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DOMINATION IN CARTESIAN PRODUCT OF FUZZY GRAPHS USING STRONG ARC

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Abstract

In this paper the domination in fuzzy graphs by using strong arcs are generalized and established the domination concept in Cartesian product on standard fuzzy graphs using strong arcs and non strong arcs.

Keywords: Fuzzy graph, Cartesian product, Domination number, Effective edge domination, Strong arc, Non strong arc, Strong arc domination.
AMS classification: 05C72

1. INTRODUCTION

Fuzzy graph is the generalization of the ordinary graph. Therefore it is natural that though fuzzy graph inherits many properties similar to those of ordinary graph, it deviates at many places. The earliest idea of dominating dates back to the origin of the game of chess in India over 400 years ago in which placing the minimum number of a chess piece (such as Queen, knight ext..) over chess board so as to dominate all the squares of chess board was investigated. The formal mathematical definition of domination was given by Ore. O [3] in 1962. In 1975 A.Rosenfeld [4] introduced the notion of fuzzy graph and several analogs of theoretic concepts such as path, cycle and connectedness. A. Somasundaram and S. Somasundaram [5] discussed the domination in fuzzy graph using effective arc. A. Nagoorgani and V.T. Chandrasekarann [2] discussed the domination in fuzzy graph using strong arc. Before introducing new results on domination in Cartesian product of fuzzy graphs using strong arcs , we record below some preliminary definitions and results for our use in the next sections of this paper.

2. PRELIMINARIES

Definition 2.1
A fuzzy graph $G(\sigma, \mu)$ is pair of function $V \to [0,1]$ and $\mu: V \times V \to [0,1]$ where for all $u, v \; in \; V$, we have $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$.

Definition 2.2
The fuzzy graph $H(\tau,\rho)$ is called a fuzzy subgraph of G(σ,μ) if $\tau(u) \leq \sigma(u)$ for all u in V and $\rho(u,v) \leq \mu(u,v)$ for all $u, v\ in\ V$.

Definition 2.3
A fuzzy subgraph $H(\tau,\rho)$ is said to be a spanning sub graph of G(σ,μ) if $\tau(u) = \sigma(u)$ for all u in V. In this case the two graphs have the same fuzzy node set, they differ only in the arc weights.

Definition 2.4
Let G(σ,μ) be a fuzzy graph and τ be fuzzy subset of σ, that is, $\tau(u) \leq \sigma(u)$ for all u in V. Then the fuzzy subgrpah of G(σ,μ) induced by τ is the maximal fuzzy subgraph of G(σ,μ) that has fuzzy node set τ. Evidently, this is just the fuzzy graph $H(\tau,\rho)$ where $\rho(u,v) = \tau(u) \wedge \tau(v) \wedge \mu(u,v)$ for all u,v in V.

Definition 2.5
The underlying crisp graph of a fuzzy graph G(σ,μ) is denoted by $G^* = (\sigma^*,\mu^*)$, where $\sigma^* = \{u \in V \mid \sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in V \times V \mid \mu(u,v) > 0\}$.

Definition 2.6
A fuzzy graph G(σ,μ) is a strong fuzzy graph if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in \mu^*$ and is a complete fuzzy graph if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $u, v\ in\ \sigma^*$ Two nodes u and v are said to be neighbors if $\mu(u,v) > 0$.

Definition 2.7
A fuzzy graph G = (σ, μ) is said to be Bipartite if the node set V can be Partitioned into two non empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \epsilon V_1$ or $v_1, v_2 \epsilon V_2$. Further if $\mu(v_1,v_2) > 0$ for all $v_1 \epsilon V_1$ and $v_2 \epsilon V_2$ then G is called complete bipartite graph and it is denoted by $K_{\sigma 1, \sigma 2}$ where σ_1, σ_2 are respectively the restrictions of σ to V_1 &V_2**.**

Definition 2.8
The complement of a fuzzy graph G(σ,μ) is a subgraph $\bar{G} = (\bar{\sigma},\bar{\mu})$ where $\bar{\sigma} = \sigma$ and $\bar{\mu}(u,v) = \sigma(u) \wedge \sigma(v) - \mu(u,v)$ for all $u, v\ in\ V$. A fuzzy graph is self complementary if $G = \bar{G}$.

Definition 2.9
The order p and size q of a fuzzy graph G(σ,μ) is defined as $p = \sum_{u \in V} \sigma(u)$ and $q = \sum_{(u,v) \in E} \mu(u,v)$.

Definition 2.10
The degree of the vertex u is defined as the sum of weight of arc incident at u, and is denoted by d(u).

Definition 2.11
A Path ρ of a fuzzy graph G(σ,μ) is a sequence of distinct nodes $v_1, v_2, v_3, \ldots, v_n$ such that $\mu(v_{i-1}, v_i) > 0$ where $1 \leq i \leq n$. A path is called a cycle if $u_0 = u_n$ and $n \geq 3$.

Definition 2.12
Let u, v be two nodes in G(σ,μ). If they are connected by means of a path ρ then the strength of that path is $\wedge_{i=i}^{n} \mu(u_{i-1}, v_i)$.

Definition 2.13
Two nodes that are joined by a path are said to be connected. The relation connected is reflexive, symmetric and transitive. If u and v are connected by means of length k, then $\mu^k(u,v) = \sup\{\mu(u,v_1) \wedge \mu(v_1,v_2) \wedge \ldots \wedge \mu(v_{k-1},v_k) \mid u, v_1, v_2, \ldots, v$ in such path $\rho\}$.

Definition 2.14
The Strongest path joining any two nodes u, v is a path corresponding to the maximum strength between u and v. The strength of the strongest path is denoted by $\mu^\infty(u,v)$.
$\mu^\infty(u,v) = \sup\{\mu^k(u,v) \mid k = 1,2,3\ldots\}$.

Example

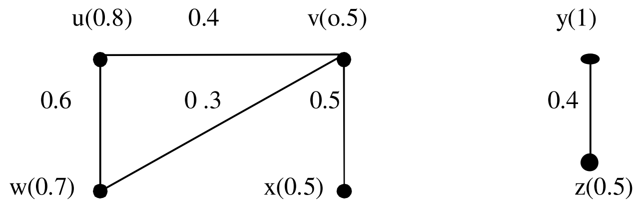


Fig (i) (a) (b)

In the fuzzy graph of Fig(i) (a), u = w, v, x is a w-x path of length 2 and strength is 0.3.
Another path of w-x is w, u, v, x of length 3 and strength is 0.4.
But the strength of the strongest path joining w and x is $\mu^{\infty}(w,x) = \sup\{0.3, 0.4\} = 0.4$

Definition 2.15
Let G(σ, μ) be a fuzzy graph. Let x, y be two distinct nodes and G' be the fuzzy subgraph obtained by deleting the arc(x,y) that is $G'(\sigma, \mu')$ where $\mu'(x,y) = 0$ and $\mu' = \mu$ for all other pairs. Then (x, y) is said to be a fuzzy bridge in G if $\mu'^{\infty}(u,v) < \mu^{\infty}(u,v)$ for some $u, v\ in\ V$.

Definition 2.16
A node is a fuzzy cut node of G(σ, μ) if the removal of it reduces the strength of the connectedness between some other pair of nodes. That is, w is a fuzzy cut node of G(σ, μ) iff there exist u,v such that w is on every strongest path from u to v.

Definition 2.17
An arc (u, v) of the fuzzy graph G(σ, μ) is called an effective edge if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ and the effective edge neighbourhood of u $\in V$ is $N_e(u) = \{v \in V: edge(u,v)\ is\ effective\}$.
$N_e[u] = N_e(u) \cup \{u\}$ is the closed neighborhood of u. The minimum cardinality of the effective neighbourhood $\delta_e(G) = \min\{|N_e(u)|: u \in V(G)\}$. The maximum cardinality of effective neighborhood $\Delta_e(G) = \max\{|N_e(u)|\ u \in V(G)\}$.

3. DOMINATION IN FUZZY GRAPHS USING STRONG ARCS

Definition 3.1
An arc (u,v) of the fuzzy graph G(σ, μ) is called a strong arc if $\mu(u,v) = \mu^{\infty}(u,v)$ else arc(u,v) is called non-strong. Strong neighborhood of u $\in V$ is $N_s(u) = \{v \in V: arc(u,v) \text{ is strong}\}$. $N_s[u] = N_s(u) \cup \{u\}$ is the closed neighborhood of u. The minimum cardinality of strong neighborhood $\delta_s(G) = \min\{|N_s(u)|: u \in V(G)\}$. Maximum cardinality of strong neighborhood $\Delta_s(G) = \max\{|N_s(u)|: u \in V(G)\}$.

Definition 3.2
Let G(σ, μ) be a fuzzy graph. Let u, v be two nodes of G(σ, μ). We say that u dominates v if edge (u,v) is a strong arc. A subset D of V is called a dominating set of G(σ, μ) if for every $v \in V - D$, there exists $u \in D$ such that u dominates v. A dominating set D is called a minimal dominating set if no proper subset of D is a dominating set. The minimum fuzzy cardinality taken over all dominating sets of a graph G is called the strong arc domination number and is denoted by $\gamma_s(G)$ and the corresponding dominating set is called the minimum strong arc dominating set. The number of elements in the minimum strong arc dominating set is denoted by n[$\gamma_s(G)$]

Example 3.3
In fig(i)(a), (u, v), (u, w), (v, x) are strong arcs and (v, w) is non strong arc.
D_1 = {u, x} ,D_2 = {w, x}, D_3 = {w, v} ,D_4 = {u,v} are dominating sets. Also D_1, D_2, D_3, D_4 are minimal dominating sets. Therefore $|D_1| = 0.8 + 0.5 = 1.3$, $|D_2| = 0.7 + 0.5 = 1.2$, $|D_3| = 0.5 + 0.7 = 1.2$ and $|D_4| = 0.8 + 0.5 = 1.3$. Therefore, min{1.3, 1.2, 1.2, 1.3 } = 1.2. Hence D_1 and D_2 are minimum dominating sets. $\gamma_s = 1.2$ and n[γ_s [G] = 2

4. CARTESIAN PRODUCT OF FUZZY GRAPH

Consider the Cartesian product G(V,X) = G_1 X G_2 of G_1 and G_2. Then V= V_1xV_2 and X= { $((u,u_2),(u,v_2))$ / u ∈ V_1, $(u_2,v_2) \in X_2$ } U {$((u_1,w),(v_1,w))$ / w ∈ V_2,$(u_2,v_1) \in X_1$ } .

Definition 3.4

Let σ_i be a fuzzy subset of V_i and let μ_i be a fuzzy subset of X_i i = 1,2.
Define the fuzzy subsets $\sigma_1 \, X \, \sigma_2$ of V and $\mu_1 \, X \, \mu_2$ of X as follows:
$(\sigma_1 \, X \, \sigma_2)(u_1, u_2) = \min\{\sigma_1(u_1), \sigma_2(u_2)\} \ \forall (u_1, u_2) \in V$
$(\mu_1 \, X \, \mu_2)((u, u_2), (u, v_2)) = \min\{\sigma_1(u_1), \mu_2(u_2, v_2)\} \ \forall u \in V_1 \ and \ \forall (u_2, v_2) \in X_2 \ and$
$(\mu_1 \, X \, \mu_2)((u_1, w), (v_1, w)) = \min\{\sigma_2(w), \mu_1(u_1, v_1)\} \ \forall w \in V_2 \ and \ \forall (u_1, v_1) \in X_1$
Then the fuzzy graph G($\sigma_1 \, X \, \sigma_2, \mu_1 \, X \, \mu_2$) is said to be the Cartesian product of $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$.

Theorem 3.1

If any two vertices of fuzzy graph G(σ, μ) are connected by exactly one path then every arc of G(σ, μ) is strong.

Proof:

Let G(σ, μ) be a connected fuzzy graph and let n be the number vertices of G.
Take n = 2 , then there must be u and v adjoined by one arc (Since G is connected fuzzy graph)
Clearly , $\mu^\infty(u.v) = \sup\{\mu(u, v)\} = \mu(u, v)$. Therefore, arc (u, v) is strong.
Assume that n > 2. In a fuzzy path, the $\mu^\infty(u.v)$ of any arc in the path will be same fuzzy value $\mu(u, v)$ of the arc (u, v) since it is connected by one path. By the above argument, evidently it is proved that $\mu^\infty(u.v) = \mu(u, v)$ for any number of arcs in a given path .Hence all the arcs are strong.

Theorems 3.2

Let G(σ, μ) be a fuzzy cycle in which lowest fuzzy value of an arc occurs more than once, then that arc must be strong.

Proof:

Let G(σ, μ) be a fuzzy cycle as in figure (ii) and let $\mu(u, v) \leq \mu(v, w) \leq \mu(u, w)$

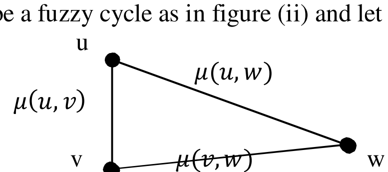
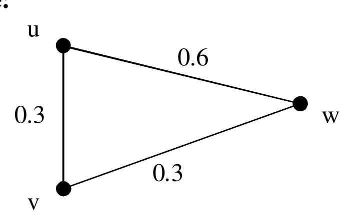


fig (ii)

Suppose , $\mu(u, v) < \mu(v, w) \leq \mu(u, w)$, then obviously arc (u, v) is non strong .
If not , $\mu(u, v) = \mu(v, w)$,then $\mu^\infty(u.v) = \sup\{\mu(u, v), \mu(v, w)\} = \mu(u, v)$
(Since there are two paths connecting u and v and $\mu(u, v) = \mu(v, w)$)
Hence , arc (u, v) is strong (by definition 3.1)

Example:



$\mu(u, v) = \mu(v, w) = 0{,}3$, then $\mu^\infty(u.v) = \sup\{\mu(u, v), \mu(v, w)\}$
$\mu^\infty(u.v) = \sup\{0.3, 0.3\} = 0.3 = \mu(u, v)$
Arc (u, v) is strong .
Note that in the above example, all arcs are strong.

Theorem 3.3

Let G($\sigma_1 \, X \, \sigma_2, \ \mu_1 \, X \, \mu_2$) be the Cartesian product of fuzzy graph $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$. If $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ doesn't have any non strong arc then G($\sigma_1 \, X \, \sigma_2, \mu_1 \, X \, \mu_2$) doesn't have any non strong arc.

In other words, if all the arcs of G_1 and G_2 are strong then all the arcs of G_1 X G_2 are strong.

Proof:

Assume that $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ are as in fig (iii) (a) ,(b)

Fig(iii)(a) (b)

The Cartesian product of fuzzy graph G_1 x G_2 of G_1 and G_2 are drawn as in fig (iv)

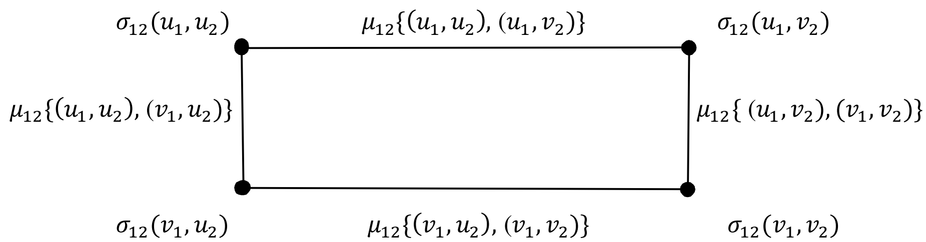


Fig (iv)

Here G_1 and G_2 are fuzzy paths (or any fuzzy graphs) . Clearly all the arcs of G_1 and G_2 are strong.(by theorem 3.1). We have to prove that all the arcs of G_1 X G_2 are strong. It is enough to show that all the four arcs of fig (iv) are strong. The proof of the above hypothesis is discussed in the four cases below.

Case 1:

Suppose $\sigma_1(u_1) < \mu_2(u_2, v_2)$ *and* $\sigma_1(v_1) < \mu_2(u_2, v_2)$ → (1)

By definition 2.1, $\mu_1(u_1, v_1) \le \sigma_1(v_1) \wedge \sigma_1(u_1)$ and $\mu_2(u_2, v_2) \le \sigma_2(u_2) \wedge \sigma_2(v_2)$

Now, $\sigma_1(v_1) < \mu_2(u_2, v_2) \le \sigma_2(u_2) \wedge \sigma_2(v_2)$ (by (1))

Hence, we must have $\sigma_1(u_1) < \sigma_2(u_2)$ *and* $\sigma_1(u_1) < \sigma_2(v_2)$ }
Similarly, $\sigma_1(v_1) < \sigma_2(u_2)$ *and* $\sigma_1(v_1) < \sigma_{s2}(v_2)$ } → (2)

By definition of Cartesian product, a vertex u_1 of $\sigma_1(u_1)$ of G_1 can contribute two new vertices of $\sigma_{12}(u_1, u_2)$, $\sigma_{12}(u_1, v_2)$ and $\mu_{12}\{(u_1, u_2), (u_1, v_2)\}$ in G_1 X G_2 .Similarly , v_1 can be done by the same method.

As in fig (iv) , following are the four arcs ,whose fuzzy values are to be found .

Now, $\mu_1 X \mu_2\{(u_1, u_2), (u_1, v_2)\} = \min\{\sigma_1(u_1), \mu_2(u_2, v_2)\} = \sigma_1(u_1)$ (by (1))

$\mu_1 X \mu_2\{(v_1, u_2), (v_1, v_2)\} = \min\{\sigma_1(v_1), \mu_2(u_2, v_2)\} = \sigma_1(v_1)$ (by (1))

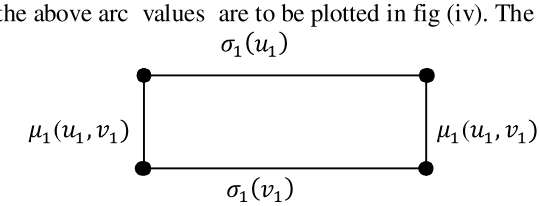
$\mu_1 X \mu_2\{(u_1, u_2), (v_1, u_2)\} = \min\{\mu_1(u_1, v_1), \sigma_2(u_2)\} = \mu_1(u_1, v_1)$

$\mu_1 X \mu_2\{(u_1, v_2), (v_1, v_2)\} = \min\{\mu_1(u_1, v_1), \sigma_2(v_2)\} = \mu_1(u_1, v_1)$

{ For, $\mu_1(u_1, v_1) \le \sigma_1(u_1) \wedge \sigma_1(v_1) < \sigma_2(u_2) \wedge \sigma_2(v_2)$ by (2)

Hence , $\mu_1(u_1, v_1) < \sigma_2(u_2)$ and $\mu_1(u_1, v_1) < \sigma_2(v_2)$}

Now, the above arc values are to be plotted in fig (iv). The fuzzy graph G_1 X G_2 becomes



Since, $\mu_1(u_1, v_1) \le \sigma_1(u_1) \wedge \sigma_1(v_1)$ and (by theorem 3.2, let $G(\sigma, \mu)$ be a fuzzy cycle in which lowest fuzzy value of an arc occurs more than once ,then it must be strong arc.)

Hence, all the arcs are strong in G_1 X G_2

Case 2:

Suppose $\sigma_1(u_1) > \mu_2(u_2, v_2)$ and $\sigma_1(v_1) > \mu_2(u_2, v_2)$ → (3)

$\mu_1 X \mu_2\{(u_1, u_2), (u_1, v_2)\} = \min\{\sigma_1(u_1), \mu_2(u_2, v_2)\} = \mu_2(u_2, v_2)$ (by (3))

$\mu_1 X \mu_2\{(v_1, u_2), (v_1, v_2)\} = \min\{\sigma_1(v_1), \mu_2(u_2, v_2)\} = \mu_2(u_2, v_2)$ (by (3))

$\mu_1 X \mu_2\{(u_1, u_2), (v_1, u_2)\} = \min\{\mu_1(u_1, v_1), \sigma_2(u_2)\}$
$\mu_1 X \mu_2\{(u_1, v_2), (v_1, v_2)\} = \min\{\mu_1(u_1, v_1), \sigma_2(v_2)\}$ } = any fuzzy value but $\geq \mu_2(u_2, v_2)$ (or) $\geq \mu_1(u_1, v_1)$.

For,

If $\mu_1(u_1, v_1) \geq \mu_2(u_2, v_2)$ then we must have $\sigma_2(u_2) \geq \mu_2(u_2, v_2)$, $\sigma_2(v_2) \geq \mu_2(u_2, v_2)$ and if $\mu_1(u_1, v_1) \leq \mu_2(u_2, v_2)$ then we must have $\sigma_2(u_2) \geq \mu_1(u_1, v_1)$, $\sigma_2(v_2) \geq \mu_1(u_1, v_1)$.

In this case $\mu_2(u_2, v_2)$ *or* $\mu_1(u_1, v_1)$ is the lowest fuzzy value of other arcs of G_1 X G_2 and it can occur more than once. By theorem 3.2, all the arcs of G_1 X G_2 are strong.

Case 3:

Suppose, $\sigma_1(u_1) < \mu_2(u_2, v_2)$ and $\sigma_1(v_1) > \mu_2(u_2, v_2)$ → (4)

$\mu_1 X \mu_2\{(u_1, u_2), (u_1, v_2)\} = \min\{\sigma_1(u_1), \mu_2(u_2, v_2)\} = \sigma_1(u_1)$ (by (4))

$\mu_1 X \mu_2\{(v_1, u_2), (v_1, v_2)\} = \min\{\sigma_1(v_1), \mu_2(u_2, v_2)\} = \mu_2(u_2, v_2)$ (by (4))

$\mu_1 X \mu_2\{(u_1, u_2), (v_1, u_2)\} = \min\{\mu_1(u_1, v_1), \sigma_2(u_2)\} = \mu_1(u_1, v_1)$

$\mu_1 X \mu_2\{(u_1, v_2), (v_1, v_2)\} = \min\{\mu_1(u_1, v_1), \sigma_2(v_2)\} = \mu_1(u_1, v_1)$

For, now, $\mu_1(u_1, v_1) \leq \sigma_1(u_1) < \mu_2(u_2, v_2)$, therefore, we must have $\mu_1(u_1, v_1) < \sigma_2(u_2)$ and $\mu_1(u_1, v_1) < \sigma_2(v_2)$. From (4), we also have $\sigma_2(u_2) > \sigma_1(u_1)$, and $\sigma_2(v_2) > \sigma_1(u_1)$.

In this case, $\mu_1(u_1, v_1)$ is the lowest value of all other arcs of G_1 X G_2 and this arc occurs more than once in fuzzy cycle. Hence all the arcs are strong.

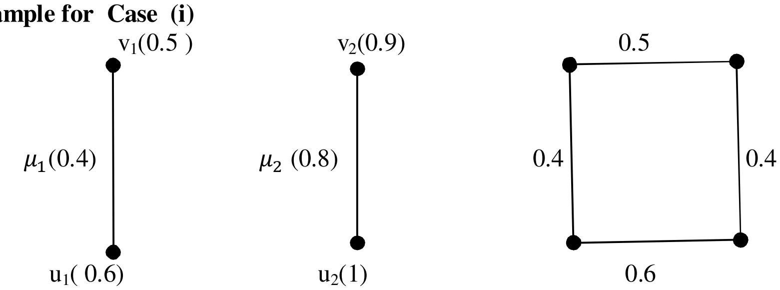
Case 4:

Suppose $\sigma_1(u_1) = \mu_2(u_2, v_2)$ *and* $\sigma_1(v_1) < \mu_2(u_2, v_2)$,
$\sigma_1(u_1) < \mu_2(u_2, v_2)$ *and* $\sigma_1(v_1) = \mu_2(u_2, v_2)$ and $\sigma_1(u_1) = \mu_2(u_2, v_2)$ *and* $\sigma_1(v_1) = \mu_2(u_2, v_2)$.

Evidently, this case can be proved by the above three cases.

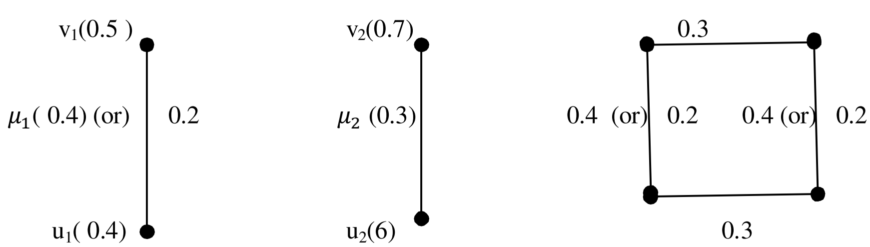
Hence, if all the arcs of G_1 and G_2 are strong then all the arcs of G_1 X G_2 are strong.

Example for Case (i)

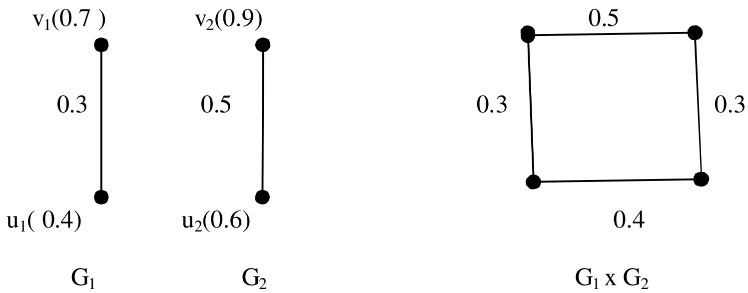


G_1 G_2 G_1XG_2

Example for Case (ii)



Example for Case (iii)



Examples of the above three cases show that if all the arcs of G_1 and G_2 are strong then all the arcs of G_1 X G_2 are also strong.

Note: The **c**onverse need not be true.
If all the arcs in G_1 X G_2 are strong, then it is not necessary that all arcs in G_1 or in G_2 must be strong. In other words, non- strong arc can exist in G_1 or in G_2.

Note: If either G_1 or G_2 have a non-strong arc then G_1 X G_2 need not have non-strong arc.
But , if both have non strong arc then G_1 X G_2 must have non strong arc.

The following theorem proves that existence of non-strong arc in G_1 X G_2.

Theorem 3. 4
Let G_1 and G_2 be two fuzzy graphs of which at least one has a non strong arc, then the existing non - strong arc in G_1 X G_2 depends on the fuzzy value of the vertices of G_1 or G_2 or it depends upon the non-strong arc that appears in G_2 or in G_1 respectively.
That is, though G_1 or G_2 have non strong arc, it is not necessary that G_1 X G_2 must have non -strong arc.

Proof:
To prove this hypothesis, we can take the minimum consideration of fuzzy graph as follows.
Let $G_1(\sigma_1, \mu_1)$ be a fuzzy graph which has one non-strong arc and let $G_2(\sigma_2, \mu_2)$ be another fuzzy graph which has no non-strong arc respectively as shown in figs. (v) (a) and (b).

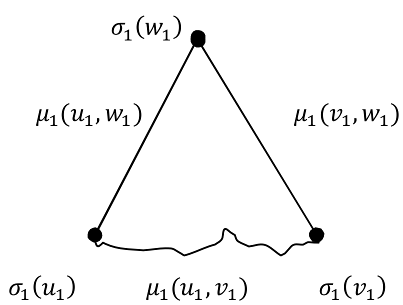
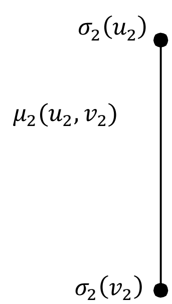


Fig. (v) (a) G_1



(b) G_2

Assume that $\mu_1(u_1, v_1) < \mu_1(v_1, w_1) \leq \mu_1(u_1, w_1)$
Therefore , $\mu_1(u_1, v_1)$ is a non-strong arc of G_1 (by theorem 3.1), there is no non-strong arc in G_2

Claim:
The existing non-strong $\mu_1(u_1, v_1)$ arc in G_1 X G_2 depends on $\sigma_2(u_2)$ and $\sigma_2(v_2)$.
The proof of this claim to be discussed in three cases as below:

Case 1:

Let $\mu_1(u_1,v_1) < \sigma_2(u_2)$ and $\mu_1(u_1,v_1) < \sigma_2(v_2)$. → (i)

Let G($\sigma_1 X \sigma_2$, $\mu_1 X \mu_2$) be the Cartesian product of fuzzy graphs $G_1(\sigma_1,\mu_1)$ and $G_2(\sigma_2,\mu_2)$ drawn in fig (vi).

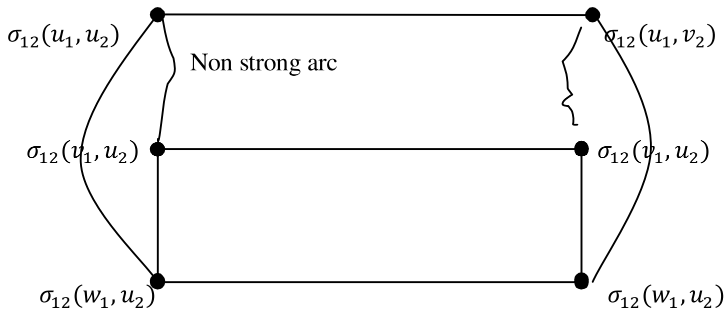


Fig. (vi)

Now, fuzzy values of the arcs of $G_1 X G_2$ are found as follows:

$\mu_1 X \mu_2\{(u_1,u_2),(v_1,u_2)\}$ = min{ $\sigma_2(u_2),\mu_1(u_1,v_1)$} = $\mu_1(u_1,v_1)$ (by (i))

$\mu_1 X \mu_2\{(v_1,u_2),(w_1,u_2)\}$ = min { $\sigma_2(u_2),\mu_1(v_1,w_1)$} = $\mu_1(v_1,w_1)$ (or) $\sigma_2(u_2)$

(Since, $\mu_1(u_1,v_1) < \mu_1(v_1,w_1) \leq \mu_1(u_1,w_1)$ and by(i))

$\mu_1 X \mu_2\{(u_1,u_2),(w_1,u_2)\}$ = min {$\mu_1(u_1,w_1)$, $\sigma_2(u_2)$} = $\mu_1(u_1,w_1)$ (or) $\sigma_2(u_2)$.

These three fuzzy arc formed in one of the fuzzy cycle in $G_1 X G_2$ as in fig (v), in which $\mu_1(u_1,v_1)$ is the lowest value and this becomes non- strong, since $\mu_1(u_1,v_1) < \sigma_2(u_2)$.

Hence, there is one non-strong arc existing in $G_1 X G_2$ when non-strong arc $\mu_1(u_1,v_1)$ in G_1 depends on $\sigma_2(u_2)$ such that $\mu_1(u_1,v_1) < \sigma_2(u_2)$.

Similarly, $\mu_1 X \mu_2\{(u_1,v_2),(v_1,v_2)\}$ = min{ $\sigma_2(v_2),\mu_1(u_1,v_1)$} = $\mu_1(u_1,v_1)$ (by (i))

$\mu_1 X \mu_2\{(v_1,v_2),(w_1,v_2)\}$ = min { $\sigma_2(v_2),\mu_1(v_1,w_1)$} = $\mu_1(v_1,w_1)$ (or) $\sigma_2(v_2)$

(Since, $\mu_1(u_1,v_1) < \mu_1(v_1,w_1) \leq \mu_1(u_1,w_1)$ and by(i))

$\mu_1 X \mu_2\{(u_1,v_2),(w_1,v_2)\}$ = min {$\mu_1(u_1,w_1)$, $\sigma_2(v_2)$} = $\mu_1(u_1,w_1)$ (or) $\sigma_2(v_2)$.

These three fuzzy arcs form another fuzzy cycle in $G_1 X G_2$ as in fig (vi), in which $\mu_1(u_1,v_1)$ is the lowest value and becomes a non-strong arc, since $\mu_1(u_1,v_1) < \sigma_2(v_2)$.

From the above argument, it is concluded that the non-strong arc existing in $G_1 X G_2$ depends on $\sigma_2(u_2)$ and $\sigma_2(v_2)$ of G_2 where non-strong $\mu_1(u_1,v_1)$ arc is in G_1 and therefore there are two non-strong arcs in $G_1 X G_2$, since non-strong arc $\mu_1(u_1,v_1) < \sigma_2(u_2)$ and $\mu_1(u_1,v_1) < \sigma_2(v_2)$.

Case 2:

Let $\mu_1(u_1,v_1) > \sigma_2(u_2)$ and $\mu_1(u_1,v_1) > \sigma_2(v_2)$ → (ii)

Now, the fuzzy value of the arc of $G_1 X G_2$ is found as follows:

$\mu_1 X \mu_2\{(u_1,u_2),(v_1,u_2)\}$ = min{ $\sigma_2(u_2),\mu_1(u_1,v_1)$} = $\sigma_2(u_2)$ (by (ii))

$\mu_1 X \mu_2\{(v_1,u_2),(w_1,u_2)\}$ = min { $\sigma_2(u_2),\mu_1(v_1,w_1)$} = $\sigma_2(u_2)$

(Since, $\mu_1(u_1,v_1) < \mu_1(v_1,w_1) \leq \mu_1(u_1,w_1)$ and by (i)

$\mu_1 X \mu_2\{(u_1,u_2),(w_1,u_2)\}$ = min {$\mu_1(u_1,w_1)$, $\sigma_2(u_2)$} = $\mu_1(u_1,w_1)$) .

These three fuzzy arcs form fuzzy cycle in $G_1 X G_2$ as in fig (vi), in which all the arc values become strong since $\mu_1(u_1,v_1) > \sigma_2(u_2)$ and $\sigma_2(u_2)$ is the lowest value that occurs more than once in $G_1 X G_2$.

Similarly, the same result provided by non-strong arc $\mu_1(u_1,v_1)$ in $G_1$by the Cartesian product with $\sigma_2(v_2)$.

Therefore, in this case no non-strong arc exists in $G_1 X G_2$, since $\mu_1(u_1,v_1) > \sigma_2(u_2)$ and $\mu_1(u_1,v_1) > \sigma_2(v_2)$.

Case 3:

Let $\mu_1(u_1,v_1) = \sigma_2(u_2)$ and $\mu_1(u_1,v_1) = \sigma_2(v_2)$.

Now, the fuzzy value of the arc of $G_1 X G_2$ is found as follows:

$\mu_1 X \mu_2\{(u_1,u_2),(v_1,u_2)\}$ = min{ $\sigma_2(u_2),\mu_1(u_1,v_1)$} = $\sigma_2(u_2) = \mu_1(u_1,v_1)$ (by (ii))

$\mu_1 X \mu_2\{(v_1,u_2),(w_1,u_2)\}$ = min { $\sigma_2(u_2),\mu_1(v_1,w_1)$} = $\sigma_2(u_2) = \mu_1(u_1,v_1)$

(Since, $\mu_1(u_1,v_1) < \mu_1(v_1,w_1) \leq \mu_1(u_1,w_1)$ and by(ii)

$\mu_1 X \mu_2\{(u_1,u_2),(w_1,u_2)\}$ = min {$\mu_1(u_1,w_1)$, $\sigma_2(u_2)$} = $\sigma_2(u_2) = \mu_1(u_1,v_1)$). These three arcs with the same fuzzy value in fuzzy cycle in $G_1 X G_2$ are strong (by theorem 3.2), similarly the same result holds for the other three arcs.

Therefore, in this case also no non-strong arc exist in G_1 X G_2 ,

Case 4:

Let $\mu_1(u_1, v_1) < \sigma_2(u_2)$ and $\mu_1(u_1, v_1) \geq \sigma_2(v_2)$.

From the above cases, in Case 1, there is one non-strong arc existing in G_1 X G_2 when $\mu_1(u_1, v_1) < \sigma_2(u_2)$ and in Cases 2, 3 and 4, all the arcs of G_1 X G_2 are strong, when $\mu_1(u_1, v_1) \geq \sigma_2(v_2)$.

From the above four cases, we conclude that non strong arc exist in G_1 X G_2 only when fuzzy value of non strong arc of G_1 (or G_2) must be greater than the degree value of the vertex of G_2 (or G_1) respectively.

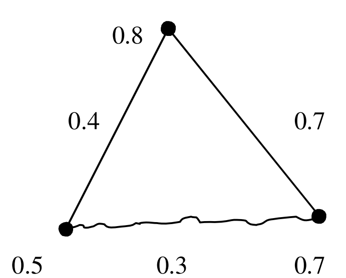
Hence, the existing non-strong arc in G_1 X G_2 depends on the fuzzy value of the vertices of G_1 or G_2 depending upon the non-strong arc that appears in G_2 or in G_1 respectively.

Corollary 3.5

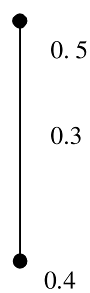
The number of non-strong arc of G_1 X G_2 is $m_1 P_2 + m_2 P_1$,

where, m_1 is the number of vertex of G_1 whose fuzzy value must greater than fuzzy value of non-strong arc in other fuzzy graph G_2 ; m_2 denotes the number of vertices of G_2 whose fuzzy value must be greater than the fuzzy value of the non-strong arc in other fuzzy graph G_1 ; P_1 is the number of non- strong arcs in G_1 and P_2 is the number of non-strong arcs in G_2.

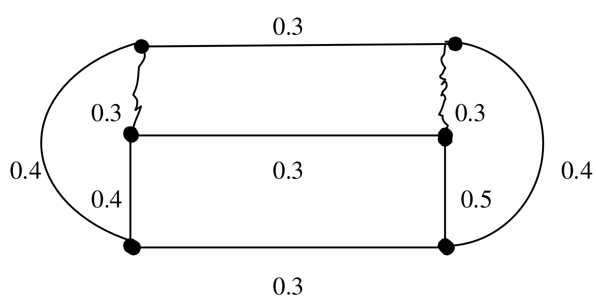
Example for Case 1:



G1

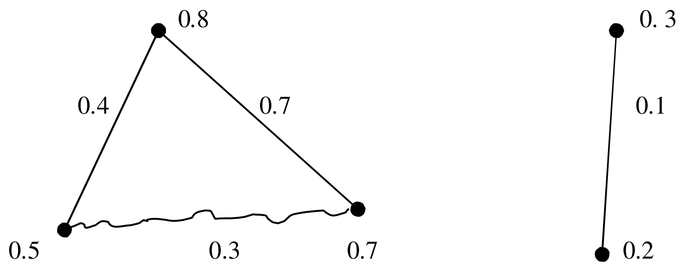


G2

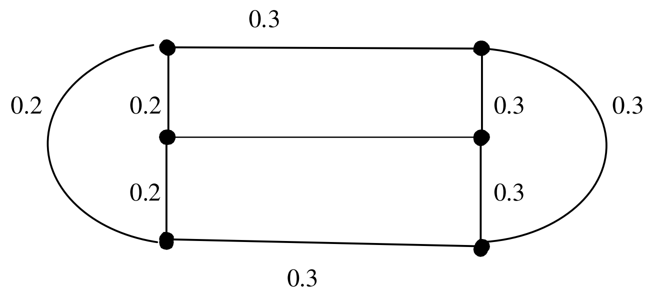


G_1 XG_2 (two non-strong arcs)

Examples for Cases 2 and 3:



G1 G2



$G_1 X G_2$ (no non-strong arc)

5. DOMINATION OF CARTESIAN PRODUCT OF TWO FUZZY PATHS

Let P_n be a fuzzy path of n distinct vertices.

Result11: 1 $n[\gamma_S(G)] = \left\lfloor \frac{2+m}{2} \right\rfloor$, where G= $P_2 x P_m$ **.**

Proof:

Let G = $P_2 x P_m$ be a Cartesian product of fuzzy graphs P_2 and P_m and its graph looks like a ladder form. We know that fuzzy path does not have a non- strong arc. Therefore, G = $P_2 x P_m$ does not have non - strong arc (by theorem). When m =1, the Cartesian product fuzzy graph $P_2 x P_1$ as in fig. (vii)(a). Oobviously $n[\gamma_S(P_2 x P_1)] = 1$. When m=2 , the Cartesian product fuzzy graph $P_2 x P_{2is}$ as in fig (i)(b).

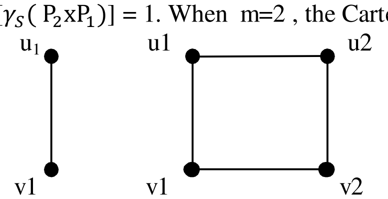
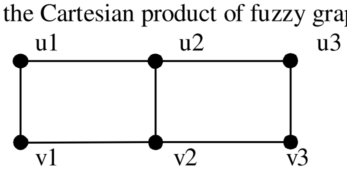


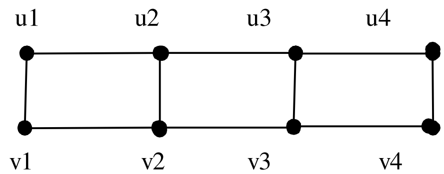
Fig. (vii) (a) (b)

Since this fuzzy graph doesn't have non -strong arc, dominating set can be found in such a way that minimum cardinality exists. Take D ={ u_1 ,u_3} , $n[\gamma_S(P_2 x P_2)] = 2$.

When m= 3 , the Cartesian product of fuzzy graph $P_2 x P_3$ is

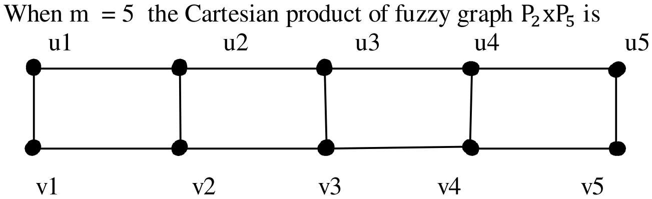


Take D = { v_1, u_3 }, therefore $n[\gamma_S(P_2 x P_3)] = 2$.
When m = 4, the Cartesian product of fuzzy graph $P_2 x P_4$ is



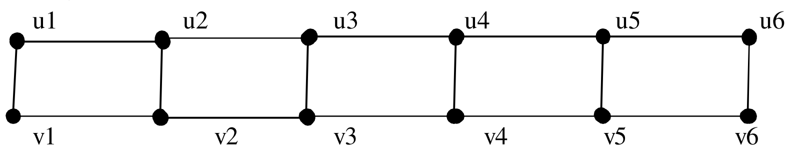
Take D = {v1, u3,v4}, therefore $n[\gamma_S(P_2 x P_4)] = 3$.

When m = 5 the Cartesian product of fuzzy graph $P_2 x P_5$ is



Take D = { v1, u3,u5 } , therefore $n[\gamma_S(P_2 x P_5)] = 3$.

When n = 6



Take D ={ v1, u3, v5 ,v5 ,u6 }, therefore $n[\gamma_S(P_2 x P_6)] = 4$.

Hence ,for different m, we have

$n[\gamma_S(P_2 x P_1)] = 1$
$n[\gamma_S(P_2 x P_2)] = 2$
$n[\gamma_S(P_2 x P_3)] = 2$
$n[\gamma_S(P_2 x P_4)] = 3$
$n[\gamma_S(P_2 x P_5)] = 3$
$n[\gamma_S(P_2 x P_6)] = 4$
$n[\gamma_S(P_2 x P_7)] = 4$
$n[\gamma_S(P_2 x P_8)] = 5$
$n[\gamma_S(P_2 x P_9)] = 5$. Thus follows the General formula $n[\gamma_S(G)] = \left\lfloor \frac{2+m}{2} \right\rfloor$, where G= $P_2 x P_m$.

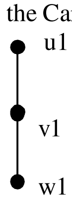
Result 2:

$$G = P_3\, x P_m, \qquad n[\gamma_S(G)] = \begin{cases} m & if \;\; m \in \{1,2,3,4\} \\ m-1 & if \;\; m \in \{5,6,7,8,9\} \\ m-2 & if \;\; m \in \{10,11,12,13,14\} \\ m-3 & if \; m \in \{15,16,17,18,19\} \\ ---\cdots & \\ ---\cdots & \end{cases}$$

Proof:

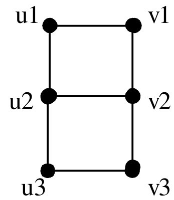
Let G = P_3 x P_m be Cartesian product fuzzy graph of P_3 and P_m which looks like a double ladder. Since a fuzzy path does not have non-strong arc, thus G = P_3 x P_m doesn't have a non-strong arc. Therefore the dominating set can be found in such a way that the minimum cardinality exists.

When m = 1 , the Cartesian product of fuzzy graph P_3 x P_1 is



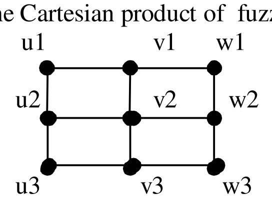
Take D = {v1 } , $n[\gamma_S(P3\ xP1)] = 1$.

When m =2 , the Cartesian product of fuzzy graph $P_3 \times P_2$ is



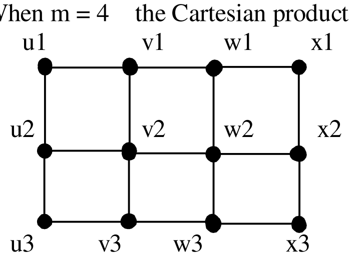
Take D ={ u1, v3} , n[γ_S(P3 xP2)] = 2

When m =3, the Cartesian product of fuzzy graph $P_3 \times P_3$ is



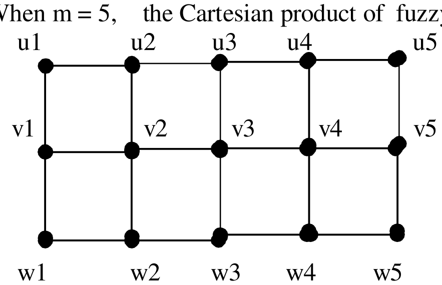
Take, D = { v1,u3,w3 } , n[γ_S(P3 xP3)] = 3.

When m = 4 the Cartesian product of fuzzy graph $P_3 \times P_4$ is



Take D = { u2, w1,w3,x2 }, n[γ_S(P3 xP4)] = 4.

When m = 5, the Cartesian product of fuzzy graph $P_3 \times P_5$ is



Take D = {v1, u3,v5,w3 } , n[γ_S(P3 xP5)] = 4.

For m = 6, 7, 8 … the domination numbers are as listed below:

n[γ_S(P3 xP1)] = 1
n[γ_S(P3 xP2)] = 2
n[γ_S(P3 xP3)] = 3
n[γ_S(P3 xP4)] = 4
n[γ_S(P3 xP5)] = 4
n[γ_S(P3 xP6)] = 5
n[γ_S(P3 xP7)] = 6
n[γ_S(P3 xP8)] = 7
n[γ_S(P3 xP9)] = 8
n[γ_S(P3 xP10)] = 8
n[γ_S(P3 xP11)] = 9
n[γ_S(P3 xP12)] = 10

From where follows the general formula $n[\gamma_S(G)] = \begin{cases} m & if \ \ m \in \{1,2,3,4\} \\ m-1 & if \ \ m \in \{5,6,7,8,9\} \\ m-2 & if \ \ m \in \{10,11,12,13,14\} \\ m-3 & if \ m \in \{15,16,17,18,19\} \\ ---\ldots \\ ---\ldots \end{cases}$

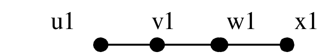
Note: Above dominating sets are found by the T –shape and the reverse of T shape (T-shape means maximum three vertices dominated by a single vertex) such that the minimum dominating set exists. Suppose if we choose + shape (+ shape means maximum four vertices dominated by a single vertex) to find the dominating set D, more isolated vertices have to be added in D and also the domination number will then increase. So in order to find a minimum cardinality set it is better to follow the T- shape process.

Result 3:

$$n[\gamma_S(G)] = \begin{cases} m \ \ if \ m \ is \ multiple \ of \ 4 \\ m+1 \quad otherwise \end{cases}$$, where G= $P_4 x P_m$

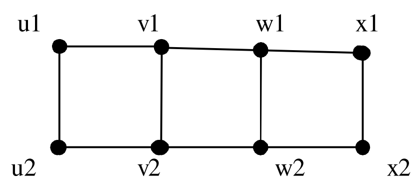
Proof: Let G =P_4x P_m be the Cartesian product of the fuzzy graphs P_4 and P_m . Since a fuzzy path does not have non- strong arc, so G = P_4 x P_m does not have a non-strong arc. Therefore the dominating set can be found in such a way that minimum cardinality exists.

When m = 1 ,Cartesian product fuzzy graph P_4XP_1 is



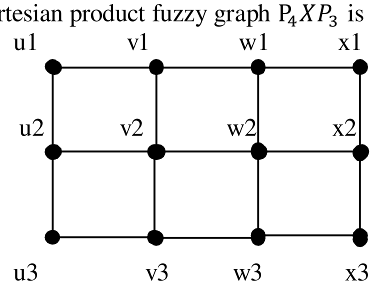
Take D = {v1 ,x1 }, $n[\gamma_S(P_4XP_1)] = 2$.

When m =2 , Cartesian product fuzzy graph P_4XP_2 is



Take D = {v1, v2,x1} , $n[\gamma_S(P_4XP_2)] = 3$.

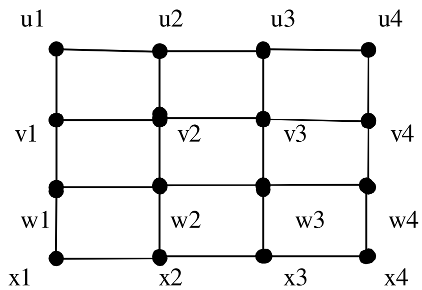
When m =3 , Cartesian product fuzzy graph P_4XP_3 is



Ttake D ={ v1, u2,v3, x2} , n[$\gamma_S(P_4XP_3)] = 4$.

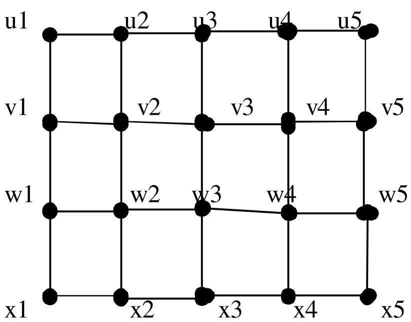
Note: Here after dominating set are found by T –shape (T-shape means maximum three vertices dominated by a single vertex) such that minimum dominating set exist. Suppose if we choose + shape (+ shape means maximum four vertices dominated by a single vertex) to find dominating set D, more isolated vertex to be added in D and domination number will be increased.

When m = 4 , Cartesian product fuzzy graph P_4XP_4 is



Take D = {u2,v4,x3,w1 } , $n[\gamma_S(P_4XP_4)] = 4$.

When m = 5, Cartesian product fuzzy graph P_4XP_5 is



Take D = { u2,v4,x3,w1,v5,x5} , $n[\gamma_S(P_4XP_5)] = 6$.

In general ,for m = 6,7,8….. the domination number

$$n[\gamma_S(P_4\,XP_6)] = 7$$
$$n[\gamma_S(P_4\,XP_7)] = 8$$
$$n[\gamma_S(P_4\,XP_8)] = 8$$
$$n[\gamma_S(P_4\,XP_9)] = 10$$
$$n[\gamma_S(P_4\,XP_{10})] = 11$$

Therefore , $n[\gamma_S(G = P4xPm)] = \begin{cases} m & if\ m\ is\ multiple\ of\ 4 \\ m+1 & otherwise \end{cases}$,
where $G= P_4xP_m$ is the Cartesian product of P_4 and P_m.

6. DOMINATION IN CARTESIAN PRODUCT OF FUZZY PATH AND FUZZY CYCLE

Result 4: (i) $n[\gamma_S(C_3\,XP_m)] \leq m + \left\lceil\frac{0}{3}\right\rceil$

(ii) $n[\gamma_S(C_4\,XP_m)] \leq m + \left\lceil\frac{m}{3}\right\rceil$

(iii) $n[\gamma_S(C_5\,XP_m)] \leq m + n[\gamma_s(P_2XP_m)]$

In general $n[\gamma_S(C_n\,XP_m)] \leq m + n[\gamma_s(P_{n-3}XP_m)]$

Note: In fuzzy cycle, at most one non -strong arc can exist and fuzzy path does not have non-strong arc.

7. DOMINATION IN CARTESIAN PRODUCT OF FUZZY CYCLE AND FUZZY CYCLE

Result 5:

Let G=($C_m\ XC_m$) be the Cartesian product of a fuzzy cycle C_m by itself,
In fuzzy cycle C_m , m is the number vertices of fuzzy cycle.
The domination number of various fuzzy cycles are listed below.

$n[\gamma_s(C_1\ XC_1)] = 1$ $\quad$ $n[\gamma_s(C_{11}\ XC_{11})] = 32$
$n[\gamma_s(C_2\ XC_2)] = 2$ $\quad$ $n[\gamma_s(C_{12}\ XC_{12})] = 36$
$n[\gamma_s(C_3\ XC_3)] = 3$ $\quad$ $n[\gamma_s(C_{13}\ XC_{13})] = 45$
$n[\gamma_s(C_4\ XC_4)] = 4$ $\quad$ $n[\gamma_s(C_{14}\ XC_{14})] = 50$
$n[\gamma_s(C_5\ XC_5)] = 7$ $\quad$ $n[\gamma_s(C_{15}\ XC_{15})] = 58$
$n[\gamma_s(C_6\ XC_6)] = 10$ $\quad$ $n[\gamma_s(C_{16}\ XC_{16})] = 64$
$n[\gamma_s(C_7\ XC_7)] = 13$ $\quad$ $n[\gamma_s(C_{17}\ XC_{17})] = 75$
$n[\gamma_s(C_8\ XC_8)] = 16$ $\quad$ $n[\gamma_s(C_{18}\ XC_{18})] = 82$
$n[\gamma_s(C_9\ XC_9)] = 22$ $\quad$ $n[\gamma_s(C_{19}\ XC_{19})] = 92$
$n[\gamma_s(C_{10}\ XC_{10})] = 26$ $\quad$ $n[\gamma_s(C_{20}\ XC_{20})] = 100$

The general formula for above series is given in tabular form as follows:

When m is divided by4 ,the remainders are 0,1,2,3 respectively . The following table is formed by the remainders 0,1,2,3, lines of the fuzzy cycle C_m..

Remaining 1 line	Remaining 2 line	Remaining 3 line	No line
$n[\gamma_s(C_1\ XC_1)] = 1$ $n[\gamma_s(C_5\ XC_5)] = 7$ $n[\gamma_s(C_9\ XC_9)] = 22$ $n[\gamma_s(C_{13}\ XC_{13})] = 45$ $n[\gamma_s(C_{17}\ XC_{17})] = 75$	$n[\gamma_s(C_2\ XC_2)] = 2$ $n[\gamma_s(C_6\ XC_6)] = 10$ $n[\gamma_s(C_{10}\ XC_{10})] = 26$ $n[\gamma_s(C_{14}\ XC_{14})] = 50$ $n[\gamma_s(C_{18}\ XC_{18})] = 82$	$n[\gamma_s(C_3\ XC_3)] = 3$ $n[\gamma_s(C_7\ XC_7)] = 13$ $n[\gamma_s(C_{11}\ XC_{11})] = 32$ $n[\gamma_s(C_{15}\ XC_{15})] = 58$ $n[\gamma_s(C_{19}\ XC_{19})] = 92$	$n[\gamma_s(C_4\ XC_4)] = 4$ $n[\gamma_s(C_8\ XC_8)] = 16$ $n[\gamma_s(C_{12}\ XC_{12})] = 36$ $n[\gamma_s(C_{16}\ XC_{16})] = 64$ $n[\gamma_s(C_{20}\ XC_{20})] = 100$
m = n +1 ,where n is the multiple of 4	m = n + 2 , where n is the multiple of 4	m = n+3 , where n is the multiple of 4	m = n , where n is the multiple of 4

$n[\gamma_S(C_m\ XC_m)] = \left\lceil\frac{n^2}{4}\right\rceil + \left\lceil\frac{2m-1}{3}\right\rceil$ for remaining one line.

$n[\gamma_S(C_m\ XC_m)] = \left\lceil\frac{n^2}{4}\right\rceil + n[\gamma_s(P_2XP_m)] + n[\gamma_s(P_2XP_{m-2})] - 1$ for remaining 2 lines

$n[\gamma_S(C_m\ XC_m)] = \left\lceil\frac{n^2}{4}\right\rceil + n[\gamma_s(P_3XP_m)] + n[\gamma_s(P_3XP_{m-3})] \quad (or)$

$= \left\lceil\frac{n^2}{4}\right\rceil + n[\gamma_s(P_3XP_m)] + n[\gamma_s(P_3XP_{m-3})] - 1$ for remaining 3 lines

$n[\gamma_S(C_m\ XC_m)] = \left\lceil\frac{n^2}{4}\right\rceil$ for no remaining line.

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