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Solution of the Diophantine equation $22^x+40^y=z^2$ *

Sudhanshu Aggarwal[1,†] and Lalit Mohan Upadhyaya[2]

1. Department of Mathematics, National Post Graduate College, Barhalganj, Gorakhpur, Uttar Pradesh-273402, India.
2. Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun, Uttarakhand -248179, India.

1. E-mail: sudhanshu30187@gmail.com
2. E-mail: lmupadhyaya@rediffmail.com , hetchres@gmail.com

Abstract Since Diophantine equations play an important role in solving important real-world problems such as business investment problems, network flow problems, pole placement problems, and data privacy problems, researchers are increasingly interested in developing new techniques for analyzing the nature and solutions of the various Diophantine equations. In this study we investigate the Diophantine problem $22^x + 40^y = z^2$, where x, y, z are non-negative integers, and discover that it does not have a non-negative integer solution.

Key words Catalan's Conjecture, Diophantine Equation, Solution.

2020 Mathematics Subject Classification 11D61.

1 Introduction

Diophantine equations (linear or non-linear) can be used to solve a wide range of problems in astronomy, algebra, and trigonometry [1]. Aggarwal et al. [2] found that the Diophantine equation $223^x+241^y = z^2$ has no solution in the set of non-negative integers. Aggarwal et al. [3] also thoroughly analyzed the Diophantine problem $181^x + 199^y = z^2$. Besides this Aggarwal and Sharma [4] also found that the non-linear Diophantine equation $379^x + 397^y = z^2$ has no non-negative integer solutions. Aggarwal [5] had also investigated the Diophantine problem $193^x + 211^y = z^2$.

The exponential Diophantine equation $\left(13^{2m}\right) + (6r + 1)^n = z^2$ was explored by Aggarwal and Kumar [6], who established that it cannot be solved in the set of non-negative integers. In one of their studies Aggarwal and Upadhyaya [7] took the Diophantine equation $8^\alpha + 67^\beta = \gamma^2$ into account and established that it had a single non-negative integer solution. The Diophantine problem $M_5{}^p + M_7{}^q = r^2$ was explored by Goel et al. [8] using the arithmetic modular technique.

The Diophantine equation $421^p + 439^q = r^2$ has no solution in non-negative integers, according to Bhatnagar and Aggarwal [9]. The non-linear exponential Diophantine equation $(x^a + 1)^m + \left(y^b + 1\right)^n = z^2$ was explored by Gupta et al. [10] who reported that this equation lacks a non-negative integer

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†Corresponding author Sudhanshu Aggarwal, E-mail: sudhanshu30187@gmail.com

solution. The non-linear exponential Diophantine equation $x^{\alpha} + (1 + my)^{\beta} = z^2$ was investigated by Gupta et al. [11].

The Diophantine equation $(p^q - 1)^x + p^{qy} = z^2$ was researched by Hoque and Kalita [12]. The Diophantine equation $601^p + 619^q = r^2$ was studied by Kumar et al. [13] for non-negative integer solutions. The Diophantine equation $\left(2^{2m+1} - 1\right) + \left(6^{r+1} + 1\right)^n = \omega^2$ has no non-negative integer solutions, according to Kumar et al. [14]. The Diophantine equation $\left(7^{2m}\right) + (6r + 1)^n = z^2$ is not solvable in the set of non-negative integers according to Kumar et al. [15].

Mishra et al. [16] analyzed the Diophantine equation $211^{\alpha} + 229^{\beta} = \gamma^2$, and they demonstrated that it cannot be solved in the set of non-negative integers. Sroysang [18–22] looked at a variety of Diophantine equations, including $323^x + 325^y = z^2$, $3^x + 45^y = z^2$, $143^x + 145^y = z^2$, $3^x + 85^y = z^2$ and $4^x + 10^y = z^2$ for non-negative integer solutions. The Diophantine equation $143^x + 45^y = z^2$ was examined by Aggarwal et al. [23]. They demonstrated that it has a single solution in the set of non-negative integers.

Another Diophantine equation $143^x + 485^y = z^2$ was examined by Aggarwal et al. [24]. The Diophantine problem $143^x + 85^y = z^2$ has just one solution in the set of non-negative integers, according to a recent study of Aggarwal et al. [25]. Recently Aggarwal et al. [26] solved the Diophantine equation $\beta^x + (\beta + 18)^y = z^2$ completely.

This primary goal of this paper is to investigate the non-negative integer solution to the Diophantine problem $22^x + 40^y = z^2$, where x, y, z are non-negative integers.

2 Preliminaries

Proposition 2.1. *Catalan's Conjecture [17]: The Diophantine equation $a^x - b^y = 1$, where $a,\ b,\ x$ and y are integers such that min$\{a,\ b,\ x, y\} > 1$, has a unique solution $(a,\ b,\ x, y) = (3,\ 2,\ 2,\ 3)$.*

Lemma 2.2. *The Diophantine equation $22^x + 1 = z^2$, where x, z are non-negative integers, has no solution in non-negative integers.*

Proof. Suppose that x, z are non-negative integers such that $22^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Now $z^2 = 22^x + 1 \geq 22^1 + 1 = 23$. Thus $z \geq 5$. Now, we consider the equation $z^2 - 22^x = 1$. By Proposition 2.1, we have $x = 1$. It follows that $z^2 = 23$. This is a contradiction. Hence, the Diophantine equation $22^x + 1 = z^2$, where x, z are non-negative integers, has no non-negative integer solution. □

Lemma 2.3. *The Diophantine equation $40^y + 1 = z^2$, where y, z are non-negative integers, has no non-negative integer solution.*

Proof. Suppose that y, z are non-negative integers such that $40^y + 1 = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Now $z^2 = 40^y + 1 \geq 40^1 + 1 = 41$. Thus $z \geq 7$. Now, we consider the equation $z^2 - 40^y = 1$. By Proposition 2.1, we have $y = 1$. It follows that $z^2 = 41$. This is a contradiction. Hence, the Diophantine equation $40^y + 1 = z^2$, where y, z are non-negative integers, has no non-negative integer solution. □

3 Main results

Theorem 3.1. *The Diophantine equation $22^x + 40^y = z^2$, where x, y, z are non-negative integers, has no solution in non-negative integers.*

Proof. Let x, y, z be non-negative integers such that $22^x + 40^y = z^2$. By Lemma 2.2 and Lemma 2.3, we have $x \geq 1,\ y \geq 1$. This implies that z is even. Then $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$. We note that $22^x \equiv 1 \pmod 3$ and $40^y \equiv 1 \pmod 3$. This implies that $z^2 \equiv 2 \pmod 3$. This is a contradiction. Hence, the Diophantine equation $22^x + 40^y = z^2$, where x, y, z are non-negative integers, has no solution in non-negative integers. □

Corollary 3.2. *The Diophantine equation $22^x + 40^y = w^4$, where $x,\ y, w$ are non-negative integers, has no non-negative integer solution.*

Proof. Let $x,\ y, w$ be non-negative integers such that $22^x + 40^y = w^4$. Let $z = w^2$. Then the equation $22^x + 40^y = w^4$ becomes $22^x + 40^y = z^2$. By Theorem 3.1, this equation has no solution in non-negative integers. Hence, the Diophantine equation $22^x + 40^y = w^4$, where $x,\ y, w$ are non-negative integers, has no non-negative integer solution. ☐

Corollary 3.3. *Let k be a positive integer. Then the Diophantine equation $22^x + 40^y = w^{2k}$, where $x,\ y, w$ are non-negative integers, has no non-negative integer solution.*

Proof. Let $x,\ y, w$ be non-negative integers such that $22^x + 40^y = w^{2k}$. Let $z = w^k$. Then the equation $22^x + 40^y = w^{2k}$ becomes $22^x + 40^y = z^2$. By Theorem 3.1, this equation has no solution in non-negative integers. Hence, the Diophantine equation $22^x + 40^y = w^{2k}$, where x, y, w are non-negative integers, has no non-negative integer solution. ☐

Corollary 3.4. *Let k be a positive integer. Then the Diophantine equation $22^x + 40^y = w^{2k+2}$, where x, y, w are non-negative integers, has no non-negative integer solution.*

Proof. Let $x,\ y, w$ be non-negative integers such that $22^x + 40^y = w^{2k+2}$. Let $z = w^{k+1}$. Then the equation $22^x + 40^y = w^{2k+2}$ becomes $22^x + 40^y = z^2$. By Theorem 3.1, this equation has no solution in non-negative integers. Hence, the Diophantine equation $22^x + 40^y = w^{2k+2}$, where x, y, w are non-negative integers, has no non-negative integer solution. ☐

4 Conclusion

With the aid of the celebrated Catalan's Conjecture, the authors in this work showed that the Diophantine equation $22^x + 40^y = z^2$, where x, y, and z are non-negative integers, does not have a solution in non-negative integers.

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