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ON FUNCTIONS OF A SINGLE MATRIX ARGUMENT - III

Lalit Mohan Upadhyaya[1,*], Ayman Shehata[2]

Author Affiliation:
[1]Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun, Uttarakhand 248179, India.
E-mail: lmupadhyaya@rediffmail.com, hetchres@gmail.com
[2]Department of Mathematics, Faculty of Science, Assiut University, Assiut 1516, Egypt.
[2]Department of Mathematics, College of Science and Arts in Unaizah, Qassim University, Qassim 10363, Kingdom of Saudi Arabia.
E-mail: drshehata2006@yahoo.com

***Corresponding Author:**
Lalit Mohan Upadhyaya, Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun, Uttarakhand 248179, India.
E-mail: lmupadhyaya@rediffmail.com, hetchres@gmail.com

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Abstract

We prove some Eulerian integrals involving the hypergeometric functions of single matrix argument. The integrals studied here involve the ${}_0F_1, {}_1F_2, {}_2F_3, {}_1F_1, {}_3F_3, {}_2F_1$ and the ${}_3F_2$ functions of matrix argument. We apply the Mathai's matrix transform technique to prove our results which provide a generalization of the corresponding results available in the literature for these functions for the case of scalar variables. At the end we also give the corresponding results when the argument matrices are complex Hermitian positive definite.

Keywords: hypergeometric functions, matrix argument, matrix transform, real positive definite, Hermitian positive definite.

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1. INTRODUCTION

The varied applications of hypergeometric functions of matrix argument in statistical distribution theory and other branches of mathematics make them a favorable topic of study. Herz [1] wrote his classic paper on Bessel functions of matrix arguments more than six decade ago. He defined the hypergeometric functions of matrix argument by a pair of integral equations using the Laplace transform technique. James [2, 3, 4] has used the zonal polynomial technique for the development of functions of single matrix argument. Dhami and Rawat [5] and Dhami [6] applied the confluent hypergeometric function of matrix argument to the study of Non-central Wishart distribution, Joshi and Joshi [7] also studied the confluent hypergeometric function of matrix argument and established a confluence in this paper. The first author has earlier proved a number of results for functions of single matrix argument in his doctoral dissertation [9, Chapter II (see also [8])] which have appeared systematically in two of his earlier papers of this series [10,11]. We now present some more results for functions of single matrix argument which involve the ${}_0F_1, {}_1F_2, {}_2F_3, {}_1F_1, {}_3F_3, {}_2F_1$ and the ${}_3F_2$ functions of matrix argument. We apply the Matahi's matrix transform technique to prove our results in this paper. Towards the end of the paper we also state the corresponding results when the argument matrices are Hermitian positive definite (see [18]). These results can be proved in a manner parallel to that adopted by us here with necessary tools as pointed out in the third section of this paper.

The preliminary material and notations of symbols used in the paper are explained below in the introductory section of the paper. The main results in the form of five theorems are stated and proved in the second section of the paper while, the third section of the paper deals with the corresponding results for functions of complex matrix arguments. We mention that all the matrices appearing in the first and second sections of this paper are real symmetric positive matrices of order $(p \times p)$ while those appearing in the third section of this paper are Hermitian positive definite of order $(p \times p)$. $A^{1/2}$ will represent the symmetric square root of the matrix A and A' denotes the transpose of the matrix A. $\text{Re}(.)$ denotes the real part of $(.)$, while $|A|$ denotes the determinant of the matrix A. Mathai [12] in 1978 defined the matrix transform (M- transform) of a function $f(X)$ of a $(p \times p)$ real symmetric positive definite matrix X as follows:

$$M_f(\rho) = \int_{X>0} |X|^{\rho-(p+1)/2} f(X) dX \tag{1.1}$$

for $X > 0$ and $\text{Re}(\rho) > (p-1)/2$, whenever $M_f(s)$ exists.

The following results and definition will be used by us at various places in this paper.

Theorem 1.1: Mathai [13, (2.24), p.23] - Let X and Y be $(p \times p)$ symmetric matrices of functionally independent real variables and A a $(p \times p)$ non singular matrix of constants. Then,

$$Y = AXA' \Rightarrow dY = |A|^{p+1} dX \tag{1.2}$$

and

$$Y = aX \Rightarrow dY = a^{p(p+1)/2} dX \tag{1.3}$$

where a is a scalar quantity.

Theorem 1.2: Gamma integral (Mathai [14, (2.1.3), p.33 and (2.1.2), p. 32]) -

$$\int_{X>0} |X|^{\alpha-(p+1)/2} e^{-tr(BX)} dX = |B|^{-\alpha} \Gamma_p(\alpha) \tag{1.4}$$

for $\text{Re}(\alpha) > (p-1)/2$, where,

$$\Gamma_p(\alpha) = \pi^{p(p-1)/4} \Gamma(\alpha)\Gamma(\alpha - \tfrac{1}{2}) \cdots \Gamma(\alpha - \tfrac{p-1}{2}) \tag{1.5}$$

for $\text{Re}(\alpha) > (p-1)/2$ and $tr(X)$ denotes the trace of the matrix X.

Theorem 1.3: Type-1 Beta Integral (Mathai [14, (2.2.2), p.34])-

$$B_p(\alpha,\beta) = \int_{0<X<I} |X|^{\alpha-(p+1)/2} |I - X|^{\beta-(p+1)/2} dX = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.6}$$

for $\text{Re}(\alpha) > (p-1)/2, \text{Re}(\beta) > (p-1)/2$.

Theorem 1.4: Type-2 Beta Integral: (Mathai [14, (2.2.4), p.36 and (2.2.2), p.34])- For a real symmetric positive definite matrix Y

$$B_p(\alpha,\beta) = \int_{Y>0} |Y|^{\alpha-(p+1)/2} |I + Y|^{-(\alpha+\beta)} dY = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.7}$$

for $\text{Re}(\alpha,\beta) > (p-1)/2$.

Definition 1.5: (Mathai [14, (2.3.5), p.38]) The ${}_rF_s$ function of matrix arguments

$${}_rF_s = {}_rF_s\left(a_1,\cdots,a_r;b_1,\cdots,b_s;-X\right)$$

is defined as that class of functions which has the following matrix transform:

$$M\left({}_rF_s\right)=\int_{X>0}\left|X\right|^{\rho-(p+1)/2}{}_rF_s\left(a_1,\cdots,a_r;b_1,\cdots,b_s;-X\right)dX$$
$$=\frac{\left\{\prod_{k=1}^{s}\Gamma_p\left(b_k\right)\right\}\left\{\prod_{m=1}^{r}\Gamma_p\left(a_m-\rho\right)\right\}}{\left\{\prod_{k=1}^{s}\Gamma_p\left(b_k-\rho\right)\right\}\left\{\prod_{m=1}^{r}\Gamma_p\left(a_m\right)\right\}}\Gamma_p\left(\rho\right) \tag{1.8}$$

for $\mathrm{Re}(\rho,a_m-\rho,b_k-\rho)>(p-1)/2$, where, $m=1,\cdots,r;k=1,\cdots,s$.

Theorem 1.6: (Mathai [14, (6.13), p. 84]- For $p=2$,

$$4^{-p\rho}\frac{\Gamma_p\left(\frac{a+1}{2}-\rho\right)\Gamma_p\left(\frac{a}{2}+\frac{1}{4}-\rho\right)}{\Gamma_p\left(\frac{a+1}{2}\right)\Gamma_p\left(\frac{a}{2}+\frac{1}{4}\right)}=\frac{\Gamma_p\left(a-2\rho\right)}{\Gamma_p\left(a\right)} \tag{1.9}$$

2. RESULTS FOR THE FUNCTIONS OF SINGLE MATRIX ARGUMENT

Now we state and prove below five results for the various hypergeometric functions of single matrix argument as stated in the previous section of the paper. We mention here that the results of the Theorems 2.2 and 2.3 below hold good only for the case of (2×2) real symmetric positive definite matrices and this has also been specifically mentioned in the statements of these two Theorems by writing that, in these theorems $p=2$. All the results of this section generalize the corresponding results available in the literature (see, for instance, [19]) for functions of scalar arguments to the case of functions of real symmetric positive definite matrices as arguments. The results presented in the Theorems 2.2 and 2.3 are different from the corresponding results in the scalar case.

Theorem 2.1:

$${}_1F_2\left[a;a+b,c;-X\right]=\frac{\Gamma_p\left(a+b\right)}{\Gamma_p\left(a\right)\Gamma_p\left(b\right)}\int_0^I\left|U\right|^{a-(p+1)/2}\left|I-U\right|^{b-(p+1)/2}{}_0F_1\left[-;c;-U^{1/2}XU^{1/2}\right]dU \tag{2.1}$$

for $\mathrm{Re}\left(a,b\right)>\left(p-1\right)/2$.

Proof: We take the matrix transform (M-transform) of the right side of (2.1) with respect to the variable X and parameter ρ to get

$$\int_{X>0}\left|X\right|^{\rho-(p+1)/2}{}_0F_1\left[-;c;-U^{1/2}XU^{1/2}\right]dX \tag{2.2}$$

Applying the transformation $Y=U^{1/2}XU^{1/2}$ to the above expression with $dY=\left|U\right|^{(p+1)/2}dX$ (from (1.2)) and $\left|Y\right|=\left|U\right|\left|X\right|$ we get

$$\left|U\right|^{-\rho}\int_{Y>0}\left|Y\right|^{\rho-(p+1)/2}{}_0F_1\left[-;c;-Y\right]dY$$

which, on writing the M-transform of the ${}_0F_1$ function with the help of (1.8), yields

$$\left|U\right|^{-\rho}\frac{\Gamma_p\left(c\right)\Gamma_p\left(\rho\right)}{\Gamma_p\left(c-\rho\right)} \tag{2.3}$$

Substituting this expression on the right side of (2.1) gives

$$\frac{\Gamma_p\left(a+b\right)\Gamma_p\left(c\right)\Gamma_p\left(\rho\right)}{\Gamma_p\left(a\right)\Gamma_p\left(b\right)\Gamma_p\left(c-\rho\right)}\int_0^I\left|U\right|^{a-\rho-(p+1)/2}\left|I-U\right|^{b-(p+1)/2}dU$$

in which the variable U can be integrated out with the help of (1.6) to give

$$\frac{\Gamma_p(a+b)\Gamma_p(c)\Gamma_p(\rho)\Gamma_p(a-\rho)}{\Gamma_p(a)\Gamma_p(a+b-\rho)\Gamma_p(c-\rho)} \tag{2.4}$$

Now taking the M-transform of the left side of (2.1) with respect to the variable X and the parameter ρ and writing the M-transform of the ${}_1F_2$ function with the help of (1.8) produces the same result as in (2.4) above. ■

Theorem 2.2: For $p=2$,

$${}_2F_3\left[a,b;\frac{a+b+1}{2},\frac{a+b}{2}+\frac{1}{4},c;-\frac{X}{4}\right]=\frac{\Gamma_p(a+b)}{\Gamma_p(a)\Gamma_p(b)}\int_0^I |U|^{a-(p+1)/2}|I-U|^{b-(p+1)/2}\times$$
$${}_0F_1\left[-;c;-(I-U)^{1/2}U^{1/2}XU^{1/2}(I-U)^{1/2}\right]dU \tag{2.5}$$

for $\operatorname{Re}(a,b)>(p-1)/2$.

Proof: Taking the M-transform of the right side of (2.5) with respect to the variable X and the parameter ρ we have

$$\int_{X>0}|X|^{\rho-(p+1)/2}\,{}_0F_1\left[-;c;-(I-U)^{1/2}U^{1/2}XU^{1/2}(I-U)^{1/2}\right]dX \tag{2.6}$$

Applying the transformation $Y=(I-U)^{1/2}U^{1/2}XU^{1/2}(I-U)^{1/2}$, so that, $dY=|I-U|^{(p+1)/2}|U|^{(p+1)/2}dX$ and $|Y|=|I-U||U||X|$, which renders (2.6) with the utilization of (1.8) as

$$|I-U|^{-\rho}|U|^{-\rho}\frac{\Gamma_p(c)\Gamma_p(\rho)}{\Gamma_p(c-\rho)} \tag{2.7}$$

Substituting this expression on the right side of (2.5) and integration U in the resulting expression by the help of a type-1 Beta integral (i.e. (1.6)) yields

$$\frac{\Gamma_p(a+b)\Gamma_p(c)\Gamma_p(\rho)\Gamma_p(a-\rho)\Gamma_p(b-\rho)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c-\rho)\Gamma_p(a+b-2\rho)} \tag{2.8}$$

Now taking the M-transform of the left side of (2.5) with respect to the variable X and the parameter ρ gives

$$\int_{X>0}|X|^{\rho-(p+1)/2}\,{}_2F_3\left[a,b;\frac{a+b+1}{2},\frac{a+b}{2}+\frac{1}{4},c;-\frac{X}{4}\right]dX \tag{2.9}$$

This expression, on applying the transformation $Z=\frac{X}{4}$ with $dZ=\frac{1}{4^{p(p+1)/2}}dX$ (from (1.3)) and $|Z|=\frac{1}{4^p}|X|$ followed by the use of (1.8) for writing the M-transform of a ${}_2F_3$ function produces

$$4^{p\rho}\frac{\Gamma_p\left(\frac{a+b}{2}+\frac{1}{2}\right)\Gamma_p\left(\frac{a+b}{2}+\frac{1}{4}\right)\Gamma_p(c)\Gamma_p(a-\rho)\Gamma_p(b-\rho)\Gamma_p(\rho)}{\Gamma_p\left(\frac{a+b}{2}+\frac{1}{2}-\rho\right)\Gamma_p\left(\frac{a+b}{2}+\frac{1}{4}-\rho\right)\Gamma_p(c-\rho)\Gamma_p(a)\Gamma_p(b)} \tag{2.10}$$

which on simplification with the help of (1.9) gives the same result as in (2.8) above. ■

Theorem 2.3: For $p=2$,

$${}_3F_3\left[c,a,b;d,\frac{a+b+1}{2},\frac{a+b}{2}+\frac{1}{4};-\frac{X}{4}\right]=\frac{\Gamma_p(a+b)}{\Gamma_p(a)\Gamma_p(b)}\int_0^I |U|^{a-(p+1)/2}|I-U|^{b-(p+1)/2}\times$$

$$ {}_1F_1\left[c;d;-(I-U)^{1/2}U^{1/2}XU^{1/2}(I-U)^{1/2}\right]dU \quad (2.11)$$

for $\mathrm{Re}(a,b)>(p-1)/2$.

Proof: Taking the M-transform of the right side of (2.11) with respect to the variable X and the parameter ρ we get

$$\int_{X>0}|X|^{\rho-(p+1)/2}\,{}_1F_1\left[c;d;-(I-U)^{1/2}U^{1/2}XU^{1/2}(I-U)^{1/2}\right]dX \quad (2.12)$$

which on the application of the transformation $Y=(I-U)^{1/2}U^{1/2}XU^{1/2}(I-U)^{1/2}$ as in the Theorem 2.2 and writing the M-transform of the ${}_1F_1$ function by appealing to (1.8) generates

$$\frac{\Gamma_p(a+b)\Gamma_p(c-\rho)\Gamma_p(d)\Gamma_p(\rho)\Gamma_p(a-\rho)\Gamma_p(b-\rho)}{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c)\Gamma_p(d-\rho)\Gamma_p(a+b-2\rho)} \quad (2.13)$$

Again taking the M-transform of the left side of (2.11) with respect to the variable X and the parameter ρ gives

$$\int_{X>0}|X|^{\rho-(p+1)/2}\,{}_3F_3\left[c,a,b;d,\frac{a+b+1}{2},\frac{a+b}{2}+\frac{1}{4};-\frac{X}{4}\right]dX \quad (2.14)$$

which on applying the transformation $Z=\frac{X}{4}$ as in the Theorem 2.2 and writing the M-transform of the ${}_3F_3$ function by the help of (1.8) yields

$$4^{p\rho}\frac{\Gamma_p(d)\Gamma_p\left(\frac{a+b}{2}+\frac{1}{2}\right)\Gamma_p\left(\frac{a+b}{2}+\frac{1}{4}\right)\Gamma_p(c-\rho)\Gamma_p(a-\rho)\Gamma_p(b-\rho)\Gamma_p(\rho)}{\Gamma_p(d-\rho)\Gamma_p\left(\frac{a+b}{2}+\frac{1}{2}-\rho\right)\Gamma_p\left(\frac{a+b}{2}+\frac{1}{4}-\rho\right)\Gamma_p(c)\Gamma_p(a)\Gamma_p(b)} \quad (2.15)$$

This expression on simplification with the help of (1.9) returns the same result as in (2.13). ■

Theorem 2.4:

$$|I+X|^{-a}=\frac{\Gamma_p(b)}{\Gamma_p(c)\Gamma_p(b-c)}\int_0^I|U|^{c-(p+1)/2}|I-U|^{b-c-(p+1)/2}\,{}_2F_1\left[a,b;c;-U^{1/2}XU^{1/2}\right]dU \quad (2.16)$$

for $\mathrm{Re}(c,b-c)>(p-1)/2$.

Proof: Taking the M-transform of the right side of (2.16) with respect to the variable X and the parameter ρ we have

$$\int_{X>0}|X|^{\rho-(p+1)/2}\,{}_2F_1\left[a,b;c;-U^{1/2}XU^{1/2}\right]dX \quad (2.17)$$

which on the application of the transformation $Y=U^{1/2}XU^{1/2}$ and on writing the M-transform of a ${}_2F_1$ function by the use of (1.8) generates

$$|U|^{-\rho}\frac{\Gamma_p(c)\Gamma_p(a-\rho)\Gamma_p(b-\rho)\Gamma_p(\rho)}{\Gamma_p(c-\rho)\Gamma_p(a)\Gamma_p(b)} \quad (2.18)$$

Substituting this expression on the right side of (2.16) and integrating U with the help of (1.6) gives

$$\frac{\Gamma_p(a-\rho)\Gamma_p(\rho)}{\Gamma_p(a)} \quad (2.19)$$

Now taking the M-transform of the left side of (2.16) with respect to the variable X and the parameter ρ gives

$$\int_{X>0}|X|^{\rho-(p+1)/2}|I+X|^{-(\rho+(a-\rho))}dX \quad (2.20)$$

which on integrating X by the help of the type-2 Beta integral ((1.7)) gives the same result as in (2.19).

■

Theorem 2.5:

$$ {}_3F_2\left[a,b,e;c,d+e;-X\right]=\frac{\Gamma_p\left(e+d\right)}{\Gamma_p\left(e\right)\Gamma_p\left(d\right)}\int_0^I \left|U\right|^{e-(p+1)/2}\left|I-U\right|^{d-(p+1)/2}\times {}_2F_1\left[a,b;c;-U^{1/2}XU^{1/2}\right]dU \qquad (2.21) $$

for $\text{Re}\left(e,d\right)>\left(p-1\right)/2$.

Proof: We take the M-transform of the right side of (2.21) with respect to the variable X and the parameter ρ to get the same expression as in (2.17). The application of the transformation $Y=U^{1/2}XU^{1/2}$ in this expression produces the same expression as in (2.18). Finally substituting the expression thus obtained in the right side of (2.21) and integrating U in the consequent expression by (1.6) produces

$$ \frac{\Gamma_p\left(e+d\right)\Gamma_p\left(c\right)\Gamma_p\left(a-\rho\right)\Gamma_p\left(b-\rho\right)\Gamma_p\left(e-\rho\right)\Gamma_p\left(\rho\right)}{\Gamma_p\left(e\right)\Gamma_p\left(c-\rho\right)\Gamma_p\left(a\right)\Gamma_p\left(b\right)\Gamma_p\left(d+e-\rho\right)} \qquad (2.22) $$

which can be seen to be $M\left({}_3F_2\left[a,b,e;c,d+e;-X\right]\right)$ with the help of (1.8). ■

3. CORRESPONDING RESULTS FOR THE FUNCTIONS OF COMPLEX MATRIX ARGUMENT

In this section we proceed to give the corresponding results for the above cases when the argument matrices are Hermitian positive definite (see also section 4, pp. 213-215 of [18]). In this section of the paper, all the matrices are Hermitian positive definite matrices of order $\left(p\times p\right)$. We use the same notation for matrices (complex matrices) here as are used in the previous sections of this paper, unlike Mathai [16], where the matrices having complex entries are shown by placing a tilde (~) sign over the notation of the matrix concerned. For results concerning the Jacobians of matrix transformations in the case of matrices when their elements are complex quantities, we refer the reader to Chapter 3 of Mathai [16]. The analogues of all the results mentioned in (1.1) to (1.8) of this paper for the corresponding case of complex matrices can be found in Chapter 3 and Chapter 6 of Mathai [16]. We also mention here that for the results developed in the Theorems 2.2 and 2.3, which hold explicitly only for $\left(2\times 2\right)$ matrices, the complex analogue of the result in (1.9) (in fact, the complex case of the analogue of Lemma 5.4 of Chapter 5 p. 340 of Mathai [16]) has a different structure (see also Mathai [16], Chapter 6, p.399) therefore the corresponding complex analogues of the results given in the Theorems (2.2) and (2.3) will have different structures hence, they are not given here. We also find it worth mentioning here that the complex analogues of the results in the Theorems 2.1, 2.4 and 2.5 can be most easily written down with the help of the results in (2.1), (2.16) and (2.21) merely by replacing the expression $\left(p+1\right)/2$ appearing in the power of the determinant of the matrix by p and in the condition of convergence of the integral, the expression $\text{Re}\left(.\right)>\left(p-1\right)/2$ has to be replaced by $\text{Re}\left(.\right)>\left(p-1\right)$ (see Mathai [16], pp. 364-365 and see also Mathai and Provost [17]). Using these observations, which are already very well established in the literature before hand (e.g., Mathai [16] and Mathai and Provost [17]), we mention below, without proofs, the complex analogues of the results in (2.1), (2.16) and (2.21) in the form of the statements of the Theorem 3.1, 3.2 and 3.3 respectively. All these results can be easily proved by using the complex analogues of the corresponding results in (1.1) to (1.8) which are available from the Chapters 3 and 6 of Mathai [16] and following the same parallel steps as we have done for establishing these results in Section 2 of this paper (see Chapters 5 and 6 of Mathai [16]). We also mention here that in this Section of the paper $\left|A\right|$ now represents the absolute value of the determinant of the matrix A of complex elements.

Theorem 3.1:

$$ {}_1F_2\left[a;a+b,c;-X\right]=\frac{\Gamma_p\left(a+b\right)}{\Gamma_p\left(a\right)\Gamma_p\left(b\right)}\int_0^I \left|U\right|^{a-p}\left|I-U\right|^{b-p}\,{}_0F_1\left[-;c;-U^{1/2}XU^{1/2}\right]dU \qquad (3.1) $$

for $\text{Re}\left(a,b\right)>\left(p-1\right)$.

Theorem 3.2:

$$\left|I+X\right|^{-a}=\frac{\Gamma_p(b)}{\Gamma_p(c)\Gamma_p(b-c)}\int_0^I\left|U\right|^{c-p}\left|I-U\right|^{b-c-p}\ {}_2F_1\left[a,b;c;-U^{1/2}XU^{1/2}\right]dU \tag{3.2}$$

for $\operatorname{Re}(c,b-c)>(p-1)$.

Theorem 3.3:

$$ {}_3F_2\left[a,b,e;c,d+e;-X\right]=\frac{\Gamma_p(e+d)}{\Gamma_p(e)\Gamma_p(d)}\int_0^I\left|U\right|^{e-p}\left|I-U\right|^{d-p}\times \\ {}_2F_1\left[a,b;c;-U^{1/2}XU^{1/2}\right]dU \tag{3.3}$$

for $\operatorname{Re}(e,d)>(p-1)$.

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