alpha+2$	$2\alpha^2$
$\alpha+1$	$2\alpha^2+2\alpha+2$	$2\alpha^2+\alpha+2$	$\alpha^2+2\alpha$	α^2+1	$2\alpha^2+2$	$2\alpha^2+\alpha$	$2\alpha^2+2\alpha+2$	2	$\alpha^2+\alpha+1$
$\alpha+2$	$\alpha^2+\alpha+1$	$2\alpha^2+1$	$2\alpha^2+2\alpha+1$	$2\alpha^2$	$\alpha^2+2\alpha$	$\alpha^2+\alpha+2$	2	$\alpha^2+2\alpha+2$	$\alpha+2$
$2\alpha+1$	α^2+2	α^2+2	$\alpha^2+\alpha+2$	α^2	$2\alpha^2$	$2\alpha^2+2\alpha$	1	$2\alpha^2+\alpha+1$	$2\alpha+2$
$2\alpha+2$	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+1$	$2\alpha^2+\alpha$	$2\alpha^2+2$	α^2+1	$\alpha+2$	α	$2\alpha+1$	$\alpha^2+2\alpha+1$
α^2+1	α^2	$2\alpha^2+\alpha$	$2\alpha+1$	$2\alpha^2+2\alpha+1$	$2\alpha^2+\alpha+1$	$\alpha^2+2\alpha+1$	$\alpha+1$	$\alpha+1$	α^2+1
α^2+2	$2\alpha^2$	$2\alpha^2+2$	$\alpha^2+2\alpha+2$	$\alpha^2+\alpha+2$	$\alpha^2+\alpha+2$	1		$\alpha+2$	$2\alpha^2+\alpha+2$
$2\alpha^2+1$	$2\alpha^2+2\alpha+1$	α^2+1	$2\alpha^2+\alpha+1$	$\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$	α^2+2	2α	$2\alpha^2+\alpha+1$
$2\alpha^2+2$	1	$\alpha+2$	$\alpha^2+2\alpha+1$	$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha$	2	$2\alpha^2+2\alpha+2$
$\alpha^2+\alpha$	$2\alpha^2+\alpha+1$	2	$\alpha^2+\alpha$	$\alpha+1$	$\alpha+1$	$\alpha+1$	$2\alpha^2+2\alpha+1$	$2\alpha+2$	α^2+2
$\alpha^2+2\alpha$	$\alpha+2$	$\alpha^2+\alpha+1$	$2\alpha+1$	α^2+1	$\alpha+2$	$2\alpha^2$	$\alpha^2+\alpha$	$2\alpha^2+2\alpha+1$	$2\alpha^2+\alpha+2$
$\alpha^2+\alpha+1$	$2\alpha+2$	$2\alpha+2$	$\alpha^2+2\alpha+1$	$\alpha^2+\alpha$	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha+1$	$\alpha^2+2\alpha+1$	$2\alpha^2+1$	$2\alpha^2+1$
$\alpha^2+\alpha+2$	α^2+2	α	$\alpha^2+2\alpha$	$\alpha^2+\alpha+2$	$2\alpha^2$	$\alpha^2+\alpha$	$2\alpha^2+2\alpha+2$	$2\alpha+1$	$\alpha^2+\alpha$
$\alpha^2+2\alpha+1$	$2\alpha+1$	$2\alpha^2+2$	α^2+1	α^2	$2\alpha^2+2\alpha$	$2\alpha^2+\alpha$	1	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+1$
$\alpha^2+2\alpha+2$	2	1	α^2+2	$2\alpha+2$	2α	$\alpha^2+\alpha+2$	2	$2\alpha^2+\alpha+1$	$2\alpha^2+2\alpha$
$2\alpha^2+\alpha$	α^2+1	$2\alpha^2+2\alpha$	$\alpha+2$	$\alpha^2+\alpha+1$	$\alpha+1$	1	α^2	$\alpha^2+\alpha+1$	2
$2\alpha^2+\alpha+1$	α	$2\alpha^2+2$	$2\alpha^2+2\alpha+1$	α^2+2	$\alpha^2+\alpha+1$	α^2+2	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+1$	α
$2\alpha^2+\alpha+2$	$2\alpha^2+2$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha$	$\alpha+1$	$2\alpha^2+\alpha+2$	$\alpha+2$	2α	$\alpha^2+2\alpha+2$	$2\alpha^2$
$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha+2$	$2\alpha^2$	α	$2\alpha^2+2\alpha+1$	α	2	$2\alpha^2+1$	$\alpha+1$	1
$2\alpha^2+2\alpha+1$	$2\alpha^2+\alpha+2$	α^2	$\alpha^2+\alpha+2$	2	$\alpha^2+\alpha$	$\alpha^2+2\alpha$	$\alpha^2+2\alpha$	$2\alpha^2$	α^2
$2\alpha^2+2\alpha+2$	$\alpha+1$	$\alpha^2+2\alpha+2$	$2\alpha^2+\alpha+2$	$\alpha^2+2\alpha$	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+1$	α	$2\alpha^2+2\alpha+1$

Table 12(c):

$\times_3$	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+2$	$2\alpha^2+\alpha$	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha+2$	$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$
0	0	0	0	0	0	0	0	0	0
1	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+2$	$2\alpha^2+\alpha$	$2\alpha^2+\alpha+1$	$2\alpha^2+\alpha+2$	$2\alpha^2+2\alpha$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha+2$
2	$2\alpha^2+2\alpha+1$	$2\alpha^2+\alpha+2$	$2\alpha^2+\alpha+1$	$\alpha^2+2\alpha$	$\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+1$	$\alpha^2+\alpha$	$\alpha^2+\alpha+2$	$\alpha^2+\alpha+1$
α	$\alpha2+2$	$2\alpha^2+2\alpha+2$	$2\alpha^2+2$	$\alpha^2+2\alpha+1$	α^2+1	$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+1$	$2\alpha^2+1$	$2\alpha^2+\alpha+1$
2α	$2\alpha^2+1$	$\alpha^2+\alpha+1$	α^2+1	$2\alpha^2+\alpha+2$	$2\alpha^2+2$	$2\alpha^2+2\alpha+2$	$\alpha^2+\alpha+2$	α^2+2	$\alpha^2+2\alpha+2$
α^2	2	$2\alpha^2+\alpha+1$	$\alpha+1$	$2\alpha^2+2\alpha+2$	$2\alpha+2$	$\alpha^2+2\alpha+2$	$2\alpha^2+1$	1	α^2+1
$2\alpha^2$	1	$\alpha^2+2\alpha+2$	$2\alpha+2$	$\alpha^2+\alpha+1$	$\alpha+1$	$2\alpha^2+\alpha+1$	α^2+2	2	$2\alpha^2+2\alpha+2$
$\alpha+1$	$2\alpha^2+\alpha+1$	α	$2\alpha+1$	1	$\alpha+2$	2α	$\alpha^2+\alpha+1$	$\alpha^2+2\alpha+2$	α^2
$\alpha+2$	$2\alpha+2$	$\alpha+2$	α^2	$\alpha^2+\alpha+2$	$\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha+1$	$2\alpha+2$	$\alpha+1$	α^2+1
$2\alpha+1$	$2\alpha+1$	α^2+2	$2\alpha+1$	2	α^2+1	α	$2\alpha^2+2$	$2\alpha^2+2\alpha+2$	$\alpha+1$
$2\alpha+2$	α	$2\alpha^2+2$	1	$2\alpha^2+2\alpha$	$2\alpha^2+2$	$2\alpha2+2\alpha+1$	$2\alpha^2$	α^2	$\alpha^2+2\alpha+2$
α^2+1	$\alpha^2+2\alpha$	α^2+1	α^2+2	$\alpha+2$	$2\alpha^2+\alpha+1$	$2\alpha^2+2\alpha$	α	$\alpha^2+\alpha+2$	$2\alpha^2+\alpha+2$
α^2+2	$\alpha^2+\alpha$	$\alpha^2+\alpha+2$	α^2	$2\alpha+2$	$\alpha^2+\alpha+1$	α^2+2	$\alpha+1$	$2\alpha^2+2\alpha$	$\alpha^2+2\alpha$
$2\alpha^2+1$	$2\alpha^2$	$2\alpha^2+2\alpha+2$	2α	$\alpha+1$	$\alpha^2+\alpha+2$	$2\alpha^2+\alpha+2$	2	$\alpha^2+\alpha$	$\alpha^2+\alpha+2$
$2\alpha^2+2$	α^2	$2\alpha^2+\alpha$	$\alpha^2+\alpha+2$	1	α^2+2	$\alpha+2$	1	$\alpha^2+2\alpha$	$\alpha^2+2\alpha+2$
$\alpha^2+\alpha$	2	α	$\alpha^2+\alpha$	α	α^2+1	$2\alpha+2$	$2\alpha+1$	$\alpha^2+\alpha+1$	$2\alpha^2$
$\alpha^2+2\alpha$	1	α^2	$2\alpha^2+1$	$\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha+1$	α^2	α^2+1	$\alpha^2+2\alpha+1$	$\alpha+1$
$\alpha^2+\alpha+1$	$\alpha^2+\alpha+2$	2α	$\alpha^2+2\alpha+2$	α^2	$\alpha^2+2\alpha+2$	$2\alpha^2$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	$\alpha+2$
$\alpha^2+\alpha+2$	$2\alpha^2+1$	$\alpha+1$	$2\alpha^2+1$	$2\alpha^2+2\alpha+2$	$\alpha^2+2\alpha+2$	$\alpha^2+\alpha+2$	$\alpha^2+\alpha+2$	1	$2\alpha^2+\alpha+1$
$\alpha^2+2\alpha+1$	α^2+1	α^2+1	α^2	α^2	$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha+1$	2	$2\alpha^2+2$
$\alpha^2+2\alpha+2$	2α	$\alpha^2+2\alpha+1$	2	$\alpha^2+2\alpha+1$	α^2	α^2+2	$2\alpha+1$	α^2	2
$2\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α	$\alpha^2+2\alpha+1$	$\alpha^2+\alpha+2$	$\alpha^2+\alpha+1$	$2\alpha^2+2$	$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+1$	$2\alpha^2+1$
$2\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+1$	$2\alpha^2+2\alpha$	α^2+2	$2\alpha^2+1$	$2\alpha^2+2\alpha+1$	$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+1$	α^2+1	$2\alpha^2+2$
$2\alpha^2+\alpha+2$	$2\alpha^2$	$\alpha^2+\alpha+1$	$2\alpha^2+2\alpha+1$	α^2+1	α	$2\alpha^2+2\alpha+1$	$\alpha^2+2\alpha+2$	$2\alpha^2+2\alpha+2$	$\alpha^2+\alpha$
$2\alpha^2+2\alpha$	α	$\alpha^2+2\alpha+2$	α^2	$2\alpha^2+2\alpha$	1	2α	$\alpha^2+2\alpha$	α^2+2	α^2
$2\alpha^2+2\alpha+1$	1	α^2	α	$2\alpha^2+\alpha+1$	2	1	2α	2α	1
$2\alpha^2+2\alpha+2$	2	$\alpha^2+2\alpha$	1	$\alpha^2+\alpha+1$	$2\alpha^2$	α	α^2	$\alpha^2+2\alpha$	$\alpha^2+\alpha+1$

From the composition tables, it is clear that set F_{3^3} satisfies all the axioms of a field and hence it forms a field containing $3^3 = 27$ elements. Since $\left(F_{27}^{*},\times_3\right)$ is a cyclic group of order 26, it has $\phi(26)=12$ generators. Since the order of an element of a group divides the order of the group, we check powers 2, 13, of all elements of the cyclic group as they are only the divisors of 26. Using the condition (F) we get

$$o(1)=1;o(2)=2;\ (\alpha)^{13}=2\Rightarrow(\alpha)^{26}=1\Rightarrow o(\alpha)=26;(2\alpha)^2=\alpha^2;\ (2\alpha)^{13}=1\Rightarrow o(2\alpha)=13;$$

$$\left(\alpha^2\right)^2=\alpha^2+2\alpha;\left(\alpha^2\right)^{13}=1\Rightarrow o\left(\alpha^2\right)=13;\left(2\alpha^2\right)^2=\alpha^2+2\alpha;\ \left(2\alpha^2\right)^{13}=2\Rightarrow\left(2\alpha^2\right)^{26}=1\Rightarrow o\left(2\alpha^2\right)=26;$$

$$(\alpha+1)^2=\alpha^2+2\alpha+1;(\alpha+1)^{13}=2\Rightarrow(\alpha+1)^{26}=1\Rightarrow o(\alpha+1)=26;$$

$$(\alpha+2)^2=\alpha^2+\alpha+1;\ (\alpha+2)^{13}=2\Rightarrow(\alpha+2)^{26}=1\Rightarrow o(\alpha+2)=26;$$

$$(2\alpha+1)^2=\alpha^2+\alpha+1;(2\alpha+1)^{13}=1\Rightarrow o(2\alpha+1)=13;(2\alpha+2)^2=\alpha^2+2\alpha+1;(2\alpha+2)^{13}=1\Rightarrow o(2\alpha+2)=13;$$

$$\left(\alpha^2+1\right)^2=2\alpha+1;\left(\alpha^2+1\right)^{13}=2\Rightarrow\left(\alpha^2+1\right)^{26}=1\Rightarrow o\left(\alpha^2+1\right)=26;$$

$$\left(\alpha^2+2\right)^2=2\alpha^2+2\alpha+1;\ \left(\alpha^2+2\right)^{13}=1\Rightarrow o\left(\alpha^2+2\right)=13;$$

$$\left(2\alpha^2+1\right)^2=2\alpha^2+2\alpha+1;\ \left(2\alpha^2+1\right)^{13}=2\Rightarrow\left(2\alpha^2+1\right)^{26}=1\Rightarrow o\left(2\alpha^2+1\right)=26;$$

$(2\alpha^2+2)^2 = 2\alpha+1; (2\alpha^2+2)^{13} = 1 \Rightarrow o(2\alpha^2+2) = 13;$

$(\alpha^2+\alpha)^2 = 2\alpha^2+\alpha+1; (\alpha^2+\alpha)^{13} = 1 \Rightarrow o(\alpha^2+\alpha) = 13;$

$(\alpha^2+2\alpha)^2 = 2\alpha^2+2; (\alpha^2+2\alpha)^{13} = 1 \Rightarrow o(\alpha^2+2\alpha) = 13;$

$(\alpha^2+\alpha+1)^2 = \alpha^2+2; (\alpha^2+\alpha+1)^{13} = 1 \Rightarrow o(\alpha^2+\alpha+1) = 13;$

$(\alpha^2+\alpha+2)^2 = 2\alpha+2; (\alpha^2+\alpha+2)^{13} = 2 \Rightarrow (\alpha^2+\alpha+2)^{26} = 1 \Rightarrow o(\alpha^2+\alpha+2) = 26;$

$(\alpha^2+2\alpha+1)^2 = \alpha^2+\alpha; (\alpha^2+2\alpha+1)^{13} = 1 \Rightarrow o(\alpha^2+2\alpha+1) = 13;$

$(\alpha^2+2\alpha+2)^2 = 2\alpha; (\alpha^2+2\alpha+2)^{13} = 2 \Rightarrow o(\alpha^2+2\alpha+2)^{26} = 1 \Rightarrow o(\alpha^2+2\alpha+2) = 26;$

$(2\alpha^2+\alpha)^2 = 2\alpha^2+\alpha; (2\alpha^2+\alpha)^{13} = 2 \Rightarrow (2\alpha^2+\alpha)^{26} = 1 \Rightarrow o(2\alpha^2+\alpha)^2 = 26;$

$(2\alpha^2+\alpha+1)^2 = 2\alpha; (2\alpha^2+\alpha+1)^{13} = 1 \Rightarrow o(2\alpha^2+\alpha+1) = 13;$

$(2\alpha^2+\alpha+2)^2 = \alpha^2+\alpha; (2\alpha^2+\alpha+2)^{13} = 2 \Rightarrow (2\alpha^2+\alpha+2)^{26} = 1 \Rightarrow o(2\alpha^2+\alpha+2) = 26;$

$(2\alpha^2+2\alpha)^2 = 2\alpha^2+\alpha+1; (2\alpha^2+2\alpha)^{13} = 2 \Rightarrow (2\alpha^2+2\alpha)^{26} = 1 \Rightarrow o(2\alpha^2+2\alpha) = 26;$

$(2\alpha^2+2\alpha+1)^2 = 2\alpha+2; (2\alpha^2+2\alpha+1)^{13} = 1 \Rightarrow o(2\alpha^2+2\alpha+1) = 13;$

$(2\alpha^2+2\alpha+2)^2 = \alpha^2+2; (2\alpha^2+2\alpha+2)^{13} = 2 \Rightarrow (2\alpha^2+2\alpha+2)^{26} = 1 \Rightarrow o(2\alpha^2+2\alpha+2) = 26;$

There are 12 elements of order 26 and their orders are equal to the order of the group. Therefore, the generators are

$\alpha, 2\alpha^2, \alpha+1, \alpha+2, \alpha^2+1, 2\alpha^2+1, \alpha^2+\alpha+2, \alpha^2+2\alpha+2, 2\alpha^2+\alpha, 2\alpha^2+\alpha+2, 2\alpha^2+2\alpha, 2\alpha^2+2\alpha+2;$

Similarly, we can construct the higher order fields also.

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