Original Article

Bulletin of Pure and Applied Sciences.
Vol. 37E (Math & Stat.), No.1, 2018.P.178-183
Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2018.00018.8

Some Fixed Point Theorems for Mappings on Complete S_b –Metric Space

Manju Rani[1,*], Nawneet Hooda[2]

Author Affiliation:
[1,2] Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonipat, Haryana, India-131039.

***Corresponding Author:**
Manju Rani, Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonipat, Haryana, India-131039.

E-mail: manjuantil7@gmail.com

Received on 20.04.2018, Accepted on 18.05.2018

Abstract

In this paper, we prove some common fixed point theorems in S_b-metric space using an increasing function $\phi : [0,\infty) \to [0,\infty)$ *with* $\lim_{n\to\infty} \phi^n(t) = 0$ *and* $\phi(t) < t$ *for each fixed* $t > 0$. Our results extend the result of Savitri & Hooda [8] of S-metric space.

1. INTRODUCTION

Mustafa and Sims [10] introduced the notion of G-metric spaces.

Definition 1.1: Let X be a non-empty set and $G : X \times X \times X \to R^+$ be a function satisfying the following conditions:

1. $G(x,y,z) = 0 \; if \; x = y = z,$
2. $0 < G(x,x,y), for \; all \; x,y \in X \; and \; x \neq y,$
3. $G(x,x,y) \leq G(x,y,z), for \; all \; x,y,z \in X \; and \; z \neq y,$
4. $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots\cdots$ (symmetry in all three variables)
5. $G(x,y,z) \leq G(x,a,a) + G(a,y,z), for \; all \; x,y \leq z,a \in X.$ (rectangle inequality)

Then the function G is called a generalized metric or more specifically a G-metric on X and the pair (X,G) is called a G-metric space.

Sedghi et al. [5] introduced the concept of S-metric space by modifying G-metric space. The definition of S-metric space is as follows:

Definition 1.2: Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions, for each $x,y,z,a \in$ X,

1. $S(x,y,z) \geq 0$
2. $S(x,y,z) = 0$ *if and only if* $x = y = z$
3. $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$

The pair (X,S) is called S-metric space. The space (X,S) is a generalization of G-metric space.

Lemma 1.3: ([5]) In a S-metric space , we have

$$S(x,x,y) = S(y,y,x) \text{ for all } x,y \in X .$$

Sedghi and Dung [6] remarked that every S-metric space is topologically equivalent to a metric space.

Souayah and Mlaiki [2] introduced the concept of S_b-metric space as follows:

Definition 1.4: ([2]) Let X be a nonempty set. A function $S_b : X^3 \to [0,\infty)$ is said to be S_b–metric if and only if for all $x,y,z,t \in X$, the following conditions hold:

1. $S_b(x,y,z) = 0$ *if and only if* $x = y = z$,
2. $S_b(x,x,y) = S_b(y,y,x)$ *for all* $x,y \in X$,
3. $S_b(x,y,z) \leq s[S_b(x,x,t) + S_b(y,y,t) + S_b(z,z,t)]$, where $s \geq 1$ be a given number .

The pair (X,S_b) is called an S_b-metric space. See also ([7], Definition 1.7).

Shatanawi [9] proved the following theorems for the existence of fixed points in G-metric space:

Theorem 1.5: Let X be a complete G-metric space. Suppose the mappings $T : X \to X$ satisfies the condition:

$$G(Tx,Ty,Tz) \leq \phi\big(G(x,y,z)\big), \text{for all } x,y,z \in X.$$

Then T has a unique fixed point.

Theorem 1.6: Let X be a complete G-metric space. Suppose there is $k \in [0,1)$ such that the mappings $T : X \to X$ satisfies the condition:

$$G(Tx,Ty,Tz) \leq k\phi\big(G(x,y,z)\big), \text{for all } x,y,z \in X.$$

Then T has a unique fixed point.

Savitri [8] proved the following theorems for the existence of fixed points in S-metric space :

Theorem 1.7: Let X be a complete S-metric space. Suppose that the mapping $T : X \to X$ satisfies the condition:

$$S(Tx,Ty,Tz) \leq \phi\big(S(x,y,z)\big), \text{for all } x,y,z \in X,$$

where $\phi : [0,\infty) \to [0,\infty)$ *is an increasing function such that* $\lim_{n\to\infty} \phi^n(t) = 0$ *and* $\phi(t) < t$ *for each fixed* $t > 0$, then T has a unique fixed point .

Theorem 1.8: Let (X,S) be a complete S-metric space and T be a continuous self mapping on X satisfy the condition:

$$S(Tx,Ty,Tz) \leq \phi[\max\{S(x,y,z),\ S(Tx,Tx,x),\ S(Ty,Ty,y),\ S(Tz,Tz,x)\}] \text{for all } x,y,z \in X,$$

where $\phi : [0,\infty) \to [0,\infty)$ *is an increasing function such that* $\lim_{n\to\infty} \phi^n(t) = 0$ *and* $\phi(t) < t$ *for each fixed* $t > 0$, then T has a unique fixed point in X.

2. MAIN RESULTS

In this section, we prove Theorem 1.7 and Theorem 1.8 in S_b –metric space.

Theorem 2.1: Let X be a complete S_b-metric space. Suppose that the mapping $T : X \to X$ satisfies the condition:

$$S_b(Tx,Ty,Tz) \leq \phi\big(S_b(x,y,z)\big), \text{for all } x,y,z \in X, \tag{2.1}$$

where $\phi : [0,\infty) \to [0,\infty)$ *is an increasing function such that* $\lim_{n\to\infty} \phi^n(t) = 0$ *and* $\phi(t) < t$ *for each fixed* $t > 0$, then T has a unique fixed point .

Proof . For arbitrary point $x_0 \in$ X, construct a sequence $\{x_n\}$ *such that* $x_n = Tx_{n-1}$, n ∈ N.
Assume $x_n \neq x_{n-1}$, for each n ∈ N.

We claim that $\{x_n\}$ is a Cauchy sequence in X. For n ∈ N, we have

$$\begin{aligned} S_b(x_n, x_n, x_{n+1}) &= S_b(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\le \phi(S_b(x_{n-1}, x_{n-1}, x_n)) \quad (2.2) \\ &\vdots \\ &\vdots \\ &\le \phi^n(S_b(x_0, x_0, x_1)). \end{aligned}$$

Given $\varepsilon > 0$, since $\lim_{n\to\infty} \phi^n(S_b(x_0, x_0, x_1)) = 0$ and $\phi(\varepsilon) < \varepsilon$, there is an integer n_0 such that

$$\phi^n(S_b(x_0, x_0, x_1)) < \frac{\varepsilon}{2s} - \frac{\phi(\varepsilon)}{2}, \text{ for all } n \ge n_0$$

This implies

$$S_b(x_n, x_n, x_{n+1}) < \frac{\varepsilon}{2s} - \frac{\phi(\varepsilon)}{2}, \textit{for all } n \ge n_0 \quad (2.3)$$

For $m, n \in N$ *with* $m > n$,we claim that

$$S_b(x_n, x_n, x_{n+1}) < \varepsilon, \text{for all } m > n \ge n_0 . \quad (2.4)$$

We prove inequality (2.4) by induction on m .

Inequality (2.4) holds for $m = n + 1$ by using inequality (2.3) and the fact that $\frac{\varepsilon}{s} - \phi(\varepsilon) < \varepsilon$.
Assume inequality (2.4) holds for $m = k$.

For $m = k + 1$, we have

$$S_b(x_n, x_n, x_{k+1}) \le s[S_b(x_n, x_n, x_{n+1}) + S_b(x_n, x_n, x_{n+1}) + S_b(x_{k+1}, x_{k+1}, x_{n+1})]$$
$$= s[2S_b(x_n, x_n, x_{n+1}) + S_b(x_{k+1}, x_{k+1}, x_{n+1})] .$$

using equations (2.2), (2.3) and $[S_b(x, x, y) = S_b(y, y, x) \textit{ for all } x, y \in X]$, we get

$$\begin{aligned} S_b(x_n, x_n, x_{k+1}) &\le s\left[\frac{\varepsilon}{s} - \phi(\varepsilon) + \phi(S_b(x_k, x_k, x_n))\right] \\ &\le s\left[\frac{\varepsilon}{s} - \phi(\varepsilon) + \phi(S_b(x_n, x_n, x_k))\right] \\ &< s\left[\frac{\varepsilon}{s} - \phi(\varepsilon) + \phi(\varepsilon)\right] \\ &< s\left(\frac{\varepsilon}{s}\right) \\ &= \varepsilon. \end{aligned}$$

By induction on m , we conclude that inequality (2.4) holds for all $m > n \ge n_0$. So$\{x_n\}$ is Cauchy sequence in complete S_b-metric space and hence $\{x_n\}$ converges to some $w \in X$.

For n∈ N , we have

$$\begin{aligned} S_b(w, w, Tw) &\le s[S_b(w, w, x_{n+1}) + S_b(w, w, x_{n+1}) + S_b(Tw, Tw, x_{n+1})] \\ &\le s[2S_b(w, w, x_{n+1}) + \phi(S_b(w, w, x_n))] . \end{aligned}$$

Since $\phi(t) < t$, we have $S_b(w, w, Tw) \le s[2S_b(w, w, x_{n+1}) + S_b(w, w, x_n)]$.

Letting $n \to \infty$ and using the fact that S_b is continuous in its variables, we get that $S_b(w, w, Tw) = 0$.
Hence $T(w) = w$. So w is a fixed point of T. Now, Let v be another fixed point of T with $v \neq w$.
Since$\phi(t) < t$, we have

$$\begin{aligned} S_b(w, w, v) &= S_b(Tw, Tw, Tv) \\ &\le \phi(S_b(w, w, v)) \end{aligned}$$
$$< S_b(w, w, v) .$$

which is not possible . So $v = w$ and hence T has a unique fixed point .

Corollary 2.2: Let X be a complete S_b-metric space . Suppose that the mapping$T : X \to X$ satisfies the condition:

$$S_b(T^m(x), T^m(y), T^m(z)) \le \phi(S_b(x, y, z)), \textit{for all } x, y, z \in X \textit{ and } m \in N.$$

Then T has a unique fixed point .

Proof. From Theorem 2.1, we obtain that T^m has a unique fixed point say w.

Since $T^m(Tw) = T^{m+1}(w) = T(T^m w)$, we get Tw is also a fixed point of T^m. But w is a unique fixed point of T^m, so we have $Tw = w$.

Hence w is a unique fixed point of T.

Corollary 2.3: Let X be a complete S_b-metric space . Suppose that the mapping $T : X \to X$ satisfies the condition:

$$S_b(T(x), T(x), T(z)) \leq \phi(S_b(x, x, z)), for\ all\ x, z \in X .$$

Then T has a unique fixed point .

Proof. We obtain the result by taking $y = x$ in Theorem 2.1 .

Corollary 2.4: Let X be a complete S_b-metric space . Suppose that there is $k \in [0,1)$the mapping$T : X \to X$ satisfies the condition:

$$S_b(T(x), T(y), T(z)) \leq k(S_b(x, y, z)), for\ all\ x, y, z \in X .$$

Then T has a unique fixed point .

Proof. Define$\phi: [0, \infty) \to [0, \infty)\ by\ \phi(t) = kt$. Then clearly ψ is a non-decreasing function with $\lim_{n\to\infty} \phi^n(t) = 0, for\ all\ t > o$. Using given condition and by virtue of ϕ , we have

$$S_b(T(x), T(y), T(z)) \leq \phi(S_b(x, y, z)), for\ all\ x, y, z \in X .$$

Now the results follows from Theorem 2.1 .

Corollary 2.5: Let X be a complete S_b-metric space .Suppose that the mapping$T : X \to X$ satisfies the condition:

$$S_b(T(x), T(y), T(z)) \leq \frac{S_b(x, y, z)}{1 + S_b(x, y, z)}, for\ all\ x, y, z \in X .$$

Then T has a unique fixed point .

Proof. Define $\phi: [0, \infty) \to [0, \infty)\ by\ \phi(t) = \frac{w}{1+w}$. Then clearly ϕ is a non-decreasing function with $\lim_{n\to\infty} \phi^n(t) = 0, for\ all\ t > o$. Using given condition and by virtue of ϕ , we have

$$S_b(T(x), T(y), T(z)) \leq \phi(S_b(x, y, z)), for\ all\ x, y, z \in X .$$

Now the results follows from Theorem 2.1 .

Theorem 2.6: Let (X, S_b) be a complete S_b-metric space and T be a continuous self mapping on X satisfying the condition:

$$S_b(Tx, Ty, Tz) \leq \phi[max\{S_b(x, y, z), S_b(Tx, Tx, x), S_b(Ty, Ty, y), S_b(Tz, Tz, x)\}], \quad (2.5)$$

for all x, y, z $\in$ X, where $\phi : [0, \infty) \to [0, \infty)$ is an increasing function such that $\lim_{n\to\infty} \phi^n(t) = 0$ and $\phi(t) < t\ for\ each\ fixed\ t > 0$ then T has a unique fixed point .

Proof. For arbitrary point $x_0 \in$ X, construct a sequence$\{x_n\}\ such\ that\ x_n = Tx_{n-1}$, for all n $\in$ N.

Assume $x_n \neq x_{n-1}$, for each n $\in$ N.

Thus for n$\in$ N , we have$S_b(x_{n+1}, x_{n+1}, x_n) = S_b(Tx_n, Tx_n, Tx_{n-1})$

$$\leq \phi[max\{S_b(x_n, x_n, x_{n-1}), S_b(x_{n+1}, x_{n+1}, x_n), S_b(x_{n+1}, x_{n+1}, x_n), S_b(x_n, x_n, x_n)\}]$$

$$\leq \phi[max\{S_b(x_n, x_n, x_{n-1}), S_b(x_{n+1}, x_{n+1}, x_n)\}]$$

If $max\{S_b(x_n, x_n, x_{n-1}), S_b(x_{n+1}, x_{n+1}, x_n)\} = S_b(x_{n+1}, x_{n+1}, x_n)$ then

$$S_b(x_{n+1}, x_{n+1}, x_n) \leq \phi(S_b(x_{n+1}, x_{n+1}, x_n))$$

$< S_b(x_{n+1}, x_{n+1}, x_n)$,

which is impossible.

So $ax\{S_b(x_n, x_n, x_{n-1}), S_b(x_{n+1}, x_{n+1}, x_n)\} = S_b(x_n, x_n, x_{n-1})$.

Thus for n$\in$ N , we have

$$S_b(x_{n+1}, x_{n+1}, x_n) \leq \phi(S_b(x_n, x_n, x_{n-1}))$$

$$\leq \phi^2(S_b(x_{n-1}, x_{n-1}, x_{n-2}))$$

$$\vdots$$
$$\vdots$$
$$\leq \phi^n(S_b(x_1, x_1, x_0)).$$

This implies

$$S_b(x_{n+1}, x_{n+1}, x_n) \leq \phi^n(S_b(x_1, x_1, x_0)) .$$

using $[S_b(x,x,y) = S_b(y,y,x) \text{ for all } x, y \in X]$, we get

$$S_b(x_n, x_n, x_{n+1}) \leq \phi^n(S_b(x_0, x_0, x_1)) .$$

By similar arguments as in Theorem 2.1, we get $\{x_n\}$ is a Cauchy sequence in complete S_b-metric space and hence $\{x_n\}$ converges to some $w \in X$.

For $n \in N, we\ have$

$$S_b(w,w,Tw) \leq s[S_b(w,w,x_{n+1}) + S_b(w,w,x_{n+1}) + S_b(Tw,Tw,x_{n+1})]$$
$$= s[2S_b(w,w,x_{n+1}) + S_b(Tw,Tw,x_{n+1})]$$
$$\leq s\,[2S_b(w,w,x_{n+1}) + \phi(\max\{S_b(w,w,x_n),\ S_b(Tw,Tw,w),\ S_b(Tw,Tw,w),\ S_b(x_{n+1},x_{n+1},w)\})]$$
$$= s\,[2S_b(w,w,x_{n+1}) + \phi(\max\{S_b(w,w,x_n), S_b(Tw,Tw,w),\ S_b(x_{n+1},x_{n+1},w)\})] .$$

Case I.

If $\max\{S_b(w,w,x_n), S_b(Tw,Tw,w),\ S_b(x_{n+1},x_{n+1},w)\} = S_b(w,w,x_n), then$

$$S_b(w,w,Tw) \leq s[2S_b(w,w,x_{n+1}) + \phi(S_b(w,w,x_n))]$$
$$< s[2S_b(w,w,x_{n+1}) + S_b(w,w,x_n)] .$$

letting $n \to \infty, we\ have\ Tw = w$.

Case II.

If $\max\{S_b(w,w,x_n), S_b(Tw,Tw,w),\ S_b(x_{n+1},x_{n+1},w)\} = S_b(Tw,Tw,w), then$

$$S_b(w,w,Tw) \leq s[2S_b(w,w,x_{n+1}) + \phi(S_b(Tw,Tw,w))]$$
$$< s[2S_b(w,w,x_{n+1}) + S_b(Tw,Tw,w)].$$

using $[S_b(x,x,y) = S_b(y,y,x) \text{ for all } x, y \in X]$, we get

$$S_b(w,w,Tw) < s[2S_b(w,w,x_{n+1}) + S_b(w,w,Tw)].$$

letting $n \to \infty, we\ have\ Tw = w$.

Case III.

If $\max\{S_b(w,w,x_n), S_b(Tw,Tw,w),\ S_b(x_{n+1},x_{n+1},w)\} = S_b(x_{n+1},x_{n+1},w), then$

$$S_b(w,w,Tw) \leq s[2S_b(w,w,x_{n+1}) + \phi(S_b(x_{n+1},x_{n+1},w))]$$
$$< s[2S_b(w,w,x_{n+1}) + S_b(x_{n+1},x_{n+1},w)].$$

using $[S_b(x,x,y) = S_b(y,y,x) \text{ for all } x, y \in X]$, we get

$$S_b(w,w,Tw) < s[2S_b(w,w,x_{n+1}) + S_b(w,w,x_{n+1})].$$

letting $n \to \infty, we\ have\ Tw = w$.

Hence , we can say that w is a fixed point of T.

If v is another fixed point of $T, then$

$$S_b(w,w,v) = S_b(Tw,Tw,Tv)$$
$$\leq \phi[max\{S_b(w,w,v), S_b(Tw,Tw,w), S_b(Tw,Tw,w), S_b(Tv,Tv,w)\}]$$
$$\leq \phi[max\{S_b(w,w,v), S_b(w,w,w), S_b(w,w,w), S_b(v,v,w)\}]$$
$$\leq \phi[max\{S_b(w,w,v),\ S_b(v,v,w)\}]$$
$$= \phi\{S_b(w,w,v)\} \quad (\text{by } S_b(v,v,w) = S_b(w,w,v))$$
$$< S_b(w,w,v), \quad (\text{because} \phi(t) < t)$$

which is not possible and hence w is a unique fixed point of T.

Corollary 2.7: Let X be a complete S_b-metric space . Suppose that there is $k \in [0,1)$and the mapping$T : X \to X$ satisfies the condition:

$$S_b(Tx, Ty, Tz) \leq k[\max\{S_b(x,y,z), S_b(Tx,Tx,x), S_b(Ty,Ty,y), S_b(Tz,Tz,x)\}],$$

for all $x, y, z \in X$. Then T has a unique fixed point .

Proof. Define $\phi: [0,\infty) \to [0,\infty)\ by\ \phi(w) = kw$. Then clearly ϕ is a non-decreasing function with $\lim_{n\to\infty} \phi^n(t) = 0, for\ all\ t > o$. Using given condition and by virtue of ϕ , we have

$$S_b(Tx, Ty, Tz) \leq \phi[max\{S_b(x, y, z), S_b(Tx, Tx, x), S_b(Ty, Ty, y), S_b(Tz, Tz, x)\}]\ , for\ all\ x, y, z \in X.$$

Now the results follows from Theorem 2.6.

Corollary 2.8: Let X be a complete S_b-metric space. Suppose the mapping $T : X \to X$ satisfies the condition:

$$S_b(Tx, Tx, Tz) \leq k[max\{S_b(x, x, z), S_b(Tx, Tx, x),\ S_b(Tz, Tz, x)\}],$$
$$for\ all\ x, z \in X.$$

Then T has a unique fixed point.

Proof. We obtain the result by taking $y = x$ in Theorem 2.7.

REFERENCES

[1]. I.A. Bakhtin: The contraction mapping principle in quasi metrics spaces, Functional Analysis, 30 (1989) 26-37.

[2]. N. Souayah and N. Mlaiki, A fixed point theorem in S_b-metric spaces, J. Math. Computer Sci., 16 (2016) 131-139.

[3]. S. Czerwik: Contraction mapping in b-metric spaces, Acta Math. Inform. Univ. Ostrav., 1 (1993) 5-11.

[4]. S. Czerwik: Nonlinear set-valued contraction mappings in b-metric spaces., AttiSemin. Mat. Fis. Univ. Modena, 46 (1998) 263-276.

[5]. S. Sedghi and N. Shobe, N, A. Aliouche: A generalization of fixed point theorems in S-metric spaces, Matematiˇ ckiVesnik, 64(3) (2012) 258-266.

[6]. S. Sedghi and Nguyen Van Dung, Fixed point theorems on S -metric spaces, Matematiˇ cki Vesnik, 66(1) (2014) 113-124.

[7]. S. Sedghi, A. Gholidahneh, T. Doˇ senovi´ c, J. Esfahani and S. Radenovi´ c: Common fixed point of four maps in S_b- metric spaces, J.LinearTopol. Algebra, 5 (2) 2016, 93–104.

[8]. Savitri and Nawneet Hooda, Fixed Point Theorems for Mappings in Various Spaces [Ph. D. thesis], Department of Mathematics, Faculty of Science and Technology Interface, Deenbandhu Chhotu RamUniversity of Science and Technology,Murthal,2017.

[9]. W. Shatanawi, Fixed point theory for contractive mappings satisfying ϕ-map in G-metric spaces, Fixed Point Theory and Applications, 2010(2010), Article ID 181650.

[10]. Z. Mustafa and B. Sims, A new approach to generalized metric space , J. Nonlinear Convex Anal., 7(2) (2006), 289-297.