

## Generalized Record Values from the Power-Exponential Hazard Rate distribution and Characterizations

<sup>1</sup>Zaki Anwar, <sup>2</sup>Abdul Nasir Khan\* and <sup>3</sup>Rafiqullah Khan

---

### Author Affiliation:

<sup>1,2,3</sup>Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh-202002, India

<sup>2</sup>Department of Mathematics and Statistics, Dr. Vishwanath Karad MIT World Peace University, Pune-411038, Maharashtra

\*Corresponding Author: Abdul Nasir Khan, Department of Mathematics and Statistics, Dr. Vishwanath Karad MIT World Peace University, Pune-411038, Maharashtra.

E-mail: [nasirgd4931@gmail.com](mailto:nasirgd4931@gmail.com)

---

### ABSTRACT

In this study we derive recurrence relations for the single and product moments of generalized record values (GRV's) based on the Power-Exponential hazard rate distribution. Additionally, these relations are formulated for the moments of upper record value (RV's). Furthermore, the study characterizes this distribution by utilizing recurrence relations for single and product moments, conditional expectation, and truncated moments.

**Keyword:** Order statistics, generalized record values, Power-exponential hazard rate distribution, single moments, product moments, recurrence relations and characterization.

**AMS 2020 Subject Classifications:** 62G30, 62E10, 60E05.

---

**How to cite this article:** Anwar Z., Khan A.N. and h Khan R. Generalized Record Values from the Power-Exponential Hazard Rate distribution and Characterizations. *Bulletin of Pure and Applied Sciences- Math & Stat.*, 2025;44E (2):70-81.

---

Received on 22.06.2025, Revised on 03.07.2025, Accepted on 20.08.2025

---

### 1. INTRODUCTION

The  $RV$ 's hold significant importance in everyday life. People are often interested in various records, including those related to sports, weather, crime statistics, economic trends and so on. Almost every measurable phenomenon is systematically recorded. The concept of  $RV$ 's was initially introduced by Chandler (1952) as a model for analyzing successive extreme values within a sequence of independent and identically distributed random variables [1]. Later, Dziubdziela and Kopociński (1976) expanded this concept by applying it to a broader class of random variables, referring to them as  $k$ -th  $RV$ 's. [2] Subsequently, Minimol and Thomas (2013) termed these as  $GRV$ 's, as the  $k$ -th term in the sequence of ordinary  $RV$ 's is also recognized as the  $k$ -th record value. By setting  $k = 1$ , one can obtain ordinary  $RV$ 's. [3]

Let  $\{X_n, n \geq 1\}$  be a sequence of independently identically distributed (*iid*) continuous random variables with distribution function (*df*)  $F(x)$  and probability density function (*pdf*). For fixed integer  $k \geq 1$  we define the sequence  $\{U_n^{(k)}, n \geq 1\}$  of  $k$ -th upper record times of  $\{X_n, n \geq 1\}$  as follows:

$$U_n^{(k)} = 1$$

$$U_{n+1}^{(k)} = \min\{j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1}\}$$

where  $X_{j:n}$  denotes the  $j$ -th order statistics in a sample of size  $n$ .

For  $k=1$  and  $n=1,2,\dots$  we write  $U_n^{(1)}=U_n$ . Then  $\{U_n, n \geq 1\}$  is the sequence of record times of  $\{X_n, n \geq 1\}$ . The sequence  $\{Y_n^{(k)}, n \geq 1\}$ , where  $Y_n^{(k)} = X_{U_n^{(k)}}$  is called the sequence of  $k$ -th upper RV's of  $\{X_n, n \geq 1\}$ . Note that for  $k=1$  we have  $Y_n^{(1)} = X_{U_n}, n \geq 1$ , which are the record values of  $\{X_n, n \geq 1\}$  (Ahsanullah (1995)). Moreover, we see that  $Y_1^{(k)} = \min\{X_1, X_2, \dots, X_n\} = X_{1:k}$ . Then the *pdf* of  $Y_n^{(k)}$  and joint *pdf* of  $Y_m^{(k)}$  and  $Y_n^{(k)}$  are as follows:

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x), \quad n \geq 1 \tag{1.1}$$

$$f_{Y_m^{(k)}, Y_n^{(k)}}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \times [\bar{F}(y)]^{k-1} f(y), \quad x < y, \quad 1 \leq m < n, \quad n \geq 2, \tag{1.2}$$

where  $\bar{F}(x) = 1 - F(x)$ .

The various developments on record values and related topics are extensively studied in the literature [4, 5, 6, 7, 8, 9, 10].

Various probability distributions, including the Weibull, Rayleigh, exponential, and Gompertz, are extensively utilized for analyzing lifetime data. While these distributions exhibit monotonic hazard rate functions, they fail to adequately model bathtub-shaped hazard rate functions, which are frequently encountered in reliability and biological studies. To address this limitation, researchers have introduced extended versions of lifetime distributions that effectively capture bathtub-shaped hazard rates. These modified distributions offer greater flexibility and improved fitting capabilities. In this context, Tarvirdizade and Nematollahi (2017) proposed the Power-Exponential hazard rate distribution, which integrates the power hazard rate function with the exponential hazard rate function, thereby preserving the bathtub-shaped property [11]. A random variable  $X$  follows Power-exponential hazard rate distribution (Tarvirdizade and Nematollahi (2017)), if it's *pdf* is of the form

$$f(x) = (\alpha x^\beta + \lambda e^{\gamma x}) e^{-\left(\frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma x} - 1)\right)}, \quad x \geq 0, \quad \alpha, \beta, \gamma, \lambda > 0 \tag{1.3}$$

with *df*

$$F(x) = 1 - e^{-\left(\frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma x} - 1)\right)}, \quad x \geq 0, \quad \alpha, \beta, \gamma, \lambda > 0. \tag{1.4}$$

From (1.3) and (1.4), we have obtained

$$f(x) = (\alpha x^\beta + \lambda - \frac{\alpha \gamma}{1 + \beta} x^{\beta+1} + \gamma [-\ln \bar{F}(x)]) \bar{F}(x). \tag{1.5}$$

The relation (1.5) will be used to establish some simple recurrence relations for moments of generalized upper record values from the Power-exponential hazard rate distribution.

Some sub-models of power-exponential hazard rate distribution are as follows:

Table 1: Sub-Models of Power-Exponential Hazard Rate Distribution

Parameters				Distribution
$\alpha$	$\beta$	$\gamma$	$\lambda$	
-	0	-	0	exponential
-	1	-	0	Rayleigh
-	1	0		Linear Hazard Rate
-	-	-	0	power hazard rate
$\mu/\theta$	$\mu-1$	-	0	Weibull
$\mu\theta$	$\mu-1$	-	0	modified Weibull
0	-	-	-	Gompertz
-	0	-	-	Gompertz-Makeham

2. RELATIONS FOR SINGLE MOMENTS

**Theorem 2.1.** For the distribution given in (1.4) and  $1 \leq k \leq n, j = 0, 1, 2, \dots$

$$\begin{aligned} & \frac{\alpha \gamma k}{(\beta + 1)(j + \beta + 2)} \{E(Y_n^{(k)})^{j+\beta+2} - E(Y_{n-1}^{(k)})^{j+\beta+2}\} \\ &= \frac{\alpha k}{(j + \beta + 1)} \{E(Y_n^{(k)})^{j+\beta+1} - E(Y_{n-1}^{(k)})^{j+\beta+1}\} + \frac{\lambda k}{(j + 1)} \{E(Y_n^{(k)})^{j+1} - E(Y_{n-1}^{(k)})^{j+1}\} \\ & \quad + \frac{n\gamma}{(j + 1)} \{E(Y_{n+1}^{(k)})^{j+1} - E(Y_n^{(k)})^{j+1}\} - E(Y_n^{(k)})^j. \end{aligned} \tag{2.1}$$

**Proof.** From (1.1), we have

$$E(Y_n^{(k)})^j = \frac{k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx. \tag{2.2}$$

From (1.5) and (2.2), we have

$$\begin{aligned} E(Y_n^{(k)})^j &= \frac{\alpha k^n}{\Gamma(n)} \int_0^\infty x^{j+\beta} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx \\ &+ \frac{\lambda k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx - \frac{\alpha \gamma k^n}{(1 + \beta)\Gamma(n)} \int_0^\infty x^{j+\beta+1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx \\ & \quad + \frac{\gamma k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^n [\bar{F}(x)]^k dx. \end{aligned} \tag{2.3}$$

On Integrating (2.3) by parts, treating  $x^j$ 's for integration and rest of the integrand for differentiation and simplifying, we get

$$\begin{aligned} E(Y_n^{(k)})^j &= \frac{\alpha k}{(j + \beta + 1)} \{E(Y_n^{(k)})^{j+\beta+1} - E(Y_{n-1}^{(k)})^{j+\beta+1}\} \\ &+ \frac{\lambda k}{(j + 1)} \{E(Y_n^{(k)})^{j+1} - E(Y_{n-1}^{(k)})^{j+1}\} - \frac{\alpha \gamma k}{(\beta + 1)(j + \beta + 2)} \{E(Y_n^{(k)})^{j+\beta+2} - E(Y_{n-1}^{(k)})^{j+\beta+2}\} \\ & \quad + \frac{n\gamma}{(j + 1)} \{E(Y_{n+1}^{(k)})^{j+1} - E(Y_n^{(k)})^{j+1}\}. \end{aligned}$$

On rearranging the terms of the above expression, we get the required expression.

**Remark**

i) For  $k=1$  (2.1), the recurrence relation for single moments of upper record values from the Power-exponential hazard rate distribution has the form

$$\begin{aligned} & \frac{\alpha \gamma k}{(\beta+1)(j+\beta+2)} \{E(X_{U(n)})^{j+\beta+2} - E(X_{U(n-1)})^{j+\beta+2}\} \\ &= \frac{\alpha k}{(j+\beta+1)} \{E(X_{U(n)})^{j+\beta+1} - E(X_{U(n-1)})^{j+\beta+1}\} + \frac{\lambda k}{(j+1)} \{E(X_{U(n)})^{j+1} - E(X_{U(n-1)})^{j+1}\} \\ & \quad + \frac{n\gamma}{(j+1)} \{E(X_{U(n+1)})^{j+1} - E(X_{U(n)})^{j+1}\} - E(X_{U(n)})^j. \end{aligned}$$

ii) Setting  $\beta = \lambda = \gamma = 0$  in (2.1), the result for single moments of generalized record values is deduced for exponential distribution as obtained by Balakrishnan and Ahsanullah (1995) [4].

iii) Setting  $\lambda = \gamma = 0$ ,  $\beta = 1$  in (2.1), we deduced the recurrence relation for single moments of generalized record values from Rayleigh distribution as established by Khan *et al.* (2015) [7].

iv) Setting  $\lambda = \gamma = 0$  and replace  $\beta$ ,  $\alpha$  by  $\mu - 1$  and  $\mu\theta$  respectively in (2.1), we deduced the result for moments of generalized record values from Weibull distribution with shape  $\mu$  and scale  $\theta$  parameters as obtained by Pawlas and Szynal (2000) [5].

v) Putting  $\lambda = \gamma = 0$  in (2.1), we get the recurrence relation for single moments of power hazard rate distribution as established by Khan and Khan (2019) [12].

vi) Putting  $\alpha = 0$  in (2.1), the result for single moments of generalized record values is deduced for Gompertz distribution as established by Minimol and Thomas (2014) [8].

vii) Setting  $\beta = 0$  in (2.1), we obtained the recurrence relation for moments of generalized record values from Gompertz-Makeham distribution.

### 3. RELATIONS FOR PRODUCT MOMENTS

The joint *pdf* of  $Y_m^{(k)}$  and  $Y_n^{(k)}$  is given by

$$\begin{aligned} f_{Y_m^{(k)}, Y_n^{(k)}}(x, y) &= \frac{k^n}{\Gamma(m)\Gamma(n-m)} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \\ & \quad \times [\bar{F}(y)]^{k-1} f(y), \quad x > y, \quad 1 \leq m < n, \quad n \geq 2. \end{aligned} \tag{3.1}$$

**Theorem 3.1.** For the distribution given in (1.2) and  $k \geq 1$ ,  $1 \leq m < n$ ,  $n \geq 2$ ,  $i, j = 0, 1, 2, \dots$

$$\begin{aligned} & \frac{\alpha \gamma k}{(\beta+1)(i+\beta+2)} \{E[(Y_m^{(k)})^{i+\beta+2} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta+2} (Y_{n-1}^{(k)})^j]\} \\ &= \frac{\alpha k}{(i+\beta+1)} \{E[(Y_m^{(k)})^{i+\beta+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta+1} (Y_{n-1}^{(k)})^j]\} \\ & \quad + \frac{\lambda k}{(i+1)} \{E[(Y_m^{(k)})^{i+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+1} (Y_{n-1}^{(k)})^j]\} \\ & \quad + \frac{\gamma m}{(i+1)} \{E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j]\} - E[(Y_m^{(k)})^i (Y_n^{(k)})^j]. \end{aligned} \tag{3.2}$$

**Proof.** From (3.1), we have

$$\begin{aligned} E[(Y_m^{(k)})^i (Y_n^{(k)})^j] &= \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^y x^i y^j [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} \\ & \quad \times [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dx dy. \end{aligned} \tag{3.3}$$

From (1.5) and (3.3), we have

$$\begin{aligned}
 E[(Y_m^{(k)})^i (Y_n^{(k)})^j] &= \frac{\alpha k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^y x^{i+\beta} y^j [-\ln \bar{F}(x)]^{m-1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \\
 &\times [\bar{F}(y)]^{k-1} f(y) dx dy + \frac{\lambda k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_y^\infty x^i y^j [-\ln \bar{F}(x)]^{m-1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \\
 &\times [\bar{F}(y)]^{k-1} f(y) dx dy - \frac{\alpha \gamma k^n}{(\beta+1)\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_y^\infty x^{i+\beta+1} y^j [-\ln \bar{F}(x)]^{m-1} \\
 &\times [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^k f(y) dx dy + \frac{\gamma k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^y x^i y^j [-\ln \bar{F}(x)]^m \\
 &\times [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [F(y)]^{k-1} f(y) dx dy. \tag{3.4}
 \end{aligned}$$

On Integrating (3.4) by parts, treating  $x^i$ 's for integration and rest of the integrand for differentiation and simplifying, we get

$$\begin{aligned}
 E(Y_n^{(k)})^j &= \frac{\alpha k}{(i+\beta+1)} \{E[(Y_m^{(k)})^{i+\beta+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta+1} (Y_{n-1}^{(k)})^j]\} \\
 &+ \frac{\lambda k}{(i+1)} \{E[(Y_m^{(k)})^{i+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+1} (Y_{n-1}^{(k)})^j]\} \\
 &- \frac{\alpha \gamma k}{(\beta+1)(i+\beta+2)} \{E[(Y_m^{(k)})^{i+\beta+2} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta+2} (Y_{n-1}^{(k)})^j]\} \\
 &+ \frac{\gamma m}{(i+1)} \{E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j]\}
 \end{aligned}$$

On rearranging the terms of the above expression we get the required expression.

**Remark**

(i) For  $k=1$  in (3.2), the recurrence relation for product moments of upper record values from the Power-exponential hazard rate distribution has the form

$$\begin{aligned}
 &\frac{\alpha \gamma k}{(\beta+1)(i+\beta+2)} \{E[(X_{U(m)})^{i+\beta+2} (X_{U(n-1)})^j] - E[(X_{U(m-1)})^{i+\beta+2} (X_{U(n-1)})^j]\} \\
 &= \frac{\alpha k}{(i+\beta+1)} \{E[(X_{U(m)})^{i+\beta+1} (X_{U(n-1)})^j] - E[(X_{U(m-1)})^{i+\beta+1} (X_{U(n-1)})^j]\} \\
 &+ \frac{\lambda k}{(i+1)} \{E[(X_{U(m)})^{i+1} (X_{U(n-1)})^j] - E[(X_{U(m-1)})^{i+1} (X_{U(n-1)})^j]\} \\
 &+ \frac{\gamma m}{(i+1)} \{E[(X_{U(m+1)})^{i+1} (X_{U(n)})^j] - E[(X_{U(m)})^{i+1} (X_{U(n)})^j]\} - E[(X_{U(m)})^i (X_{U(n)})^j].
 \end{aligned}$$

(ii) Setting  $\beta = \lambda = \gamma = 0$  in (3.2), the result for product moments of generalized record values is deduced for exponential distribution.

(iii) Setting  $\lambda = \gamma = 0, \beta = 1$  in (3.2), we deduced the recurrence relation for product moments of generalized record values from Rayleigh distribution.

(iv) Putting  $\lambda = \gamma = 0$  in (3.2), we obtained the recurrence relation for product moments of power hazard rate distribution.

(v) Putting  $\alpha = 0$  in (3.2), the result for product moments of generalized record values is deduced for Gompertz distribution as established by Minimol and Thomas (2014) [8].

(vi) Setting  $\beta = 0$  in (3.2), we obtained the recurrence relation for product moments of generalized record values from Gompertz-Makeham distribution.

**Generalized Record Values from the Power-Exponential Hazard Rate distribution and Characterizations**

(vii) Setting  $\lambda = \gamma = 0$  and replace  $\beta, \alpha$  by  $\mu - 1$  and  $\mu\theta$  respectively in (3.2), we deduced the result for product moments of generalized record values from Weibull distribution with shape  $\mu$  and scale  $\theta$  parameters.

**Table 2: Moments of record values**

n	$\alpha = 1, \beta = 1.5, \gamma = 2$				$\alpha = 2, \beta = 1.5, \gamma = 2$			
	$\lambda = 2$				$\lambda = 2$			
	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	0.29201	0.12679	0.067263	0.04064	0.28636	0.12128	0.06265	0.036847
2	0.48778	0.28310	0.18421	0.13041	0.47663	0.26970	0.17098	0.11794
3	0.63221	0.44136	0.33185	0.26468	0.61682	0.41981	0.30770	0.23930
4	0.74584	0.59353	0.49768	0.43596	0.72734	0.56442	0.46160	0.39454
5	0.83923	0.73757	0.67402	0.63710	0.81848	0.70173	0.62582	0.57755
n	$\alpha = 2, \beta = 2.5, \gamma = 3.5$				$\alpha = 3, \beta = 2.5, \gamma = 3.5$			
	$\lambda = 3$				$\lambda = 3$			
	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	0.18828	0.05204	0.01743	0.00661	0.18801	0.05185	0.01732	0.00655
2	0.31142	0.11433	0.04675	0.02069	0.31079	0.11381	0.04640	0.02048
3	0.40047	0.17587	0.08274	0.04112	0.39950	0.17496	0.08207	0.04066
4	0.46944	0.23388	0.12225	0.06651	0.46817	0.23256	0.12118	0.06573
5	0.52544	0.28789	0.16344	0.09568	0.52391	0.28618	0.16196	0.09451

**4. CHARACTERIZATION**

**Theorem 4.1.** Let  $k$  and  $j$  are integers such that  $k \geq 1, j \geq 0$ . A necessary and sufficient condition for a random variable  $X$  to be distributed with *pdf* given by (1.4) is that

$$\begin{aligned} & \frac{\alpha \gamma k}{(\beta + 1)(j + \beta + 2)} \{E(Y_n^{(k)})^{j+\beta+2} - E(Y_{n-1}^{(k)})^{j+\beta+2}\} \\ &= \frac{\alpha k}{(j + \beta + 1)} \{E(Y_n^{(k)})^{j+\beta+1} - E(Y_{n-1}^{(k)})^{j+\beta+1}\} + \frac{\lambda k}{(j + 1)} \{E(Y_n^{(k)})^{j+1} - E(Y_{n-1}^{(k)})^{j+1}\} \\ & \quad + \frac{n\gamma}{(j + 1)} \{E(Y_{n+1}^{(k)})^{j+1} - E(Y_n^{(k)})^{j+1}\} - E(Y_n^{(k)})^j. \end{aligned} \tag{4.1}$$

**Proof.** The necessary part follows from Theorem 4.1. On the other hand if the recurrence relation (4.1) is satisfied, then on using Khan *et al.* (2017), we have

$$\begin{aligned} & \frac{n \gamma k^n}{(\beta + 1)\Gamma(n)} \int_0^\infty x^{j+\beta+1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx = \frac{\lambda k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx \\ & + \frac{\alpha k^n}{\Gamma(n)} \int_0^\infty x^{j+\beta} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx - \frac{k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx \\ & + \frac{\gamma k^{n+1}}{(j + 1)\Gamma(n)} \int_0^\infty x^{j+1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) \left\{ -\ln \bar{F}(x) - \frac{n}{k} \right\} dx. \end{aligned} \tag{4.2}$$

let

$$h(x) = -\frac{1}{k}[-\ln \bar{F}(x)]^n [\bar{F}(x)]^k \tag{4.3}$$

Differentiating both the sides of (4.3), we get

$$\begin{aligned} h'(x) &= [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) \left\{ -\ln \bar{F}(x) - \frac{n}{k} \right\} \\ \frac{n\gamma k^n}{(\beta+1)\Gamma(n)} \int_0^\infty x^{j+\beta+1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx &= \frac{\lambda k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx \\ + \frac{\alpha k^n}{\Gamma(n)} \int_0^\infty x^{j+\beta} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx &- \frac{k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx \\ &+ \frac{\gamma k^{n+1}}{(j+1)\Gamma(n)} \int_0^\infty x^{j+1} h'(x) dx. \end{aligned} \tag{4.4}$$

Integrating by parts last terms of (4.4) in the right hand side and using (4.3), we find that

$$\frac{k^n}{\Gamma(n)} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k \left\{ \frac{f(x)}{\bar{F}(x)} - \alpha x^\beta - \lambda + \frac{\alpha\gamma}{(\beta+1)} x^{\beta+1} - \gamma [-\ln \bar{F}(x)] \right\} dx = 0. \tag{4.5}$$

Now applying a generalization of the Müntz-Szász theorem (see for example Hwang and Lin, 1984) to (4.5), we get

$$\frac{f(x)}{\bar{F}(x)} = \alpha x^\beta + \lambda - \frac{\alpha\gamma}{(\beta+1)} x^{\beta+1} + \gamma [-\ln \bar{F}(x)]$$

Which is the characterizing equation for the *pdf* as given in (1.3).

Hence the sufficient part proved.

**Theorem 4.2.** For fix a positive integer  $k \geq 1$  and  $i, j$  are non-negative integers. A necessary and sufficient condition for a random variable  $X$  to be distributed with *pdf* given by (1.3) is that

$$\begin{aligned} &\frac{\alpha\gamma k}{(\beta+1)(i+\beta+2)} \{E[(Y_m^{(k)})^{i+\beta+2} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta+2} (Y_{n-1}^{(k)})^j]\} \\ &= \frac{\alpha k}{(i+\beta+1)} \{E[(Y_m^{(k)})^{i+\beta+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta+1} (Y_{n-1}^{(k)})^j]\} \\ &\quad + \frac{\lambda k}{(i+1)} \{E[(Y_m^{(k)})^{i+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+1} (Y_{n-1}^{(k)})^j]\} \\ &\quad + \frac{\gamma m}{(i+1)} \{E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j]\} - E[(Y_m^{(k)})^i (Y_n^{(k)})^j]. \end{aligned} \tag{4.6}$$

**Proof.** The necessary part follows from Theorem 2.1. On the other hand if the relation in (4.6) is satisfied, then to prove the sufficient part, we have consider

$$\begin{aligned} &E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j] \\ &= -\frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^y x^{i+1} y^j [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2} \\ &\quad \times [\bar{F}(y)]^{k-1} f(y) \left\{ \ln \bar{F}(x) - \ln \bar{F}(y) - \frac{(n-m-1)[-\ln \bar{F}(x)]}{m} \right\} dx dy. \end{aligned} \tag{4.7}$$

let

$$h(x, y) = \frac{1}{m} [-\ln \bar{F}(x)]^m [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \tag{4.8}$$

Differentiating both the side of (4.8) with respect to  $x$ , we get

$$\begin{aligned} \frac{\partial}{\partial x} h(x, y) &= [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2} \\ &\quad \times \left\{ \ln \bar{F}(x) - \ln \bar{F}(y) - \frac{(n-m-1)[-\ln \bar{F}(x)]}{m} \right\} \end{aligned} \tag{4.9}$$

From (4.7) and (4.9), we have

$$\begin{aligned} E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] \\ = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \left\{ \int_0^y x^{i+1} \frac{\partial}{\partial x} h(x, y) dx \right\} y^j [\bar{F}(y)]^{k-1} f(y) dy. \end{aligned} \tag{4.10}$$

Integrating (4.10) with respect to  $x$  and using (4.8), we get

$$\begin{aligned} E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] \\ = \frac{(i+1)k^n}{\Gamma(m+1)\Gamma(n-m)} \int_0^\infty \int_0^y x^i y^j [-\ln \bar{F}(x)]^m [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \\ \times [\bar{F}(y)]^{k-1} f(y) dx dy. \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} E[(Y_{m+1}^{(k)})^{i+1} (Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+1} (Y_n^{(k)})^j] \\ = -\frac{k^{n-1}}{\Gamma(m-1)\Gamma(n-m)} \int_0^\infty \int_0^y x^{i+1} y^j [-\ln \bar{F}(x)]^{m-2} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2} \\ [\bar{F}(y)]^{k-1} f(y) \left\{ \ln \bar{F}(x) - \ln \bar{F}(y) - \frac{(n-m-1)[-\ln \bar{F}(x)]}{m-1} \right\} dx dy. \end{aligned} \tag{4.12}$$

let

$$h(x, y) = \frac{1}{(m-1)} [-\ln \bar{F}(x)]^{m-1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1}. \tag{4.13}$$

Differentiating both the side of (4.13) with respect to  $x$ , we get

$$\begin{aligned} \frac{\partial}{\partial x} h(x, y) &= [-\ln \bar{F}(x)]^{m-2} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2} \\ &\quad \times \left\{ \ln \bar{F}(x) - \ln \bar{F}(y) - \frac{(n-m-1)[-\ln \bar{F}(x)]}{m-1} \right\}. \end{aligned} \tag{4.14}$$

From (4.12) and (4.14), we have

$$\begin{aligned} E[(Y_m^{(k)})^{i+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+1} (Y_{n-1}^{(k)})^j] \\ = \frac{k^{n-1}}{\Gamma(m-1)\Gamma(n-m)} \int_0^\infty \left\{ \int_0^y x^{i+1} \frac{\partial}{\partial x} h(x, y) dx \right\} y^j [\bar{F}(y)]^{k-1} f(y) dy. \end{aligned} \tag{4.15}$$

Integrating (4.15) with respect to  $x$  and using (4.14), we get

$$\begin{aligned} E[(Y_m^{(k)})^{i+1} (Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+1} (Y_{n-1}^{(k)})^j] \\ = \frac{(i+1)k^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^y x^i y^j [-\ln \bar{F}(x)]^{m-1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \\ \times [\bar{F}(y)]^{k-1} f(y) dx dy. \end{aligned} \tag{4.16}$$

Using (4.11) and (4.16), (4.6) can be written as

$$\frac{k^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^y x^i y^j [-\ln \bar{F}(x)]^{m-1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) \times \left\{ -\frac{f(x)}{\bar{F}(x)} + \alpha x^\beta + \lambda - \frac{\alpha \gamma x^{\beta+1}}{(\beta+1)} + \gamma [-\ln \bar{F}(x)] \right\} dx dy. \tag{4.17}$$

Now applying a generalization of the Müntz-Szász theorem (see for example Hwang and Lin, 1984) to (4.17), we get

$$\frac{f(x)}{\bar{F}(x)} = \alpha x^\beta + \lambda - \frac{\alpha \gamma}{(\beta+1)} x^{\beta+1} + \gamma [-\ln \bar{F}(x)]$$

Which is the characterizing equation for the *pdf* as given in (1.3).

Hence the sufficient part proved.

**Theorem 4.3.** Let  $X$  be an absolutely continuous non-negative random variable having *df*  $F(x)$ , with  $F(0) = 0$  and  $0 \leq F(x) \leq 1 \quad \forall \quad 0 < x < \infty$ , then

$$E \left( \left\{ \frac{\alpha}{\beta+1} (Y_n^{(k)})^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma Y_n^{(k)}} - 1) \right\} \mid (Y_m^{(k)}) = x \right) = \frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma x} - 1) + \frac{n-m}{k}. \tag{4.18}$$

if and only if

$$F(x) = 1 - e^{-\left( \frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma x} - 1) \right)}, \quad x \geq 0, \quad \alpha, \beta, \gamma, \lambda > 0.$$

**Proof.** From (1.1) and (1.2), we have

$$\begin{aligned} & E \left( \left\{ \frac{\alpha}{\beta+1} (Y_n^{(k)})^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma Y_n^{(k)}} - 1) \right\} \mid (Y_m^{(k)}) = x \right) \\ &= \frac{k^{n-m}}{\Gamma(n-m)} \int_x^\infty \left\{ \frac{\alpha}{(\beta+1)} y^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma y} - 1) \right\} \left( -\ln \frac{\bar{F}(y)}{\bar{F}(x)} \right)^{n-m-1} \left( \frac{\bar{F}(y)}{\bar{F}(x)} \right)^{k-1} \frac{f(y)}{\bar{F}(x)} dy. \end{aligned} \tag{4.19}$$

By setting  $z = \frac{\bar{F}(y)}{\bar{F}(x)} = \frac{e^{-\left( \frac{\alpha}{\beta+1} y^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma y} - 1) \right)}}{e^{-\left( \frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma x} - 1) \right)}}$  in (4.19), we have

$$\begin{aligned} & E \left( \left\{ \frac{\alpha}{\beta+1} (Y_n^{(k)})^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma Y_n^{(k)}} - 1) \right\} \mid (Y_m^{(k)}) = x \right) \\ &= \frac{k^{n-m}}{\Gamma(n-m)} \left( \left\{ \frac{\alpha}{(\beta+1)} x^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma x} - 1) \right\} \int_0^1 (-\ln z)^{n-m-1} z^{k-1} dz - \int_0^1 (-\ln z)^{n-m} z^{k-1} dz \right). \end{aligned} \tag{4.20}$$

In view of Gradshteyn and Ryzhik (2007, p-551), note that

$$\int_0^1 (-\ln x)^{\mu-1} x^{\nu-1} dx = \frac{\Gamma \mu}{\nu^\mu}, \quad \mu > 0, \nu > 0. \tag{4.21}$$

Using (4.21) in (4.20), we obtain the result given in (4.18).

To prove the sufficient part, we have

$$\frac{k^{n-m}}{\Gamma(n-m)} \int_x^\infty \left\{ \frac{\alpha}{\beta+1} y^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma y} - 1) \right\} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} \times [\bar{F}(y)]^{k-1} f(y) dy = [\bar{F}(x)]^k g_{n|m}(x), \quad (4.22)$$

where

$$g_{s|r}(x) = \frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma x} - 1) + \frac{n-m}{k}.$$

Differentiating both sides of (4.22) with respect to  $x$ , we get

$$-\frac{k^{n-m} f(x)}{\Gamma(n-m-1) \bar{F}(x)} \int_x^\infty \left\{ \frac{\alpha}{\beta+1} y^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma y} - 1) \right\} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-2} \times [\bar{F}(y)]^{k-1} f(y) dy = g'_{n|m}(x) [\bar{F}(x)]^k + k g_{n|m}(x) [\bar{F}(x)]^{k-1} f(x)$$

or

$$-k g_{n|m+1}(x) [\bar{F}(x)]^{k-1} f(x) = g'_{n|m}(x) [\bar{F}(x)]^k - k g_{n|m}(x) [\bar{F}(x)]^{k-1} f(x).$$

Therefore

$$\begin{aligned} \frac{f(x)}{\bar{F}(x)} &= -\frac{g'_{n|m}(x)}{k [g_{n|m+1}(x) - g_{n|m}(x)]} \\ &= \alpha x^\beta + \lambda e^{\gamma x}, \end{aligned} \quad (4.23)$$

where

$$\begin{aligned} g'_{n|m}(x) &= \alpha x^\beta + \lambda e^{\gamma x} \\ g_{n|m+1}(x) - g_{n|m}(x) &= -\frac{1}{k}. \end{aligned}$$

Integrating both the sides of (4.23) with respect to  $x$  between  $(0, y)$ , Hence the sufficiency part has proved.

**Theorem 4.4.** Suppose that  $X$  be an absolutely continuous (with respect to Lebesgue measure) random variable with the *df*  $F(x)$  and *pdf*  $f(x) \forall 0 < x < \infty$ , such that  $f'(x)$  and  $E(X | X \leq x)$ , exist for all  $x, 0 < x < \infty$ , then

$$E(X | X \leq x) = g(x) \eta(x), \quad (4.24)$$

where

$$\eta(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = -\frac{x}{\alpha x^\beta + \lambda e^{\gamma x}} + \frac{e^{\left(\frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma x} - 1)\right)}}{\alpha x^\beta + \lambda e^{\gamma x}} \int_0^x e^{\left(\frac{\alpha}{\beta+1} u^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma u} - 1)\right)} du$$

if and only if

$$f(x) = (\alpha x^\beta + \lambda e^{\gamma x}) e^{-\left(\frac{\alpha}{\beta+1} x^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma x} - 1)\right)}, \quad x \geq 0, \quad \alpha, \beta, \gamma, \lambda > 0.$$

**Proof.** From (1.3), we have

$$E(X | X \leq x) = \frac{1}{F(x)} \int_0^x u (\alpha u^\beta + \lambda e^{\gamma u}) e^{-\left(\frac{\alpha}{\beta+1} u^{\beta+1} + \frac{\lambda}{\gamma} (e^{\gamma u} - 1)\right)} du. \quad (4.25)$$

Integrating (4.25) by parts, treating  $(\alpha x^\beta + \lambda e^{\gamma x})e^{-\left(\frac{\alpha}{\beta+1}x^{\beta+1} + \frac{\lambda}{\gamma}(e^{\gamma x}-1)\right)}$  for integration and rest of the integrand for differentiation, we get

$$E(X | X \leq x) = \frac{1}{F(x)} \left\{ -x e^{-\left(\frac{\alpha}{\beta+1}x^{\beta+1} + \frac{\lambda}{\gamma}(e^{\gamma x}-1)\right)} + \int_0^x e^{-\left(\frac{\alpha}{\beta+1}u^{\beta+1} + \frac{\lambda}{\gamma}(e^{\gamma u}-1)\right)} du \right\}. \quad (2.26)$$

Now dividing and multiplying (4.27) by  $f(x)$ , we obtain the result as given in (4.24).

For proving sufficient part, we have from (4.24)

$$\int_0^x u f(u) du = g(x)f(x). \quad (4.27)$$

Differentiating (4.27) on both sides with respect to  $x$ , we find that

$$xf(x) = g'(x)f(x) + g(x)f'(x).$$

Therefore,

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)} \quad [\text{Ahsanuallah, et. al (2016)}]$$

$$\frac{f'(x)}{f(x)} = -(\alpha x^\beta + \lambda e^{\gamma x}) + \frac{\alpha\beta x^{\beta-1} + \lambda\gamma e^{\gamma x}}{\alpha x^\beta + \lambda e^{\gamma x}}, \quad (4.28)$$

where

$$g'(x) = x - g(x) \left( -(\alpha x^\beta + \lambda e^{\gamma x}) + \frac{\alpha\beta x^{\beta-1} + \lambda\gamma e^{\gamma x}}{\alpha x^\beta + \lambda e^{\gamma x}} \right).$$

On integrating (4.28) both sides with respect to  $x$ , we get

$$f(x) = C(\alpha x^\beta + \lambda e^{\gamma x}) e^{-\left(\frac{\alpha}{\beta+1}x^{\beta+1} + \frac{\lambda}{\gamma}e^{\gamma x}\right)}.$$

Further, To obtain the value of C (constant of integration), we have used the property of *pdf*, note that

$$\int_0^\infty f(x) dx = 1.$$

Thus

$$\frac{1}{C} = \int_0^\infty (\alpha x^\beta + \lambda e^{\gamma x}) e^{-\left(\frac{\alpha}{\beta+1}x^{\beta+1} + \frac{\lambda}{\gamma}e^{\gamma x}\right)} dx = \frac{1}{e^{\lambda/\gamma}},$$

which proves that

$$f(x) = (\alpha x^\beta + \lambda e^{\gamma x}) e^{-\left(\frac{\alpha}{\beta+1}x^{\beta+1} + \frac{\lambda}{\gamma}(e^{\gamma x}-1)\right)}, \quad x > 0, \quad \alpha, \beta, \lambda > 0.$$

## REFERENCES

1. Chandler KN. The distribution and frequency of record values. J R Stat Soc Ser B. 1952;14:220-228.
2. Dziubdziela W, Kopociński B. Limiting properties of the  $r$ th record value. Appl Math. 1976;15:187-190.
3. Minimol S, Thomas PY. On some properties of Makeham distribution using generalized record values and its characterization. Braz J Probab Stat. 2013;27(4):487-501.
4. Balakrishnan N, Ahsanuallah M. Relations for single and product moments of record values from exponential distribution. J Appl Stat Sci. 1995;2:73-87.
5. Pawlas P, Szynal D. Recurrence relations for single and product moments of k-th record values from Weibull distributions and a characterization. J Appl Stat Sci. 2000;10:17-26.

6. Sultan KS. Record values from the modified Weibull distribution and applications. *Int Math Forum*. 2007;2(41):2045–2054.
7. Kumar D, Khan MI. Recurrence relations for moments of k record values from generalized beta II distribution and a characterization. *Selcuk J Appl Math*. 2012;13(1):75–82.
8. Minimol S, Thomas PY. On characterization of Gompertz distribution by properties of generalized record values. *J Stat Theory Appl*. 2014;13:38–45.
9. MirMostafaei ST, Asgharzadeh A, Fallah A. Record values from NH distribution and associated inference. *Metron*. 2016;74:37–59.
10. Athar H, Fawzy MA. Moment properties of record values from Rayleigh Lomax distribution and characterization. *J Mod Appl Stat Methods*. 2023;22(2).
11. Tarvirdizade B, Nematollahi N. A new flexible hazard rate distribution: application and estimation of its parameters. *Commun Stat Simul Comput*. 2019;48:882–899.
12. Khan MI, Khan MAR. Generalized record values from distributions having power hazard function and characterization. *J Stat Appl Probab*. 2019;8:103–111.

\*\*\*\*\*