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A Condition on the Masses of Composite Particles

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Abstract

Based on the recent formulation of the standard model with the spinor space given by a direct sum of tensor products of modules isomorphic to division algebras, these algebras are used to derive an expression for masses of elementary particles. It is shown that the formula derived for the tensor product of states is consistent with the Lagrangian model of composite particles.

Keywords: composite mass formula, bound states

MSC: 11E25, 17C60, 55R25

1. INTRODUCTION

In one of the formulations of the standard model, the spinor space is modelled on $T = \mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ for each generation [15]. There are 32 complex degrees of freedom in T and 64 complex degrees of freedom in T^2, which are sufficient for the inclusion of the fermions and the antifermions in four dimensions, since the two quarks with three colours would be described by 6 Dirac spinors, each having 4 components, and the lepton doublet has 8 components, when each of the leptons is described by a Dirac spinor. In a theory with massless neutrinos, described by Weyl spinors, two components of the Dirac spinor would be set equal to zero. Upon generation of mass at two loops, however, the massive neutrino can be represented by a Dirac spinor. Since the mixing between three generations is determined by a unitary Cabibbo-Kobayashi-Maskawa matrix, the states may be defined such that the spinor space $\oplus_{i=1}^{3} T_i^2$ is a direct sum $\oplus_{i=1}^{3} [\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}]^2$ [11].

Given the identification of the division algebras with the modules in the spinor space of the standard model, the squared absolute value of a wavefunction representing a linear combination of basis elements in each module shall be represented as the sum of n squares for n=1, 2, 4 or 8, thereby implying positive-definiteness.

It will be demonstrated that the mass formula is consistent with the description of the particles provided that an equality holds. The validity of this equality follows from a theorem on sums of squares. Consequently, a Lagrangian for bound states of more than one particle exists in this formulation. The properties of the bound state have been formulated in terms of Green functions which satisfy the Bethe-Salpeter equation. For sufficiently large separations and weak coupling, the composite state is given by a tensor product of the fundamental states. If there is a strong coupling between the particles, the description of the composite state must be modified.

An example of a generalized relation which would be based on a direct sum of states in the spinor space, with the scalar state in a theory with broken supersymmetry having an expectation value proportional to the square root of the mass, is the Koide relation for the masses of the charged leptons. A theoretical basis is suggested in the last section.

2. DIVISION ALGEBRAS AND THE FERMIONS IN THE STANDARD MODEL

The number of pairwise orthogonal matrices $U_i : \mathbb{R}^n \to \mathbb{R}^n$ such that $U_i^2 = -I$ and $U_i U_j = -U_j U_i$ is known to be less than $\rho(n) = 2^{c(n)} + 8d(n)$ by the Hurwitz-Radon-Eckmann theorem [18, 27, 28, 40], where $n = (2\alpha(n)+1)2^{b(n)}$, with $\alpha(n) \geq 0$ and $b(n) = c(n) + 4d(n)$, $0 \leq c(n) \leq 3$. It may be verified that $\rho(n) \leq n$ since the inequality

$$2^{-c(n)} d(n) \leq (2\alpha(n)+1).2^{4d(n)-3} - \frac{1}{8} \tag{2.1}$$

or

$$\begin{aligned} 0 &\leq \frac{\alpha(n)}{4} & d(n) &= 0 \\ 2^{-c(n)} &\leq 2(2\alpha(n)+1) - \frac{1}{8} & d(n) &= 1 \\ 2^{-c(n)} d(n) &\leq (2\alpha(n)+1).2^{3(d(n)-1)}.2^{d(n)} - \frac{1}{8} & d(n) &\geq 2. \end{aligned} \tag{2.2}$$

The equality $\rho(n) = n$, or equivalently,

$$2^{c(n)} + 8d(n) = (2\alpha(n)+1).2^{c(n)+4d(n)} \tag{2.3}$$

is valid only when $d(n) = 0$ and $\alpha(n) = 0$ such that $n = 2^{c(n)}$, with the values $n = 1, 2, 4, 8$ representing division algebras over $\mathbb{R}$.

The maximal number of nonvanishing smooth vector fields on S^1, S^3 and S^7 is known to be related to the existence normed real division algebras [26] only in the dimensions 1, 2, 4 and 8 [1,2,7,8, 32]. Together with the unit radial vector, the tangent vectors form an orthonormal set.
A basis of tangent vector fields on S^3 would be given by

$$\begin{aligned} T_1 &= -X_1 \frac{\partial}{\partial X_0} + X_0 \frac{\partial}{\partial X_1} - X_3 \frac{\partial}{\partial X_2} + X_2 \frac{\partial}{\partial X_3} \\ T_2 &= -X_2 \frac{\partial}{\partial X_0} + X_3 \frac{\partial}{\partial X_1} + X_0 \frac{\partial}{\partial X_2} - X_1 \frac{\partial}{\partial X_3} \\ T_3 &= -X_3 \frac{\partial}{\partial X_0} - X_2 \frac{\partial}{\partial X_1} + X_1 \frac{\partial}{\partial X_0} + X_0 \frac{\partial}{\partial X_1} \end{aligned} \tag{2.4}$$

such that $|T_1|^2 = |T_2|^2 = |T_3|^2 = 1$.

The coefficients of the triad can be nonlinear since the transformation induced by $X = X_0 + X_1 i + X_2 j + X_3 k \to X^2$ yields

$$\left(X_0, X_1, X_2, X_3\right) \to \left(X_0^2 - X_1^2 - X_2^2 - X_3^2, 2X_0X_1, 2X_0X_2, 2X_0X_3\right) \tag{2.5}$$

and generates a new set of tangent vector fields

$$\begin{aligned} T_1^{'} &= -2X_0X_1\frac{\partial}{\partial X_0} + \left(X_0^2 - X_1^2 - X_2^2 - X_3^2\right)\frac{\partial}{\partial X_1} - 2X_0X_3\frac{\partial}{\partial X_2} + 2X_0X_1\frac{\partial}{\partial X_3} \\ T_2^{'} &= -2X_0X_2\frac{\partial}{\partial X_0} + 2X_0X_3\frac{\partial}{\partial X_1} + \left(X_0^2 - X_1^2 - X_2^2 - X_3^2\right)\frac{\partial}{\partial X_2} - 2X_0X_1\frac{\partial}{\partial X_3} \\ T_3^{'} &= -2X_0X_3\frac{\partial}{\partial X_0} - 2X_0X_2\frac{\partial}{\partial X_1} + 2X_0X_1\frac{\partial}{\partial X_2} + \left(X_0^2 - X_1^2 - X_2^2 - X_3^2\right)\frac{\partial}{\partial X_1} \end{aligned} \tag{2.6}$$

that have unit norms, $\left|T_1^{'}\right|^2 = \left|T_2^{'}\right|^2 = \left|T_3^{'}\right|^2 = 1$, and each pair is orthogonal, $\left\langle T_1^{'}T_2^{'}\right\rangle = \left\langle T_1^{'}T_3^{'}\right\rangle = \left\langle T_2^{'}T_3^{'}\right\rangle = 0$.

The existence of orthonormal matrices preserving the norm of the orthonormal basis of linear vector fields, such as those given in Eq.(2.4), is generally not relevant for fields with nonlinear coefficients unless these are induced by multiplication in a division algebra. For the sphere S^2, it may be noted that a highly nonlinear function of the coordinates in the stereographically projected plane would yield a divergence in the inverse image at the point $\left(X, Y, Z\right) = \left(0,0,1\right)$. The coordinates in the chart of the northern hemisphere under the stereographic projection are

$$\left(x, y\right) = \left(\frac{X}{1-Z}, \frac{Y}{1-Z}\right) \tag{2.7}$$

It follows that

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{1-Z}\frac{\partial}{\partial X}\Big|_{TS^2} + \frac{X}{\left(1-Z\right)^2}\frac{\partial}{\partial Z}\Big|_{TS^2} \\ \frac{\partial}{\partial y} &= \frac{1}{1-Z}\frac{\partial}{\partial Y}\Big|_{TS^2} + \frac{Y}{\left(1-Z\right)^2}\frac{\partial}{\partial Z}\Big|_{TS^2} \end{aligned} \tag{2.8}$$

Since, $\frac{\partial}{\partial Z}\Big|_{TS^2}\left(Z = 1\right) = 0$,

$$\begin{aligned} & g_1\left(x\left(X,Y,Z\right), y\left(X,Y,Z\right)\right)\frac{\partial}{\partial x} + g_2\left(x\left(X,Y,Z\right), y\left(X,Y,Z\right)\right)\frac{\partial}{\partial y}\Big|_{TS^2} \\ & \overrightarrow{Z=1}\, \frac{g_1\left(x\left(X,Y,Z\right), y\left(X,Y,Z\right)\right)}{1-Z}\frac{\partial}{\partial X} + \frac{g_2\left(x\left(X,Y,Z\right), y\left(X,Y,Z\right)\right)}{1-Z}\frac{\partial}{\partial Y} \end{aligned} \tag{2.9}$$

Finiteness requires $g_1\left(x\left(X,Y,Z\right), y\left(X,Y,Z\right)\right), g_2\left(x\left(X,Y,Z\right), y\left(X,Y,Z\right)\right) \sim 1-Z$ as $x, y \to \infty$.

At other points on the sphere, the radial vector is $\frac{\partial}{\partial r}\Big|_{r=1} = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}$ and two perpendicular vectors would be

$$\begin{aligned} \vec{v}_1 &= -Y\frac{\partial}{\partial X} + X\frac{\partial}{\partial Y} \\ \vec{v}_2 &= -Z\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial Z} \end{aligned} \tag{2.10}$$

The projection of a vector $\vec{w}$ onto the hyperplane spanned by $\vec{v}_1$ and $\vec{v}_2$ would be

$$\overrightarrow{w}_{proj.}=\frac{\left[\left(\left(\overrightarrow{w}\cdot\overrightarrow{v_1}\right)\left|\overrightarrow{v_2}\right|^2-\left(\overrightarrow{w}\cdot\overrightarrow{v_2}\right)\left(\overrightarrow{v_1}\cdot\overrightarrow{v_2}\right)\right)\overrightarrow{v_1}+\left(\left(\overrightarrow{w}\cdot\overrightarrow{v_2}\right)\left|\overrightarrow{v_1}\right|^2-\left(\overrightarrow{w}\cdot\overrightarrow{v_1}\right)\left(\overrightarrow{v_1}\cdot\overrightarrow{v_2}\right)\right)\overrightarrow{v_2}\right]}{\left|\overrightarrow{v_1}\right|^2\left|\overrightarrow{v_2}\right|^2-\left(\overrightarrow{v_1}\cdot\overrightarrow{v_2}\right)^2} \tag{2.11}$$

If

$$\overrightarrow{w}=g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)\left[\frac{1}{1-Z}\frac{\partial}{\partial X}+\frac{X}{\left(1-Z\right)^2}\frac{\partial}{\partial Z}\right]+g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)\left[\frac{1}{1-Z}\frac{\partial}{\partial Y}+\frac{Y}{\left(1-Z\right)^2}\frac{\partial}{\partial Z}\right],$$

$$\overrightarrow{w}_{proj.}=\left[-\frac{g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)}{Y\left(1-Z\right)}+\frac{g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)X^2}{Y\left(1-Z\right)^2}+\frac{g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)X}{\left(1-Z\right)^2}\right]\cdot\left(-Y\frac{\partial}{\partial X}+X\frac{\partial}{\partial Y}\right)$$
$$+\frac{g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)X+g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)Y}{Y\left(1-Z\right)}\left(-Z\frac{\partial}{\partial Y}+Y\frac{\partial}{\partial Z}\right) \tag{2.12}$$

Consider the form of $g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)$ and $g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)$ required for a non-zero mass. Let

$$\begin{aligned} g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)&\sim\frac{\left(1-Z\right)^2}{X}\\ g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)&\sim\frac{\left(1-Z\right)^2}{Y}\end{aligned} \tag{2.13}$$

The coefficient of $\frac{\partial}{\partial X}$ is

$$\frac{g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)}{1-Z}\sim\frac{\left(1-Z\right)^2}{X}\frac{1}{1-Z}=\frac{1-Z}{X} \tag{2.14}$$

The coefficient of $\frac{\partial}{\partial Y}$ would be

$$\begin{aligned}&-\frac{g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)X}{Y\left(1-Z\right)}\\ &+\frac{g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)X+g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)Y}{Y\left(1-Z\right)}\left(-Z\right)\\ &\sim-O\left(\frac{\left(1-Z\right)^2}{Y}\right)\end{aligned} \tag{2.15}$$

while it equals

$$\frac{g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)X+g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)Y}{1-Z}\sim1-Z \tag{2.16}$$

for $\frac{\partial}{\partial Z}$, and vanishes in the limit $Z\to1$.

Similarly, if

$$\begin{aligned} g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right) &\sim 1-Z \\ g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right) &\sim 1-Z \end{aligned} \tag{2.17}$$

The coefficient of $\frac{\partial}{\partial X}$ would be

$$\frac{g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)}{1-Z} \sim O(1), \tag{2.18}$$

and the coefficient of $\frac{\partial}{\partial Y}$ is

$$-\frac{g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)X}{1-Z} \sim -(1-Z)\frac{X}{Y(1-Z)} = -\frac{X}{Y} \tag{2.19}$$

For the coefficient of $\frac{\partial}{\partial Z}$ is

$$\begin{aligned} \frac{g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)X}{1-Z} &\sim \frac{1-Z}{1-Z}\cdot X = X \\ \frac{g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)Y}{1-Z} &\sim \frac{1-Z}{1-Z}\cdot Y = Y \end{aligned} \tag{2.20}$$

which again vanishes in the limit $Z \to 1$.

Amongst the analytic functions of X, Y and Z, the dependence of the kind given in Eqs.(2.13) or (2.17) is selected for a nonvanishing vector field at $Z = 1$.However, a singularity in the vector field will develop at another point on the sphere. When $X \neq 0, Y = 0$ and $Z \neq 0$, there is a divergence in the first expression. Since the point $\left(X,0,Z\right)$ can exist in the northern hemisphere, the vector field would not be regular everywhere in this region. Inclusion of an additional factor of Y in $g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)$ and $g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)$ would render the vector field

$$g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)\frac{\partial}{\partial X} + g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)\frac{\partial}{\partial Y}$$

to be vanishing in the limit $Z \to 1$. Alternatively, the denominators in Eqs.(2.14) and (2.15) can be perturbed to be $X + \delta_X$ and $Y + \delta_Y$ respectively. For the second choice of $g_1\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)$ and $g_2\left(x\left(X,Y,Z\right),y\left(X,Y,Z\right)\right)$, it would be sufficient to replace Y by $Y + \delta_Y$ in Eq.(2.19).

This technique will be developed in the theorem for spheres in arbitrary dimensions.

Theorem 1. Specific nonlinear functions of the coefficients of the stereographically projected vector field are allowed in a general dimension n. A divergence in the vector fields elsewhere in a chart of the sphere will require a perturbation in $n-1$ of the embedding coordinates. It can be shown that a singularity will arise generally because any multiplicative identity between sums of squares requires a number of terms less than n. The spheres S^1, S^3 and S^7 admit the maximal number of smooth, nonvanishing vector fields, and the coefficients of the coordinate derivatives can be nonlinear functions, such that there will be many vector fields on the stereographically projected plane that images of the orthonormal sets of tangent vectors.

Proof. In n dimensions, the radial vector is $\frac{\partial}{\partial r}\Big|_{r=1} = X_0\frac{\partial}{\partial X_0} + \cdots + X_{n-1}\frac{\partial}{\partial X_{n-1}}$, while the perpendicular vectors are

$$\begin{aligned}\overrightarrow{v_0} &= -X_1 \frac{\partial}{\partial X_0} + X_0 \frac{\partial}{\partial X_1} \\ \overrightarrow{v_1} &= -X_2 \frac{\partial}{\partial X_1} + X_1 \frac{\partial}{\partial X_2} \\ &\vdots \\ \overrightarrow{v_{n-2}} &= -X_{n-1} \frac{\partial}{\partial X_n} + X_n \frac{\partial}{\partial X_{n-1}}\end{aligned} \tag{2.21}$$

The projection of the vector

$$\begin{aligned}\vec{w} &= g_0\left(x_0,\ldots,x_{n-2}\right)\frac{\partial}{\partial x_0}+\ldots+g_{n-2}\left(x_0,\ldots,x_{n-2}\right)\frac{\partial}{\partial x_{n-2}} \\ &= g_0\left(x_0\left(X_0,\ldots,X_{n-1}\right),\ldots,x_{n-2}\left(X_0,\ldots,X_{n-1}\right)\right)\left(\frac{1}{1-X_{n-1}}\frac{\partial}{\partial X_0}+\frac{X_0}{\left(1-X_{n-1}\right)^2}\frac{\partial}{\partial X_{n-1}}\right) \\ &+ g_{n-2}\left(x_0\left(X_0,\ldots,X_{n-1}\right),\ldots,x_{n-2}\left(X_0,\ldots,X_{n-1}\right)\right)\left(\frac{1}{X_{n-1}}\frac{\partial}{\partial X_{n-2}}+\frac{X_{n-2}}{\left(1-X_{n-1}\right)^2}\frac{\partial}{\partial X_{n-1}}\right)\end{aligned} \tag{2.22}$$

is

$$\overrightarrow{w}_{proj.} = \frac{\det\begin{pmatrix} \left|\overrightarrow{w_0}\right|^2 & \overrightarrow{v_0}\cdot\overrightarrow{v_1} & \ldots & \overrightarrow{v_0}\cdot\overrightarrow{v_{n-2}} \\ \ldots & \ldots & \ldots & \ldots \\ \overrightarrow{w_0}\cdot\overrightarrow{v_{n-2}} & \overrightarrow{v_1}\cdot\overrightarrow{v_{n-2}} & \ldots & \left|\overrightarrow{v_{n-2}}\right|^2 \end{pmatrix}}{\det\begin{pmatrix} \left|\overrightarrow{v_0}\right|^2 & \ldots & \ldots & \overrightarrow{v_0}\cdot\overrightarrow{v_{n-2}} \\ \ldots & \ldots & \ldots & \ldots \\ \overrightarrow{v_0}\cdot\overrightarrow{v_{n-2}} & \ldots & \ldots & \left|\overrightarrow{v_{n-2}}\right|^2 \end{pmatrix}}\overrightarrow{v_0} + \ldots + \frac{\det\begin{pmatrix} \left|\overrightarrow{v_0}\right|^2 & \ldots & \ldots & \vec{w}\cdot\overrightarrow{v_0} \\ \ldots & \ldots & \ldots & \ldots \\ \overrightarrow{v_0}\cdot\overrightarrow{v_{n-2}} & \ldots & \ldots & \vec{w}\cdot\overrightarrow{v_{n-2}} \end{pmatrix}}{\det\begin{pmatrix} \left|\overrightarrow{v_0}\right|^2 & \ldots & \ldots & \overrightarrow{v_0}\cdot\overrightarrow{v_{n-2}} \\ \ldots & \ldots & \ldots & \ldots \\ \overrightarrow{v_0}\cdot\overrightarrow{v_{n-2}} & \ldots & \ldots & \left|\overrightarrow{v_{n-2}}\right|^2 \end{pmatrix}}\overrightarrow{v_{n-2}} \tag{2.23}$$

The order of the denominator in n dimensions is $O\left(X_i^{2(n-1)}\right)$. The divergence that occurs at $X_0 \neq 0,\ldots,X_{i_1-1} \neq 0, X_{i_1} = 0, X_{i_1+1} \neq 0,\ldots,X_{i_l-1} \neq 0, X_{i_l} = 0, X_{i_l+1} \neq 0,\ldots,X_{n-1} \neq 0$ that can be circumvented if $\left\{X_{i_1},\ldots,X_{i_l}\right\}$ is replaced by $\left\{X_{i_1}+\delta_{i_1},\ldots,X_{i_l}+\delta_{i_l}\right\}$. If $\delta_i \in [-1,1]$, there will be divergences in the coefficients of $\frac{\partial}{\partial X_0},\ldots,\frac{\partial}{\partial X_{n-1}}$ at $X_{i_1} = -\delta_{i_1},\ldots X_{i_l} = -\delta_{i_l}$. However, a singularity in the vector fields on the sphere does not arise if $\delta_i \neq [-1,1]$.

The multiplicative equivalent of the closure of the commutator algebra of tangent vector fields is the n-square identity

$$\left(X_0^2+\ldots+X_{n-1}^2\right)\left(X_0^{'2}+\ldots+X_{n-1}^{'2}\right)=X_0^{"2}+\ldots+X_{n-1}^{"2} \qquad (2.24)$$

When $n=2,4,8$, the identity is valid generally, and the values of $\delta_0,\ldots,\delta_{n-2}$ can be chosen to be beyond the range $[-1,1]$. This identity does not hold for $n\neq 2,4,8$.
For these values of n, the factors must have the form

$$X_0^2+\ldots+\left(X_{i_1}+\delta_{i_1}\right)^2+\ldots+\left(X_{i_l}+\delta_{i_l}\right)^2+\ldots+X_{n-1}^2 \qquad (2.25)$$

The introduction of coordinates such that δ_i has a value producing a cancellation in the factor would be characteristic of dimensions other than $2,4$ and 8. Then, on the spheres $S^{n-1}, n\neq 2,4,8$, there must be a value of $\delta_i\in[-1,1]$ generating divergences in the tangent vector fields.

The multiplication in the complex, quaternion and octonion algebras yield the following transformations of the coefficents of the tangent vector fields

$$\begin{aligned}
&\left(X_0,X_1\right)\to\left(X_0^2-X_1^2,2X_0X_1\right)\\
&\left(X_0,X_1,X_2,X_3\right)\to\left(X_0^2-X_1^2-X_2^2-X_3^2,2X_0X_1,2X_0X_2,2X_0X_3\right)\\
&\left(X_0,X_1,X_2,X_3,X_4,X_5,X_6,X_7\right)\to\left(X_0^2-X_1^2-X_2^2-X_3^2-X_4^2-X_5^2-X_6^2-X_7^2,\right.\\
&\qquad\left.2X_0X_1,2X_0X_2,2X_0X_3,2X_0X_4,2X_0X_5,2X_0X_6,2X_0X_7\right)
\end{aligned} \qquad (2.26)$$

Iteration of these transformations yields an infinite number of orthonormal sets of tangent vector fields on the spheres S^1,S^3 and S^7. There will be an equal number of vector fields on the stereographically projected hyperplanes.

q.e.d.

The division algebras are modules for Clifford algebras, and, in particular, the adjoint algebra of the spinor space is $T_L\sim R_{0,9}$, the equivalent of the Pauli algebra in three dimensions, where $R_{p,q}$ is defined by the relations $\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha=2\eta_{\alpha\beta}I_{p+q},\eta_{\alpha\beta}=diag(1,\ldots,1,-1,\ldots,-1)$. The Dirac algebra for T_L is a complexification of $R_{1,9}$, the Clifford algebra in ten dimensions, with the spinors having 32 components. The matrices $\gamma_{[\alpha\beta]}=\frac{1}{2}\left(\gamma_\alpha\gamma_\beta-\gamma_\beta\gamma_\alpha\right)$ generate the ten-dimensional Lorentz algebra. Embedded in this Lorentz group is $SU(4)$, which upon intersection with G_2 yields $SU(3)$ [11], the gauge group of quantum chromodynamics. Similarly, the $R_{1,3}$ subalgebra of $R_{1,9}$, which acts on $\mathbb{C}\otimes\mathbb{H}$, corresponds to the $SU(2)\times U(1)$ subgroup of the standard model gauge group, representing the electroweak interactions. These symmetries are considered internal, however, because the Lorentz group may be viewed as independent of the gauge group as transformations of these modules, through right and left adjoint actions. It has been verified that both the fermion and boson content and the gauge groups of the standard model may be constructed after the appropriate identification of the quarks and leptons with the spinor space degrees of freedom.

3. THE EXPRESSION FOR THE MASS

Given the identification of fermions with elements in the spinor space consisting of the division algebra modules, it follows that the wavefunctions generally should be linear combinations of the tensor products of bases of these algebras.

The operator $P^0=H$ is Hermitian with real eigenvalues and

$$\left\langle \psi \left| P^0 \right| \psi \right\rangle = M \left\langle \psi | \psi \right\rangle = M \tag{3.1}$$

This is not a Lorentz-invariant result, but it would hold in a ground state at rest.

The Schrodinger equation is a linear equation in $|\psi\rangle$ with eigenvalue E

$$i\hbar \frac{d|\psi\rangle}{dt} = E|\psi\rangle \tag{3.2}$$

Therefore, a linear relation between the energy of a composite state and the energies of the component states can be maintained during time evolution of the nonrelativistic system.

For relativistic fields, the Klein-Gordon equation

$$\left(\Box + m^2\right)\phi = 0 \tag{3.3}$$

and its generalizations are used. The spin $-\frac{1}{2}$ and spin -1 fields satisfy the equations

$$\begin{aligned} &\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0 \\ &\left(\Box + m^2\right)V^\mu = 0 \end{aligned} \tag{3.4}$$

which are linear and quadratic in m respectively. Expectation values can be computed through integrals of the type

$$\left\langle P^\mu P_\mu \right\rangle = \int d^3x \phi^* P^\mu P_\mu \phi \tag{3.5}$$

which would be the equivalent of the definition in terms of state vectors. The Dirac operator is the square root of the Klein-Gordon equation and the relation between the derivatives is clarified only when the equation is squared. The characteristics of the d'Alembertian operator are given by light cones. Along null directions, a linear relation between the squared masses of a composite state and the square masses of the component states would be preserved. For composite particles propagating near the speed of light, this linear relation should be approximately conserved along the null cones. Furthermore, if the field is studied as a configuration in a region of a spatial hypersurface, the Cauchy development of this region would cover increasingly larger domains in hypersurfaces at later times.

If $P^\mu \psi = p^\mu \psi$,

$$\begin{aligned} &M^2 \psi = P^\mu P_\mu \psi \\ &M^2 \int \psi^* \psi d^3x = \left\langle \psi \left| P_\mu P^\mu \right| \psi \right\rangle \end{aligned} \tag{3.6}$$

or equivalently

$$M = \left[\left\langle \psi \left| P_\mu P^\mu \right| \psi \right\rangle\right]^{\frac{1}{2}}. \tag{3.7}$$

When $P^\mu \psi$ is defined in terms of basis elements in the spinor space of the standard model.

Division algebras are known to preserve norms

$$|x \cdot y| = |x||y| \qquad x, y \in \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \tag{3.8}$$

Given that x, y are $n-$ component vectors, where $n = 1, 2, 4, 8$, this relation [24] implies that the product of the sums of squares of the components also equals a sum of square in these dimensions

$$\left(x_1^2 + \ldots + x_n^2\right)\left(y_1^2 + \ldots + y_n^2\right) = \left(z_1^2 + \ldots + z_n^2\right) \qquad n = 1, 2, 4, 8 \tag{3.9}$$

The 4-square identity was discovered by Euler [19] and Hamilton in connection with quaternions. The generalization to octonions led to the proof of the 8-square identity by Degen [13] and Cayley [9]. A general theorem restricting the identity to $n=1,2,4,8$ for fields of characteristic 0 with z_i as bilinear functions of x_i, y_i was proven by Hurwitz [27].

If the form of the variables z_i are modified to be rational functions of $\{x_i, y_i, i=1,2,\ldots,n\}$ was shown to allow 2^m-square identities for $m \geq 4$ [10, 38, 39]. The proof was based on the matrix equalities, where the presence of inverse matrices in the block form led to denominators in the equations. The 16-square identity was determined, and denominators in the expressions of the variables z_i were found [17, 41].

The mass M of an ultrarelativistic composite particle can be equated in general with the square root of a sum of n squares with $n=1,2,4$ or 8, after a restriction to each module. This equivalence is feasible essentially because any positive integer can be expressed as the sum of four squares [20] by Lagrange's theorem [37] and also eight squares. The number of representations of a positive integer N by the sum of four squares was demonstrated by Jacobi [30] to be equal to

$$\begin{array}{ll} \tau(N) & N \text{ odd} \\ 24\tau(N) & N \text{ even} \end{array} \tag{3.10}$$

This conclusion can be drawn even for spinor which are restricted to the two-dimensional module $\mathbb{C}$. For these spinors, the squares of the masses approximately equal to an odd prime p relative to a basic scale would satisfy $p \equiv 1(\bmod 4)$ by Fermat's $4n+1$ theorem [14, 21]. The formula for the mass for a complex module is similar to that found for solitons in supersymmetric Yang-Mills theory with electric and magnetic charges [22]. The presence of a non-zero mass for every charged state would follow.

4. COMPOSITE MASSES

Suppose that the composite state is regarded as the tensor product of n one-particle states

$$|\psi\rangle = |\psi_1, \ldots, \psi_n\rangle. \tag{4.1}$$

Selecting the largest division algebra module $\mathbb{O}$ in the spinor space T, an expansion of the state vector $|\psi\rangle$ with respect to the basis $\{|\psi_{ie_0}\rangle, \ldots, |\psi_{ie_7}\rangle\}$ yields

$$|\psi_i\rangle = \alpha_{i0}|\psi_{ie_0}\rangle + \ldots + \alpha_{i7}|\psi_{ie_7}\rangle \tag{4.2}$$

where $|\psi_{ie_j}\rangle$ are orthonormal states. Then

$$\begin{aligned} \langle\psi|\psi\rangle &= \left[\langle\psi_{ie_0}|\alpha_{i0}^* + \ldots + \langle\psi_{ie_7}|\alpha_{i7}^*\right]\cdot\left[\alpha_{i0}|\psi_{ie_0}\rangle + \ldots + \alpha_{i7}|\psi_{ie_7}\rangle\right] \\ &= \alpha_{i0}^*\alpha_{i0}\langle\psi_{ie_0}|\psi_{ie_0}\rangle + \ldots + \alpha_{i7}^*\alpha_{i7}\langle\psi_{ie_7}|\psi_{ie_7}\rangle \\ &= |\alpha_{i0}|^2 + \ldots + |\alpha_{i7}|^2. \end{aligned} \tag{4.3}$$

Since

$$P^{\mu_i}P_{\mu_i}|\psi_i\rangle = m_i^2|\psi_i\rangle \tag{4.4}$$

and

$$\langle\psi_i|P^{i\mu}P_{i\mu}|\psi_i\rangle = \langle\psi_i|m_i^2|\psi_i\rangle = m_i^2 \tag{4.5}$$

while

$$\left(P^{\mu_i}P_{\mu_i}\right)\left(\alpha_{i0}|\psi_{ie_0}\rangle + \ldots + \alpha_{i7}|\psi_{ie_7}\rangle\right) = \alpha_{i0}m_{ie_0}^2|\psi_{ie_0}\rangle + \ldots + \alpha_{i7}m_{ie_7}^2|\psi_{ie_7}\rangle \tag{4.6}$$

and