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# Stability results of mixed type functional equations in modular spaces and 2-Banach spaces *

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**Abstract** In this paper, we investigate the generalized Hyers-Ulam-Stability of additive and quartic functional equations in modular spaces with and without the $\triangle_2$-condition using the direct method and also in 2-Banach Spaces.

**Key words** additive and quartic functional equations, modular spaces, generalized Hyers-Ulam stability.

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## 1 Introduction

Functional equations play an important role in the study of stability problems in several structures. In 1940, Ulam [19] was the first author who suggested the stability problem of functional equations concerning the stability of group homomorphisms and this formed the foundation for the research work on stability problems. In this case if the solution exists, then the equation is stable.
Using Banach spaces, Hyers [6] solved this stability problem by considering Cauchy's functional equation. Hyers' work was expanded by Aoki [1] by assuming an unbounded Cauchy difference. Rassias [15] presented work on additive mapping which was further extended by Gavruta [3]. In 1950, Nakano [11] studied the theory of modular linear space. Recently, Senthil Kumar et al. [17] established the stability results of functional equations with both the Fatou property and the $\triangle_2$-condition together in modular spaces.

**Definition 1.1.** ( [13, 16]) Let Z be a linear space over K ($\mathbb{R}$ or $\mathbb{C}$). A generalized mapping $\sigma : \mathrm{Z} \to [0, \infty)$ is called modular if for any given $x, y \in \mathrm{Z}$, the following conditions hold:

**1.1 (i)** $\sigma(x) = 0 \Longleftrightarrow x = 0$,

**1.1 (ii)** $\sigma(\varepsilon x) = \sigma(x)$ for any scalar $\varepsilon$ with $|\varepsilon| = 1$,

**1.1 (iii)** $\sigma(\varepsilon_1 x + \varepsilon_2 y) \leq \sigma(x) + \sigma(y)$ for any scalar $\varepsilon_1,\ \varepsilon_2 \geq 0$ with $\varepsilon_1 + \varepsilon_2 = 1$.

If the condition 1.1(iii) is replaced by

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**1.1(iv)** $\sigma(\varepsilon_1 x+\varepsilon_2 y)\leq \varepsilon_1\sigma(x)+\varepsilon_2\sigma(y)$ for any scalar $\varepsilon_1$, $\varepsilon_2\geq 0$ with $\varepsilon_1+\varepsilon_2=1$,

then $\sigma$ is called a convex modular. Furthermore, the vector space induced by a modular $\sigma$,

$$\mathrm{Z}_\sigma=\{x\in \mathrm{Z}{:}\sigma(ax)\to 0 \text{ as } a\to 0\}$$

is a modular space.

**Definition 1.2.** ( [13,16]) Let $\mathrm{Z}_\sigma$ be a modular space and $\{x_n\}$ is a sequence in $\mathrm{Z}_\sigma$. Then

**1.2(i)** $\{x_n\}$ is $\sigma$-convergent to a point $x\in\mathrm{Z}_\sigma$, and we write $x_n\to x$ if $\sigma(x_n-x)\to 0$ as $n\to\infty$.

**1.2(ii)** $\{x_n\}$ is said to be $\sigma$-Cauchy if for any $\varepsilon>0$ one has $\sigma(x_n-x_m)<\varepsilon$ for sufficiently large $n,m\in\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.

**1.2(iii)** $k\subseteq\mathrm{Z}_\sigma$ is called a $\sigma$-complete if any $\sigma$-Cauchy sequence is $\sigma$-convergent in $K$.

**Definition 1.3.** ( [20]) A modular $\sigma$ has the Fatou property if $\sigma(x)\leq \operatorname{liminf}_{n\to\infty}\sigma(x_n)$, whereas the sequence $\{x_n\}$ is $\sigma$-convergent to $x$ in modular space $\mathrm{Z}_\sigma$ and conversely.

**Definition 1.4.** ( [21]) A modular $\sigma$ is said to satisfy the $\Delta_2$-condition if there exists $k>0$ such that $\sigma(2x)\leq k\sigma(x)$, for all $x\in X$.

**Proposition 1.5.** *[9] In modular spaces,*

**1.5(i)** *if $x_n\to x$ and $b$ is a constant vector, then $x_n+b\to x+b$.*

**1.5(ii)** *if $x_n\to x$ and if $y_n\to y$, then $\varepsilon_1x_n+\varepsilon_2y_n\to\varepsilon_1x+\varepsilon_2y$, where $\varepsilon_1+\varepsilon_2\leq 1$ and $\varepsilon_1$, $\varepsilon_2\geq 0$.*

**Remark 1.6.** [18] Assume that $\sigma$ satisfies a $\triangle_2$-condition with $\triangle_2$-constant $k>0$ and is convex. If $k<2$, then $\sigma(x)\leq k\sigma\left(\frac{x}{2}\right)\leq\frac{k}{2}\sigma(x)$, which indicates $\sigma=0$. So, we should have the $\triangle_2$-constant $k\geq 2$ if $\sigma$ is convex modular.

Here, it should be noted that the convergence of a sequence $\{x_n\}$ to $x$ does not imply that $\{nx_n\}$ converges to $nx$ if $n$ is selected from the equivalent scalar field with $|n|>1$ in modular spaces. Because, this is a multiple of the convergent sequence $\{x_n\}$, which naturally converges in modular spaces. Many mathematicians have researched stability without using $\triangle_2$-condition using the the fixed-point approach of quasi-contraction functions in modular spaces, which is the method introduced by Khamsi [7].
In 2019, Senthil [17] studied the stability of the equation in Banach spaces. Motivated by the method and direction of research of Senthil [17], an effort is made here to investigate the stability of the mixed-type functional equation

$$\begin{aligned}&f(2x+y)+f(2x-y)+f(x+2y)+f(x-2y)=8\left[f(x+y)+f(x-y)\right]\\&+f(2x)-5f(x)+7f(-x)+2f(2y)-5f(-y)-9f(y),\end{aligned}\tag{1.1}$$

in modular spaces with and without $\Delta_2$-condition using the direct method. We also obtain the stability of the functional equation in 2-Banach spaces.

## 2 Stability results in modular spaces

In this section, we prove the stability results of mixed-type functional equation in modular spaces by using the Direct method, which is an improved form of the methods of Wongkum et al. ( [21,22]) and Sadeghi [16]. Consider that $X$ is a linear space and $\mathrm{Z}_\sigma$ is a complete convex modular space.
We define a mapping $f:X\to\ \mathrm{Z}_\sigma$ by

$$\begin{aligned}D_{f(x,y)}=&f(2x+y)+f(2x-y)+f(x+2y)+f(x-2y)-8\left[f(x+y)+f(x-y)\right]\\&-f(2x)+5f(x)-7f(-x)-2f(2y)+5f(-y)+9f(y)\end{aligned}\tag{2.1}$$

for all $x,y\in X$.

**Theorem 2.1.** *If there exists a mapping* $\Psi$: $X^2 \to [0, \infty)$ *such that*

$$\Psi(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \Psi\left(2^{n-1}x, 2^{n-1}y\right) < \infty, \tag{2.2}$$

*and an odd mapping* $f : X \to \mathrm{Z}_\sigma$ *with* $f(0) = 0$ *and*

$$\sigma\left(D_{f(x,y)}\right) \leq \Psi(x, y), \tag{2.3}$$

*for all* $x, y \in X$*, then there exists a unique additive mapping* $Q : X \to \mathrm{Z}_\sigma$ *satisfying*

$$\sigma(f(x) - Q(x)) \leq \Psi(x, 0), \tag{2.4}$$

*for all* $x \in X$*.*

**Proof.** Putting $y = 0$ in (2.3), and $\Psi(x, 0) = \psi(x)$, we get

$$\sigma(2f(x) - f(2x)) \leq \Psi(x, 0) = \psi(x), \tag{2.5}$$

$$\sigma\left(f(x) - \frac{1}{2}f(2x)\right) \leq \frac{1}{2}\psi(x), \tag{2.6}$$

for all $x \in X$. Then, by the principle of mathematical induction, we get

$$\sigma\left(f(x) - \frac{1}{2^n}f(2^n x)\right) \leq \sum_{j=1}^{n} \frac{1}{2^j}\psi\left(2^{j-1}x\right), \tag{2.7}$$

for all $x \in X$ and all natural numbers $n = 1$. Result is true for $n = 1$ arises from (2.6). We assume that the inequality (2.7) holds for $n \in \mathbb{N}$, then we obtain

$$\begin{aligned}
\sigma\left(\frac{f\left(2^{n+1}x\right)}{2^{n+1}} - f(x)\right) &= \sigma\left(\frac{1}{2}\left(f(2x) - \frac{f(2^n x)}{2^n}\right) + \frac{1}{2}(2f(x) - f(2x))\right) \\
&\leq \frac{1}{2}\sigma\left(\frac{f(2^n x)}{2^n} - f(2x)\right) + \frac{1}{2}\sigma(f(2x) - 2f(x)) \\
&\leq \frac{1}{2}\sum_{j=1}^{n} \frac{1}{2^j}\psi\left(2^j x\right) + \frac{1}{2}\psi(x) \\
&= \sum_{j=1}^{n} \frac{1}{2^{j+1}}\psi\left(2^j x\right) + \frac{1}{2}\psi(x) \\
&= \sum_{j=1}^{n+1} \frac{1}{2^j}\psi\left(2^{j-1}x\right).
\end{aligned}$$

Therefore, inequality (2.7) holds for all $n \in \mathbb{N}$. Let $l$ and $m$be natural numbers with $m > l$. By inequality (2.7), we obtain

$$\begin{aligned}
\sigma\left(\frac{f(2^m x)}{2^m} - \frac{f\left(2^l x\right)}{2^l}\right) &= \sigma\left(\frac{1}{2^l}\left(\left(\frac{f\left(2^{m-l}2^l x\right)}{2^{m-l}}\right) - f\left(2^l x\right)\right)\right) \\
&\leq \frac{1}{2^l}\sum_{j=1}^{m-l} \frac{\psi\left(2^{j-1}2^l x\right)}{2^j} \\
&= \sum_{j=1}^{m-l} \frac{\psi\left(2^{l+j-1}x\right)}{2^{l+j}} \\
&= \sum_{n=l+1}^{m} \frac{\psi\left(2^{n-1}x\right)}{2^n}.
\end{aligned} \tag{2.8}$$

From (2.2) and (2.8), we have that the sequence $\left\{\frac{f(2^m x)}{2^m}\right\}$ is a $\sigma$-Cauchy sequence in $Z_\sigma$. The $\sigma$-completeness of $Z_\sigma$ confers its $\sigma$-convergence. Now, we define a mapping $Q\ : X \to Z_\sigma$ by

$$Q\left(x\right) = \lim_{m\to\infty} \frac{f\left(2^m x\right)}{2^m}\ , \tag{2.9}$$

for all $x \in X$.
Hereby,

$$\begin{aligned}\sigma\left(\frac{2Q\left(x\right)-Q(2x)}{2^3}\right) &= \sigma\left(\frac{1}{2^3}\left(\frac{f\left(2^{m+1}x\right)}{2^m}-Q(2x)\right)+\frac{1}{2}\left(\frac{1}{2}Q\left(x\right)-\frac{1}{2}\frac{f\left(2^{m+1}x\right)}{2^{m+1}}\right)\right)\\ &\leq \frac{1}{2^3}\sigma\left(Q\left(2x\right)-\frac{f\left(2^{m+1}x\right)}{2^m}\right)+\frac{1}{4}\sigma\left(\frac{f\left(2^{m+1}x\right)}{2^{m+1}}-Q(x)\right),\end{aligned} \tag{2.10}$$

for all $x \in X$. Then, by (2.9), the right-hand side of (2.10) tends to 0 as $m \to \infty$. Thus, we get that

$$Q\left(2x\right) = 2Q(x), \tag{2.11}$$

for all $x \in X$. We observe that for all $m \in \mathbb{N}$, by (2.11), we have

$$\begin{aligned}\sigma\left(f\left(x\right)-Q(x)\right) &= \sigma\left(\sum_{n=1}^{m}\frac{2f\left(2^{n-1}x\right)-f(2^n x)}{2^n}+\left(\frac{f\left(2^m x\right)}{2^m}-Q(x)\right)\right)\\ &= \sigma\left(\sum_{n=1}^{m}\frac{2f\left(2^{n-1}x\right)-f(2^n x)}{2^n}+\frac{1}{2}\left(\frac{f\left(2^{m-1}2x\right)}{2^{m-1}}-Q(2x)\right)\right),\end{aligned} \tag{2.12}$$

because $\sum_{n=1}^{m}\frac{1}{2^n} \leq 1$, from inequality (2.5) and (2.12), we get

$$\begin{aligned}\sigma\left(f\left(x\right)-Q(x)\right) &\leq \sum_{n=1}^{m}\frac{1}{2^n}\sigma\left(2f\left(2^{n-1}x\right)-f(2^n x)\right)+\frac{1}{2}\sigma\left(\frac{f\left(2^{m-1}2x\right)}{2^{m-1}}-Q(2x)\right)\\ &\leq \sum_{n=1}^{m}\psi\left(2^{n-1}x\right)+\frac{1}{2}\ \sigma\left(\frac{f\left(2^{m-1}2x\right)}{2^{m-1}}-Q(2x)\right)\\ &= \sum_{n=1}^{m}\frac{1}{2^n}\Psi\left(2^{n-1}x,0\right)+\frac{1}{2}\sigma\left(\frac{f\left(2^{m-1}2x\right)}{2^{m-1}}-Q(2x)\right).\end{aligned} \tag{2.13}$$

Taking the limit $m \to \infty$ in (2.13), we get

$$\sigma\left(f\left(x\right)-Q(x)\right) \leq \psi\left(x,0\right),$$

for all $x \in X$. Therefore, we get (2.4). Now, we want to show that the mapping $Q$ is additive. We observe that

$$\begin{aligned}\sigma\left(\frac{1}{2^j}D_{f\left(2^j x,2^j y\right)}\right) &\leq \frac{1}{2^j}\sigma\left(D_{f\left(2^j x,2^j y\right)}\right)\\ &\leq \frac{1}{2^j}\Psi\left(2^j x,2^j y\right) \to 0\ \text{ as } j\to\infty,\end{aligned} \tag{2.14}$$

for all $x,\ y \in X$. From the inequality (2.14), we have $\sigma(D_{Q(x,y)}) \to 0$ as $j \to \infty$. Hence, we get

$$D_{Q(x,y)} = 0.$$

Therefore, the mapping $Q$ is additive. Next, we show that the mapping $Q$ is unique. We assume that there exists another mapping $R$which satisfies (2.4). Then,

$$\begin{aligned}\sigma\left(\frac{Q\left(x\right)-R\left(x\right)}{2}\right) &= \sigma\left(\frac{1}{2}\left(\frac{Q\left(2^n x\right)}{2^n}-\frac{f\left(2^n x\right)}{2^n}\right)+\frac{1}{2}\left(\frac{f\left(2^n x\right)}{2^n}-\frac{R\left(2^n x\right)}{2^n}\right)\right)\\ &\leq \frac{1}{2}\sigma\left(\frac{Q\left(2^n x\right)}{2^n}-\frac{f\left(2^n x\right)}{2^n}\right)+\frac{1}{2}\sigma\left(\frac{f\left(2^n x\right)}{2^n}-\frac{R\left(2^n x\right)}{2^n}\right)\end{aligned}$$

$$\leq \frac{1}{22^n}\left(\sigma\left(Q\left(2^n x\right)-f\left(2^n x\right)\right)+\sigma\left(f\left(2^n x\right)-R\left(2^n x\right)\right)\right)$$

$$\leq \frac{1}{2^n}\ \psi\left(2^n x, 2^n y\right)$$

$$\leq \sum_{k=n+1}^{\infty} \Psi\left(2^{n-1} x, 0\right) \rightarrow 0 \text{ as } n \rightarrow \infty$$

. This implies that $Q = R$. □

**Corollary 2.2.** *If there exists an odd mapping $f : X \rightarrow\ Z_\sigma$ with $f(0) = 0$ and*

$$\sigma\left(D_{f(x,y)}\right) \leq \varepsilon, \tag{2.15}$$

*for all $x, y \in X$, then there exists a unique additive mapping $Q : X \rightarrow\ Z_\sigma$ satisfying*

$$\sigma\left(f(x)-Q(x)\right) \leq \frac{\varepsilon}{2},$$

*for all $x \in X$.*

**Corollary 2.3.** *If there exists an odd mapping $f : X \rightarrow\ Z_\sigma$ with $f(0) = 0$ and*

$$\sigma\left(D_{f(x,y)}\right) \leq \theta\left(\|x\|^r+\|y\|^r\right),$$

*for all $x, y \in X$, and $\theta > 0$ and $0 < r < 1$, then there exists a unique additive mapping $Q : X \rightarrow\ Z_\sigma$ satisfying*

$$\sigma\left(f(x)-Q(x)\right) \leq \frac{2\theta}{2-2^r}\|x\|^r,$$

*for all $x \in X$.*

**Theorem 2.4.** *Suppose that $Z$ is a linear space and $Z_\sigma$ satisfies the $\triangle_2$-condition with constant $k \geq 2$ and the mapping $\Psi : X^2 \rightarrow [0, \infty)$ for which there exists a mapping $f : X \rightarrow\ Z_\sigma$ such that*

$$\sigma\left(D_{f(x,y)}\right) \leq \Psi(x, y),$$

*and $\lim_{n\rightarrow\infty} k^n \Psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$ and $\sum_{j=1}^{\infty}\left(\frac{k^2}{2}\right)^j \Psi\left(\frac{x}{2^n}, 0\right) < \infty$, for all $x, y \in X$. Then, there exists a unique additive mapping $Q : X \rightarrow\ Z_\sigma$ defined by*

$$Q(x) = \lim_{n\rightarrow\infty} 2^n f\left(\frac{x}{2^n}\right)$$

*and*

$$\sigma\left(f(x)-Q(x)\right) \leq \frac{1}{2k}\sum_{j=1}^{\infty}\left(\frac{k^2}{2}\right)^j \Psi\left(\frac{x}{2^n}, 0\right),$$

*for all $x \in X$.*

**Proof.** $\sigma$ satisfies the $\triangle_2$-condition with $\eta$, therefore, the inequality (2.3) implies that

$$\sigma\left(D_{f(x,y)}\right) \leq \eta\Psi(x, y),$$

for all $x, y \in X$. Then the required result follows from the proof of Theorem 2.1. □

## 3 Stability results for the even case

In this section, we investigate the Ulam stability of the quartic functional in modular spaces $Z_\sigma$, without using the Fatou property.

**Theorem 3.1.** *Suppose that $Z$ is a linear space and $Z_\sigma$ satisfies the $\triangle_2$-condition with a mapping $\Psi : X^2 \to [0,\infty)$ for which there exists an even mapping $f : X \to Z_\sigma$ such that*

$$\sigma\left(D_{f(x,y)}\right) \leq \Psi(x,y), \tag{3.1}$$

*and*
$\lim_{n\to\infty} k^{4n}\Psi\left(\frac{x}{2^n},\frac{y}{2^n}\right) = 0$ *and* $\sum_{j=1}^{\infty}\left(\frac{k^5}{2}\right)^j\Psi\left(\frac{x}{2^n},0\right) < \infty$,
*for all $x,y\in X$. Then, there exists a unique quartic mapping $Q : X \to Z_\sigma$ defined by*

$$Q(x) = \lim_{n\to\infty} 2^{4n} f\left(\frac{x}{2^n}\right)$$

*and*

$$\sigma\left(f(x) - Q(x)\right) \leq \frac{1}{2k}\sum_{j=1}^{\infty}\left(\frac{k^5}{2}\right)^j\Psi\left(\frac{x}{2^j},0\right), \tag{3.2}$$

*for all $x\in X$.*

**Proof.** Putting $x=y=0$ in (2.1), we obtain $f(0)=0$ so that $\Psi(0,0)=0$ along the convergence of

$$\sum_{j=1}^{\infty}\left(\frac{k^5}{2}\right)^j\Psi(0,0) < \infty.$$

Putting $y=0$ in (3.1), we get

$$\sigma\left(f(2x) - 16f(x)\right) \leq \Psi(x,0),$$

for all $x\in X$. Because $\sum_{j=1}^{\infty}\frac{1}{2^j}\leq 1$, by the $\triangle_2$-condition of $\sigma$, we obtain the next functional inequality

$$\begin{aligned}\sigma\left(f(x) - 2^{4n}f\left(\frac{x}{2^n}\right)\right) &\leq \sigma\left(\sum_{j=1}^{n}\frac{1}{2^j}\left(2^{5j-4}f\left(\frac{x}{2^{j-1}}\right)\right) - 2^{5j}f\left(\frac{x}{2^j}\right)\right)\\ &\leq \frac{1}{k^4}\sum_{j=1}^{\infty}\left(\frac{k^5}{2}\right)^j\Psi\left(\frac{x}{2^j},0\right),\end{aligned} \tag{3.3}$$

for all $x\in X$. Now, replacing $x\to 2^{-m}x$ in (3.3), we obtain the result that the series of (3.1) converges, and

$$\begin{aligned}\sigma\left(2^{4m}f\left(\frac{x}{2^m}\right) - 2^{4(m+n)}f\left(\frac{x}{2^{m+n}}\right)\right) &\leq k^{4m}\sigma\left(f\left(\frac{x}{2^m}\right) - 2^{4n}f\left(\frac{x}{2^{m+n}}\right)\right)\\ &\leq k^{4m-4}\sum_{j=1}^{n}\left(\frac{k^5}{2}\right)^j\Psi\left(\frac{x}{2^{j+m}},\frac{x}{2^{j+m}}\right)\\ &\leq \frac{2^m}{k^{m+4}}\sum_{j=m+1}^{m+n}\left(\frac{k^5}{2}\right)^j\Psi\left(\frac{x}{2^j},0\right),\end{aligned}$$

for all $x\in X$, which tends to 0 as $m$ tends to $\infty$ because $\frac{2}{k}\leq 1$. The space $Z_\sigma$ is $\sigma$-complete, for all $x\in X$ the sequence $\left\{2^{4n}f\left(\frac{x}{2^n}\right)\right\}$ is a $\sigma$-Cauchy sequence, and it is $\sigma$-convergent in $Z_\sigma$. Then we define a mapping $Q : X\to Z_\sigma$ as

$$\sigma\left(\lim_{n\to\infty}2^{4n}f\left(\frac{x}{2^n}\right)\right) = Q(x),$$

i.e, $\lim_{n\to\infty}\sigma\left(2^{4n}f\left(\frac{x}{2^n}\right) - Q(x)\right) = 0$, for all $x\in X$. Therefore, we get the inequality without using the Fatou property from the $\triangle_2$-condition that

$$\begin{aligned}\sigma\left(f(x) - Q(x)\right) &\leq \frac{1}{2}\sigma\left(2f(x) - 2\left(2^{4n}\right)f\left(\frac{x}{2^n}\right)\right) + \frac{1}{2}\sigma\left(2\left(2^{4n}\right)f\left(\frac{x}{2^n}\right) - 2Q(x)\right)\\ &\leq \frac{k}{2}\sigma\left(f(x) - 2^{4n}f\left(\frac{x}{2^n}\right)\right) + \frac{k}{2}\sigma\left(2^{4n}f\left(\frac{x}{2^n}\right) - Q(x)\right)\\ &\leq \frac{1}{2k^3}\sum_{j=1}^{n}\left(\frac{k^5}{2}\right)^j\Psi\left(\frac{x}{2^j},0\right) + \frac{k}{2}\sigma\left(2^{4n}f\left(\frac{x}{2^n}\right) - Q(x)\right),\end{aligned}$$

holds for all $x \in X$ and all natural numbers $n > 1$. Taking $n \to \infty$, we obtain the inequality (3.2) of $X$ as $Q$. Replacing $x$ by $2^{-n}x$ and $y$ by $2^{-n}y$ in (3.1), we obtain

$$2^{4n} D_{f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} \leq k^{4n} \ \Psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \to 0 \ as \ n \to \infty.$$

Therefore, $Q$ is quartic as $n \to \infty$. Now, we show that the mapping $Q$ is unique. To show the uniqueness of $Q$, we consider another quartic mapping $R \ : X \to \ Z_\sigma$ defined by

$$\sigma\left(Q\left(x\right) - R\left(x\right)\right) \leq \frac{1}{2k} \sum_{j=1}^{n} \left(\frac{k^5}{2}\right)^j \Psi\left(\frac{x}{2^j}, 0\right),$$

for all $x \in X$. Then from the equality $Q\left(2^{-n}x\right) = 2^{-4n}Q(x)$ and $R\left(2^{-n}x\right) = 2^{-4n}R(x)$, we get

$$\begin{aligned}
\sigma\left(Q\left(x\right) - R\left(x\right)\right) &\leq \frac{1}{2}\sigma\left(2\left(2^{4n}\right)Q\left(\frac{x}{2^n}\right) - 2\left(2^{4n}\right)f\left(\frac{x}{2^n}\right)\right) + \frac{1}{2}\sigma\left(2\left(2^{4n}\right)f\left(\frac{x}{2^n}\right) - 2\left(2^{4n}\right)R\left(\frac{x}{2^n}\right)\right) \\
&\leq \frac{k^{4n+1}}{2}\sigma\left(Q\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right) + \frac{k^{4n+1}}{2}\sigma\left(f\left(\frac{x}{2^n}\right) - R\left(\frac{x}{2^n}\right)\right) \\
&\leq \frac{k^{4n}}{2} \sum_{j=1}^{n} \left(\frac{k^5}{2}\right)^j \Psi\left(\frac{x}{2^{n+j}}, 0\right) \\
&\leq \frac{2^{n-1}}{k^{n+2}} \sum_{j=1}^{n} \left(\frac{k^5}{2}\right)^j \Psi\left(\frac{x}{2^j}, 0\right),
\end{aligned}$$

for all $x \in X$ and all sufficiently large natural numbers $n \to \infty$. Thus we get required result. □

**Corollary 3.2.** *Suppose that $Z_\sigma$ satisfies $\triangle_2$-condition and let $X$ be a normed space with a norm $\|.\|$. If there is a real number $\theta > 0$ and $r > \log_2 \frac{k^5}{2}$ and an even mapping $f \ : X \to \ Z_\sigma$ with $f\left(0\right) = 0$ and*

$$\sigma\left(D_{f(x,y)}\right) \leq \theta\left(\|x\|^r + \|y\|^r\right),$$

*for all $x, y \in X$, then there exists a unique quartic mapping $Q \ : X \to \ Z_\sigma$ satisfying*

$$\left(f\left(x\right) - Q\left(x\right)\right) \leq \frac{k^4\theta}{2^{r+1} - k^5},$$

*for all $x \in X$.*

The following theorem gives an alternative stability of Theorem 3.1 in modular spaces without using the Fatou property or the $\triangle_2$-condition.

**Theorem 3.3.** *Let there exists a mapping $f \ : X \to \ Z_\sigma$ that satisfies (3.1), and suppose a mapping $\Psi : X^2 \to [0, \infty)$ exists such that*

$$\lim_{n\to\infty} \frac{\Psi\left(2^n x, 2^n y\right)}{2^{4n}} = 0, \ and \ \sum_{j=1}^{\infty} \frac{\Psi\left(2^j x, 0\right)}{2^{4j}} < \infty,$$

*for all $x, y \in X$. Then, there exists a unique quartic mapping $Q : X \to \ Z_\sigma$ defined by*

$$\sigma\left(f\left(x\right) - Q\left(x\right)\right) \leq \frac{1}{2^4} \sum \frac{\Psi\left(2^j x, 0\right)}{2^{4j}}, \tag{3.4}$$

*for all $x \in X$.*

**Proof.** Putting $y = 0$ in (3.1), we get

$$\sigma\left(f\left(2x\right) - 16f\left(x\right)\right) \leq \ \Psi\left(x, 0\right).$$

By the convexity of $\sigma$ and since $\sum \frac{1}{2^{4(j+1)}} \leq 1$.

$$\left(f(x) - \frac{f(2^n x)}{2^{4n}}\right) \leq \sigma\left(\sum_{0\leq j\leq n-1} \frac{2^4 f(2^j x) - f(2^{j+1}x)}{2^{4(1+j)}}\right)$$
$$\leq \sum_{0\leq j\leq n-1} \sigma\left(\frac{2^4 f(2^j x) - f(2^{j+1}x)}{2^{4(1+j)}}\right)$$
$$\leq \frac{1}{2^4}\sum_{0\leq j\leq n-1} \frac{\Psi(2^j x, 0)}{2^{4j}},$$

for all $x \in X$ and all $n \in \mathbb{N}$. Then, one has a $\sigma$-Cauchy sequence $\left\{\frac{f(2^n x)}{2^{4n}}\right\}$ and the mapping $Q : X \to Z_\sigma$ defined as

$$\left(\lim_{n\to\infty} \frac{f(2^n x)}{2^{4n}}\right) = Q(x)$$
$$i.e, \quad \lim_{n\to\infty} \sigma\left(\frac{f(2^n x)}{2^{4n}} - Q(x)\right) = 0,$$

for all $x \in X$, without using the $\triangle_2$-condition and the Fatou property. Moreover, it is obvious that the function $Q$ satisfies the quartic functional equation in the proof that follows using the ideas from Theorem 3.1.
Now, we show that (3.4) holds with fulfilments of the Fatou property and $\triangle_2$-condition. By using the convexity of $\sigma$ and

$$\sum_{0\leq j\leq n-1} \frac{1}{2^{4(j+1)}} + \frac{1}{2^4} \leq 1,$$

we get the next inequality

$$\sigma(f(x) - Q(x)) = \sigma\left(\left(\sum_{j=0}^{n-1} \frac{2^4 f(2^j x) - f(2^{j+1}x)}{2^{4(1+j)}}\right) + \left(\frac{f(2^n x)}{2^{4n}} - \frac{Q(2x)}{2^4}\right)\right)$$
$$\leq \sum_{j=0}^{n-1} \frac{1}{2^{4(1+j)}}\sigma\left(2^4 f\left(2^j x\right) - f\left(2^{j+1}x\right)\right) + \frac{1}{2^4}\sigma\left(\frac{f(2^{n-1}2x)}{2^{4(n-1)}} - Q(2x)\right)$$
$$\leq \frac{1}{2^4}\sum_{j=0}^{n-1} \frac{1}{2^{4j}}\Psi\left(2^j x, 0\right) + \frac{1}{2^4}\left(\frac{f(2^{n-1}2x)}{2^{4(n-1)}} - Q(2x)\right),$$

for all $x \in X$ and all natural numbers $n > 1$. Taking the limit as $n \to \infty$, we obtain required result. □

**Corollary 3.4.** *Let there exist a mapping $\Psi : X^2 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{\Psi(2^n x, 2^n y)}{2^{4n}} = 0, \text{ and } \Psi(2x, 0) \leq 2^4 L\ \Psi(x, 0),$$

*for all $x, y \in X$ and for some $L \in (0, 1)$. If there exists a mapping $f : X \to Z_\sigma$ that satisfies (3.1), then, there exists a unique quartic mapping $Q : X \to Z_\sigma$ defined by*

$$\sigma(f(x) - Q(x)) \leq \frac{1}{2^4(1-L)}\Psi(x, 0),$$

*for all $x \in X$.*

**Corollary 3.5.** *Suppose that $X$ be a normed space with a norm $\|.\|$. If there is a real number $\theta > 0$, $\varepsilon > 0$ and $r \in (-\infty, 2)$ and an even mapping $f : X \to Z_\sigma$ with $f(0) = 0$ such that*

$$\sigma\left(D_{f(x,y)}\right) \leq \theta\left(\|x\|^r + \|y\|^r\right) + \varepsilon,$$

*for all $x, y \in X$, then there exists a unique quartic mapping $Q : X \to Z_\sigma$ satisfying*

$$\sigma(f(x) - Q(x)) \leq \frac{\theta}{2^4 - 2^r}\|x\|^r + \frac{\varepsilon}{3}$$

*for all $x \in X$, where $x \neq 0$ if $r < 0$.*

## 4 Stability results in 2-Banach spaces

The concept of linear 2-Banach spaces was developed by Gahler ( [4, 5]) in the 1960's.

**Definition 4.1.** Let $X$ over $\mathbb{R}$ be a linear space with dim$X > 1$ and a mapping $\|.,.\| : X^2 \to \mathbb{R}$ such that

**4.1(i)** $\|s, u\| = 0$ if and only if $s$ and $u$ are linearly dependent.

**4.1(ii)** $\|s, u\| = \|u, s\|$,

**4.1(iii)** $\|\eta s, u\| = |\eta| \, \|s, u\|$,

**4.1(iv)** $\|s, u + v\| \leq \|s, u\| + \|s, v\|$,

for all $s, u, v \in X$ and $\eta \in \mathbb{R}$. Then, the mapping $\|.,.\|$ is defined as 2-norm on $X$ and the pair $(X, \|.,.\|)$ is called a linear 2-normed space.

**Definition 4.2.** If there exists $u, v$ in a linear 2-normed space such that $X$ satisfies the condition that $u$ and $v$ are linearly independent, then the sequence $\{s_j\}$ in $X$ is known as a Cauchy sequence.

$$\text{i.e., } \lim_{i,j\to\infty} \|s_i - s_j, u\| = 0$$

and $\lim_{i,j\to\infty} \|s_i - s_j, v\| = 0$ .

**Definition 4.3.** A sequence $\{s_j\}$ in 2-normed space $X$ is called as a convergent if there exists an element $s \in X$ such that

$$\lim_{i,j\to\infty} \|s_i - s_j, u\| = 0 \ ,$$

for all $u \in X$.

If $\{s_j\}$ converges to $s$, then we denote that $s_j \to s$ as $j \to \infty$, and we call that $s$ is the limit point of $\{s_j\}$.

$$\text{i.e., } \lim_{j\to\infty} s_j = s \ .$$

**Definition 4.4.** Every Cauchy sequence is convergent in a 2-Banach space, which is linear 2-normed space.

**Lemma 4.5.** *[14] Let $(X, \|.,.\|)$ be a linear 2-normed space. If $s \in X$ and $\|s, t\| = 0$, for all $t \in X$, then $s = 0$.*

**Lemma 4.6.** *[14] For a convergent sequence $\{s_j\}$ in a linear 2-normed space $X$,*

$$\lim_{j\to\infty} \|s_j, u\| = \left\| \lim_{j\to\infty} s_j, u \right\| ,$$

*for all $u \in X$.*

W.G Park [14] studied approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in 2-Banach. In [12], Choonkil Park examined the superstability of the Cauchy functional inequality and the Cauchy-Jensen functional inequality in 2-Banach spaces under certain conditions.

In this section, we assume $X$ to be a normed space and $Z$ to be a 2-Banach space.

**Theorem 4.7.** *Let $\Psi : \ X^2 \times \mathrm{Z} \to [0, +\infty)$ exists such that*

$$\lim_{m\to\infty} \frac{1}{2^m} \Psi\left(2^m x, 2^m y, z\right) = 0 \ , \tag{4.1}$$

*for all $x, y \in X$, $z \in Z$. If there exists a mapping $f : X \to Z$ with $f(0) = 0$ such that*

$$\left\| D_{f(x,y)}, z \right\| \leq \Psi\left(x, y, z\right), \tag{4.2}$$

*and*

$$\psi(x,z)=\sum_{n=1}^{\infty}\frac{1}{2^n}\Psi\left(2^n x,2^n y,z\ \right)<\infty, \tag{4.3}$$

*for all $x,y\in X,\ z\in Z$, then there exists a unique additive mapping $A:X\to Z$ satisfying*

$$\left\|f(x)-A(x),z\right\|\leq\ \psi(x,z), \tag{4.4}$$

*for all $x,y\in X,\ z\in Z$.*

**Proof.** Putting $y=0$ in (4.2), we obtain

$$\left\|f(2x)-2f(x),z\right\|\leq\Psi(x,0,z), \tag{4.5}$$

for all $x,y\in X$ and all $z\in Z$. Interchanging $x$ into $2^n x$ in (4.5), we obtain

$$\left\|\frac{1}{2^{n+1}}f\left(2^{n+1}x\right)-\frac{1}{2^n}f\left(2^n x\right),z\right\|\leq\frac{1}{2^{n+1}}\Psi\left(2^n x,0,z\right), \tag{4.6}$$

for all $x,y\in X$ and all $z\in Z$ and $n>0$. Thus,

$$\begin{aligned}\left\|\frac{1}{2^{n+1}}f\left(2^{n+1}x\right)-\frac{1}{2^l}f\left(2^l x\right),z\right\|&\leq\sum_{m=l}^{n}\left\|\frac{1}{2^{m+1}}f\left(2^{m+1}x\right)-\frac{1}{2^m}f\left(2^m x\right),z\right\|\\&\leq\sum_{m=l}^{n}\frac{1}{2^m}\Psi\left(2^m x,0,z\right),\end{aligned} \tag{4.7}$$

for all $x\in X$ and all $z\in Z$ and $n>0$ and $l>0$ with $l\leq n$.
Thus, from (4.2) and (4.7), we conclude that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence in $Z$, for all $x\in X$. We know that $Z$ is complete, which implies that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ converges in $Z$ for all $x\in X$. Therefore, we define a mapping $A\ :X\to Z$ by

$$A(x)=\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x\right)\ , \tag{4.8}$$

for all $x\in X$. Therefore,

$$\lim_{n\to\infty}\left\|\frac{1}{2^n}f\left(2^n x\right)-A(x),z\right\|=0\ ,$$

for all $x\in X$ and all $z\in Z$. Taking the limit $n\to\infty$ and put $l=0$ in (4.7), we get (4.4). Now, we want to show that the mapping $A$ is additive. By inequalities (4.1), (4.2), (4.8) and Lemma 4.6 we get

$$\begin{aligned}\left\|D_{f(x,y)},z\right\|&=\lim_{n\to\infty}\left\|D_{f(2^n x,2^n y)},z\right\|\\&\leq\lim_{n\to\infty}\frac{1}{2^n}\Psi\left(2^n x,2^n y,z\ \right)=0\ .\end{aligned}$$

By Lemma 4.5 , we obtain $D_{A(x,y)}=0$, for all $x,y\in X$. Thus, the mapping $A$ is additive. To prove the uniqueness of the function $A$, we consider another additive mapping $A':X\to Z$ satisfying (4.4). Then,

$$\begin{aligned}\left\|A(x)-A'(x),z\right\|&=\lim_{n\to\infty}\frac{1}{2^n}\left\|A\left(2^n x\right)-f\left(2^n x\right)+f\left(2^n x\right)-A'\left(2^n x\right),z\right\|\\&\leq\lim_{n\to\infty}\frac{1}{2^n}\psi\left(2^n x,z\right)=0\ ,\end{aligned}$$

for all $x\in X$ and all $z\in Z$. By Lemma 4.5, $A(x)-A'(x)=0$, for all $x\in X$, which implies that $A(x)=A'(x)$. □

**Remark 4.8.** Theorem 4.7 can formulated, in which the sequence

$$A(x)=\lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right)$$

is defined with appropriate assumptions for $\Psi$.

**Corollary 4.9.** *Let there exist a mapping $\eta : [0, \infty) \to [0, \infty)$ such that $\eta(0) = 0$ and*

**4.9(i)** $\eta(rz) \leq \eta(r)\ \eta(z)$,

**4.9(ii)** $\eta(rz) < r$ *for all* $r > 1$.

*If a mapping $f : X \to Z$ exists with $f(0) = 0$ and*

$$\left\|D_{f(x,y)}, z\right\| \leq \eta(\|x\| + \|y\|) + \eta\|z\|, \tag{4.9}$$

*for all $x, y \in X$ and all $z \in Z$, then there exists a unique additive mapping $A : X \to Z$ satisfying*

$$\|f(x) - A(x), z\| \leq \left[\frac{(2\eta\|x\|)}{2 - \eta(2)} - \eta\|z\|\right], \tag{4.10}$$

*for all $x \in X$ and all $z \in Z$.*

**Proof.** Let

$$\Psi(x, y, z) = \eta(\|x\| + \|y\|) + \eta\|z\|,$$

for all $x, y \in X$ and all $z \in Z$, from the condition 5.3(i) above we obtain

$$\eta(2^n) \leq (\eta(2))^n$$

and

$$\Psi(2^n x, 2^n y, z) = (\eta(2))^n (\eta(\|x\| + \|y\|) + \eta\|z\|).$$

By using Theorem 4.7, we get the required result. □

**Corollary 4.10.** *Let there exists a $u \in \mathbb{R}^+$ with $u < 1$ and a homogeneous $\Omega : [0, \infty) \times [0, \infty) \to [0, \infty)$ with degree $u$. If there exists a mapping $f : X \to Z$ with $f(0) = 0$ and*

$$\left\|D_{f(x,y)}, z\right\| \leq \Omega(\|x\|, \|y\|) + \|z\|,$$

*for all $x, y \in X$ and all $z \in Z$, then there exists a unique additive mapping $A : X \to Z$ satisfying*

$$\|f(x) - A(x), z\| \leq \frac{\Omega(\|x\|, \|y\|) + \|z\|}{2 - u}, \tag{4.11}$$

*for all $x \in X$ and all $z \in Z$.*

**Corollary 4.11.** *Let there exist a homogeneous $\Omega : [0, \infty) \times [0, \infty) \to [0, \infty)$ with degree $u$. If there exists a mapping $f : X \to Z$ with $f(0) = 0$ and*

$$\left\|D_{f(x,y)}, z\right\| \leq \Omega(\|x\|, \|y\|)\|z\|,$$

*for all $x, y \in X$ and all $z \in Z$, then there exists a unique additive mapping $A : X \to Z$ satisfying*

$$\|f(x) - A(x), z\| \leq \frac{\Omega(\|x\|, 0)\|z\|}{2 - 2^t}, \tag{4.12}$$

*for all $x \in X$, $z \in Z$ and $t \in \mathbb{R}^+$ with $t < 1$.*

**Corollary 4.12.** *If there exists a mapping $f : X \to Z$ with $f(0) = 0$ and*

$$\left\|D_{f(x,y)}, z\right\| \leq \|x\|^s + \|y\|^s + \|z\|,$$

*for all $x, y \in X$ and all $z \in Z$, then there exists a unique additive mapping $A : X \to Z$ satisfying*

$$\|f(x) - A(x), z\| \leq \frac{\Omega(\|x\|^s) + \|z\|}{2 - t},$$

*for all $x \in X$,$z \in Z$and $t \in \mathbb{R}^+$ with $t < 1$.*

**Theorem 4.13.** *Let* $\Psi : X^2 \times Z \to [0, +\infty)$ *exists such that*

$$\lim_{m\to\infty} \frac{1}{2^{4m}} \Psi\left(2^n x, 2^n y, z\right) = 0 \ , \tag{4.13}$$

*for all* $x, y \in X,\ z \in Z$. *If there exists an even mapping* $f : X \to Z$ *with* $f(0) = 0$ *such that*

$$\left\|D_{f(x,y)}, z\right\| \le \Psi\left(x, y, z\right), \tag{4.14}$$

*and*

$$\psi\left(x, z\right) = \sum_{n=1}^{\infty} \frac{1}{2^{4n}} \Psi\left(2^n x, 0, z\ \right) < \infty, \tag{4.15}$$

*for all* $x, y \in X,\ z \in Z$, *then there exists a unique additive mapping* $Q : X \to Z$ *satisfying*

$$\left\|f\left(x\right) - Q\left(x\right), z\right\| \le \ \psi\left(x, z\right), \tag{4.16}$$

*for all* $x, y \in X,\ z \in Z$.

**Proof.** Putting $y = 0$ in (4.14), we obtain

$$\left\|f\left(2x\right) - 16 f\left(x\right), z\right\| \le \Psi\left(x, 0, z\right)$$

$$\left\|\frac{f\left(2x\right)}{2^4} - f\left(x\right), z\right\| \le \frac{1}{2^4} \Psi\left(x, 0, z\right), \tag{4.17}$$

for all $x, y \in X$ and all $z \in Z$. Interchanging $x$ into $2^n x$ in (4.17), we obtain

$$\left\|\frac{1}{2^{4(n+1)}} f\left(2^{n+1} x\right) - \frac{1}{2^{4n}} f\left(2^n x\right), z\right\| \le \frac{1}{2^{4(n+1)}} \Psi\left(2^n x, 0, z\right), \tag{4.18}$$

for all $x, y \in X$ and all $z \in Z$ and $n > 0$. Thus,

$$\left\|\frac{1}{2^{4(n+1)}} f\left(2^{n+1} x\right) - \frac{1}{2^{4l}} f\left(2^l x\right), z\right\| \le \sum_{m=l}^{n} \left\|\frac{1}{2^{4(m+1)}} f\left(2^{m+1} x\right) - \frac{1}{2^{4m}} f\left(2^m x\right), z\right\|$$

$$\le \sum_{m=l}^{n} \frac{1}{2^{4m}} \Psi\left(2^m x, 0, z\right), \tag{4.19}$$

for all $x \in X$ and all $z \in Z$ and $n > 0$ and $l > 0$ with $l \le n$.
Thus, from (4.14) and (4.19), we conclude that the sequence $\left\{\frac{f(2^n x)}{2^{4n}}\right\}$ is a Cauchy sequence in $Z$, for all $x \in X$. We know that $Z$ is complete, which implies that the sequence $\left\{\frac{f(2^n x)}{2^{4n}}\right\}$ converges in $Z$ for all $x \in X$. Therefore, we define a mapping $Q\ : X \to Z$ by

$$Q\left(x\right) = \lim_{n\to\infty} \frac{1}{2^{4n}} f\left(2^n x\right)\ , \tag{4.20}$$

for all $x \in X$. Therefore,

$$\lim_{n\to\infty} \left\|\frac{1}{2^{4n}} f\left(2^n x\right) - Q\left(x\right), z\right\| = 0\ ,$$

for all $x \in X$ and all $z \in Z$. Taking the limit $n \to \infty$ and put $l = 0$ in (4.18), we get (4.16). Now, we want to show that the mapping $Q$ is quartic. By inequalities (4.13), (4.14), (4.20) and Lemma 4.6 we get

$$\left\|D_{f(x,y)}, z\right\| = \lim_{n\to\infty} \left\|D_{f(2^n x, 2^n y)}, z\right\|$$

$$\le \lim_{n\to\infty} \frac{1}{2^{4n}} \Psi\left(2^n x, 2^n y, z\ \right) = 0\ .$$

By Lemma 4.5, we obtain $D_{Q(x,y)} = 0$, for all $x, y \in X$. Thus, the mapping $Q$ is quartic. To prove the uniqueness of the mapping $Q$, we consider another quartic mapping $Q' : X \to Z$ satisfying (4.16). Then,

$$\left\|Q\left(x\right) - Q'\left(x\right), z\right\| = \lim_{n\to\infty} \frac{1}{2^{4n}} \left\|Q\left(2^n x\right) - f\left(2^n x\right) + f\left(2^n x\right) - Q'\left(2^n x\right), z\right\|$$

$$\leq \lim_{n\to\infty} \frac{1}{2^{4n}} \psi\left(2^n x, z\right) = 0 \ ,$$

for all $x \in X$ and all $z \in Z$. By Lemma 4.5, $Q(x) - Q'(x) = 0$, for all $x \in X$, which implies that $Q(x) = Q'(x)$. □

**Corollary 4.14.** *Let there exists a mapping $\eta : [0,\infty) \to [0,\infty)$ such that $\eta(0) = 0$ and*

**4.14(i)** $\eta(rz) \leq \eta(r)\ \ \eta(z)$,

**4.14(ii)** $\eta(rz) < r$ *for all* $r > 1$.

*If an even mapping $f\ : X \to Z$ exists with $f(0) = 0$ and*

$$\left\|D_{f(x,y)}, z\right\| \leq \eta\left(\|x\| + \|y\|\right) + \eta\left\|z\right\|, \tag{4.21}$$

*for all $x, y \in X$ and all $z \in Z$, then there exists a unique quartic mapping $Q : X \to Z$ satisfying*

$$\|f(x) - Q(x), z\| \leq \left[\frac{(2\eta\|x\|)}{2 - \eta(2)} - \eta\|z\|\right], \tag{4.22}$$

*for all $x \in X$ and all $z \in Z$.*

**Proof.** Let

$$\Psi(x, y, z) = \eta\left(\|x\| + \|y\|\right) + \eta\left\|z\right\|,$$

for all $x, y \in X$ and all $z \in Z$, from the condition 5.8(i) above we obtain

$$\eta\left(2^n\right) \leq \left(\eta(2)\right)^{4n}$$

and

$$\Psi\left(2^n x, 2^n y, z\right) = \left(\eta(2)\right)^{4n}\left(\eta\left(\|x\| + \|y\|\right) + \eta\left\|z\right\|\right).$$

By using Theorem 4.13, we get the required result. □

**Corollary 4.15.** *Let there exist a $u \in \mathbb{R}^+$ with $u < 1$ and a homogeneous $\Omega : [0,\infty)\times[0,\infty) \to [0,\infty)$ with degree $u$. If there exists an even mapping $f : X \to Z$ with $f(0) = 0$ and*

$$\left\|D_{f(x,y)}, z\right\| \leq \Omega\left(\|x\|, \|y\|\right) + \|z\|,$$

*for all $x, y \in X$ and all $z \in Z$, then there exists a unique quartic mapping $Q : X \to Z$ satisfying*

$$\|f(x) - Q(x), z\| \leq \frac{\Omega\left(\|x\|, \|y\|\right) + \|z\|}{2 - u}, \tag{4.23}$$

*for all $x \in X$ and all $z \in Z$.*

**Proof.** Let

$$\Psi(x, y, z) = \eta\left(\|x\|, \|y\|\right) + \|z\|,$$

for all $x, y \in X$ and all $z \in Z$, then from Theorem 4.13 we obtain the result (4.23). □

**Corollary 4.16.** *Let there exist a homogeneous $\Omega : [0,\infty)\times[0,\infty) \to [0,\infty)$ of degree $u$. If there exists an even mapping $f : X \to Z$ with $f(0) = 0$ and*

$$\left\|D_{f(x,y)}, z\right\| \leq \Omega\left(\|x\|, \|y\|\right)\|z\|,$$

*for all $x, y \in X$ and all $z \in Z$, then there exists a unique quartic mapping $Q : X \to Z$ satisfying*

$$\|f(x) - Q(x), z\| \leq \frac{\Omega\left(\|x\|, 0\right)\|z\|}{2 - 2^t}, \tag{4.24}$$

*for all $x \in X$,$z \in Z$, and $t \in \mathbb{R}^+$ with $t < 1$.*

**Corollary 4.17.** *If there exists an even mapping $f : X \to Z$ with $f(0) = 0$ and*

$$\left\| D_{f(x,y)}, z \right\| \leq \|x\|^s + \|y\|^s + \|z\|,$$

*for all $x, y \in X$ and all $z \in Z$, then there exists a unique quartic mapping $Q : X \to Z$ satisfying*

$$\|f(x) - Q(x), z\| \leq \frac{\Omega(\|x\|^s) + \|z\|}{2 - t},$$

*for all $x \in X$,$z \in Z$ and $t \in \mathbb{R}^+$ with $t < 1$.*

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