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# COCHAIN-VALUED THEORY IN TOPOLOGICAL FIELD THEORY

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**Abstract**

We determine the category of boundary conditions in the case that the closed string algebra is semisimple. We find that sewing constraints - the most primitive form of worldsheet locality - already imply that D-branes are vector bundles on spacetime. This expository paper describes sewing conditions in two-dimensional open and closed topological field theory. We include a description of the G-equivariant case, where G is a finite group. We comment on extensions to cochain - valued theories and various applications.

**Keywords:** D-branes, Topological field theory, Cochain level theory
**2010 Mathematics Subject Classification:** 12Jxx

## 1. INTRODUCTION

The theory of D-branes has proven to be of great importance in the development of string theory. We will focus on certain mathematical structures central to the idea of D-branes. One might at first be tempted to declare that D-branes simply correspond to conformally invariant boundary conditions for the open string. This viewpoint is not very useful because there are too many such boundary conditions, and in general they have no geometrical description. It also neglects important restrictions imposed by sewing consistency conditions.

The drastically simpler case of two dimensional topological field theory (TFT), where the whole content of the theory is encoded in a finite-dimensional commutative Frobenius algebra. We shall find that describing the sewing conditions, and their solutions for 2d topological open and closed Topological field theory, is a tractable but not entirely trivial problem. We also extend our results to the equivariant case, where we are given a finite group G, and the world sheets are surfaces equipped with G - bundles. This is relevant to the classification of D - branes in orbifolds.

A closely related point is that in open string field theory there are different open string algebras $\mathcal{O}_{aa}$ for the different boundary conditions a. For boundary conditions with maximal support, however, they are Morita equivalent via the bimodules $\mathcal{O}_{ab}$. For some purposes it might seem more elegant to start with a single algebra. (Indeed, Witten has suggested in [1] that one should use something analogous to stabilization of $C^*$ algebras, namely one should replace the string field algebra by $\mathcal{O}_{aa}$ K where K is the algebra of compact operators.) In our framework, the single algebra is replaced by the category of boundary conditions. If one believes that a stringy spacetime is a non - commutative space, our framework is in good agreement with Kontsevich's approach to non-commutative geometry, according to which a non - commutative space is a linear category - essentially the category of modules for the ring, if the space is defined by a ring. For commutative rings the category of modules determines the ring, but in the non - commutative case the ring is determined only up to Morita equivalence.

Finally, we comment briefly on some related literature. There is a rather large literature on 2d Topological field theory and it is impossible to give comprehensive references. Here we just indicate some closely related works. The 2d closed sewing theorem is a very old result implicit in the earliest papers in string theory. The algebraic formulation was perhaps first formulated by Friedan. Accounts have been given in [1] and in the Stanford lectures by Segal. Sewing constraints in 2D open and closed string theory were first investigated in [2].Extensions to unorientable worldsheets were described in [3]. Our work -which is primarily intended as a pedagogical exposition - was first described at strings 2000 [4] and summarized briefly in [5]. It was described more completely in lectures at the KITP in 2001 and at the 2002 Clay School [6]. In [7] one can find alternative C. Lazaroiu although the emphasis in these papers is on applications to disk instanton corrections in low energy supergravity. Regarding G-equivariant theories, there is a very large literature on D - branes and orbifolds not reflected. In the context of 2D Topological field theory two relevant references are [8].Alternative discussions on the meaning of B - fields in orbifolds (in Topological Field Theory) can be found in [9] .Our treatment of cochain - level theories and $A_\infty$ algebras has been developed considerably further by Costello [10].

## 2. OPEN AND CLOSED 2D TOPOLOGICAL FIELD THEORY

Roughly speaking, a quantum field theory is a functor from a geometric category to a linear category. The simplest example is a Topological Field Theory, where we choose the geometric category to be the category whose objects are closed, oriented (d - 1) - manifolds, and whose morphisms are oriented cobordisms (two such cobordisms being identified if they are diffeomorphic by a diffeomorphism which is the identity on the incoming and outgoing boundaries). The linear category in this case is simply the category of complex vector spaces and linear maps, and the only property we require of the functor is that it takes disjoint unions to tensor products. The case d = 2 is of course especially well known and understood.

There are several natural ways to generalize the geometric category. One may, for example, consider manifolds equipped with some structure such as a Riemannian metric. The focus of this paper is on a different kind of generalization where the objects of the geometric category are oriented (d - 1) – manifolds with boundary, and each boundary component is labelled with an element of a fixed set $B_0$ called the set of boundary conditions. In this case a cobordisms from $Y_0$ to $Y_1$ means a d - manifold X whose boundary consists of three parts $\partial X = Y_0 \cup Y_1 \cup \partial_{cstr} X$, where the "constrained boundary" $\partial_{cstr} X$ is a cobordisms from $\partial Y_0\ to\ \partial Y_1$. Furthermore, we require the connected components of $\partial_{cstr} X$ to be labelled with elements of $B_0$ in agreement with the labelling of $\partial Y_0\ and\ \partial Y_1$.

Thus when d = 2 the objects of the geometric category are disjoint unions of circles and oriented intervals with labelled ends. A functor from this category to complex vector spaces which takes disjoint unions to tensor products will be called an open and closed topological field theory; such theories will give us a "baby" model of the theory of D - branes. We shall always write C for the vector space associated to the standard circles $S^1$, and $\mathcal{O}_{ab}$ for the vector space associated to the interval [0, 1] with ends labelled $a, b \in B_0$.

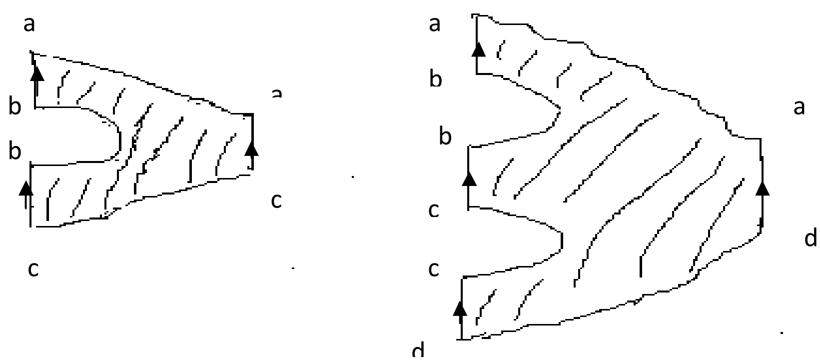



**Figure 1**: Basic cobordisms on open strings.

The cobordisms fig.1 gives us a linear map $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac}$, or equivalently a bilinear map

$$\mathcal{O}_{ab} \text{ X } \mathcal{O}_{bc} \to \mathcal{O}_{ac} \tag{2.1}$$

The cobordisms fig.2 gives us a $\mathcal{O}_{ab}\otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac}$ linear map $\mathcal{O}_{bc} \otimes\mathcal{O}_{cd} \to \mathcal{O}_{bd}$ or equivalently a bilinear map

$$\mathcal{O}_{bc} \text{ X } \mathcal{O}_{cd} \to \mathcal{O}_{bd} \tag{2.2}$$

which we think of as a composition law. In fact we have a $\mathbb{C}$ - linear category $B$ whose objects are the elements of $B_0$, and whose set of morphisms from b to a is the vector space $\mathcal{O}_{ab}$, with composition of morphisms given by (2.1).To say that B is a category means no more than that the composition (2.1) is associative in the obvious sense, and that there is an identity element $1_a \in \mathcal{O}_{aa}$ for each $a \in B_0$;

For any open and closed Topological field theory we have a map $e: C \to C$ defined by the cylindrical cobordism $S^1 \times [0,1]$, and map $e_{ab}: \mathcal{O}_{ab} \to \mathcal{O}_{ab}$ defined by the square [0, 1] X [0, 1]. Clearly$e^2 = e \; and \; e_{ab}^2 = e_{ab}$. If all these maps are identity maps we say the theory is reduced.

### 2.1 Algebraic characterization

The most general 2D open and closed Topological field theory, formulated as in the previous section, is given by the following algebraic data:

1. $(C, \theta_C, 1_C)$ is a commutative Frobenius algebra.
2. $\mathcal{O}_{ab}$ and $\mathcal{O}_{bc}$ are collection of vector spaces for $a, b \in B_0$ with an associative bilinear product $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \to \mathcal{O}_{ac}$

$$\mathcal{O}_{bc} \otimes\mathcal{O}_{cd} \to \mathcal{O}_{bd} \tag{2.3}$$

3. The $\mathcal{O}_{aa}, \mathcal{O}_{bb}, \mathcal{O}_{cc}, \mathcal{O}_{dd}$ have nondegenerate traces

$$\theta_a : \mathcal{O}_{aa} \to \mathbb{C}, \quad \theta_b : \mathcal{O}_{bb} \to \mathbb{C}$$
$$\theta_c : \mathcal{O}_{cc} \to \mathbb{C}, \quad \theta_d : \mathcal{O}_{dd} \to \mathbb{C} \tag{2.4}$$

In particular, each $\mathcal{O}_{aa}$ is a not-necessarily commutative Frobenius algebra.

4. Moreover,

$$\mathcal{O}_{ab} \otimes \mathcal{O}_{ba} \to \mathcal{O}_{aa} \xrightarrow{\theta_a} \mathbb{C}, \quad \mathcal{O}_{ba} \otimes \mathcal{O}_{ab} \to \mathcal{O}_{bb} \xrightarrow{b} \mathbb{C},$$
$$\mathcal{O}_{bc} \otimes \mathcal{O}_{cb} \to \mathcal{O}_{cc} \xrightarrow{\theta_c} \mathbb{C}, \quad \mathcal{O}_{cd} \otimes \mathcal{O}_{dc} \to \mathcal{O}_{dd} \xrightarrow{\theta_d} \mathbb{C}, \tag{2.5}$$

are perfect pairings with

$$\theta_a(\psi_1\psi_2) = \theta_b(\psi_2\psi_1), \quad \theta_c(\psi_3\psi_4) = \theta_d(\psi_4\psi_3), \tag{2.6}$$

For $\psi_1 \in \mathcal{O}_{ab}$, $\psi_2 \in \mathcal{O}_{ba}$, $\psi_3 \in \mathcal{O}_{bc}$, $\psi_4 \in \mathcal{O}_{cb}$

5. There are linear maps:

$$l_a: C \to \mathcal{O}_{aa}, l_b: C \to \mathcal{O}_{bb}, l_c: C \to \mathcal{O}_{cc}, l_d: C \to \mathcal{O}_{dd}$$
$$l^a : \mathcal{O}_{aa} \to C, \quad l^b : \mathcal{O}_{bb} \to C, \quad l^c : \mathcal{O}_{cc} \to C, \quad l^d : \mathcal{O}_{dd} \to C \tag{2.7}$$

6. $l_a$, $l_b$ are a algebra homomorphism

$$l_a(\phi_1\phi_2) = l_a(\phi_1)\, l_a(\phi_2)$$
$$l_b(\phi_1\phi_2) = l_b(\phi_1)\, l_b(\phi_2) \tag{2.8}$$

7. The identity is preserved

$$l_a(1_C) = 1_a, l_b(1_C) = 1_b, l_c(1_C) = 1_c, l_d(1_C) = 1_d, \tag{2.9}$$

8. Moreover, $l_a$ is central in the sense that

$$l_a(\phi)\psi = \psi l_a(\phi), \quad l_b(\phi)\psi = \psi l_b(\phi),$$
$$l_c(\phi)\psi = \psi l_c(\phi), \quad l_d(\phi)\psi = \psi l_d(\phi), \tag{2.10}$$

For all $\phi \in C$ and $\psi \in \mathcal{O}_{ab}, \mathcal{O}_{cd}$.

9. $l_a$ and $l^a$ , $l_b$ and $l^b$, $l_c$ and $l^c$, $l_d$ and $l^d$ are adjoints:

$\theta_C(l^a(\psi)\phi) = \theta_a(\psi l_a(\phi))$ , $\theta_C(l^b(\psi)\phi) = \theta_b(\psi l_b(\phi))$

$\theta_C(l^c(\psi)\phi) = \theta_c(\psi l_c(\phi))$ , $\theta_C(l^d(\psi)\phi) = \theta_d(\psi l_d(\phi))$ (2.11)

For all $\psi \in \mathcal{O}_{aa}$.

10. The "Cardy conditions."[3] Define $\pi_b^a: \mathcal{O}_{aa} \to \mathcal{O}_{bb}$, $\pi_c^d: \mathcal{O}_{cc} \to \mathcal{O}_{dd}$ as follows. Since $\mathcal{O}_{ab}$ and $\mathcal{O}_{ba}$ are in duality ( using $\theta_a$ and $\theta_b$ ), if we let $\psi_\mu$ be a basis for $\mathcal{O}_{ba}$ then there is a dual basis $\psi^\mu$ for $\mathcal{O}_{ab}$ . Then we define

$\pi_b^a(\psi) = \sum_\mu \psi_\mu \psi\, \psi^\mu$ , $\pi_d^c(\psi) = \sum_\mu \psi_\mu \psi\, \psi^\mu$. (2.12)

and we have the "Cardy condition":

$\pi_b^a = l_b \circ l^a, \pi_d^c = l_d \circ l^c$. (2.13)

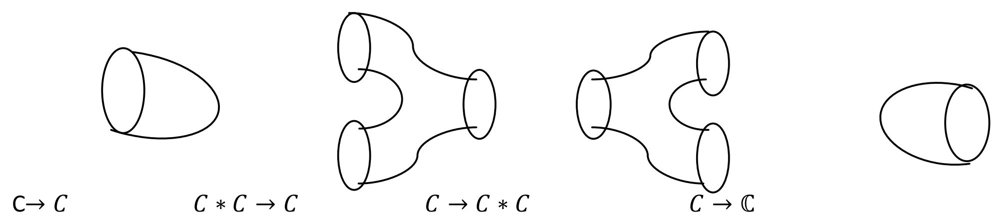



**Fig.2:** Four diagrams defining the Frobenius structure in a closed 2d TFT. It is often more convenient to represent the morphisms by the planar diagrams. In this case our convention is that a circle oriented so that the right hand points into the surface is an ingoing circle.

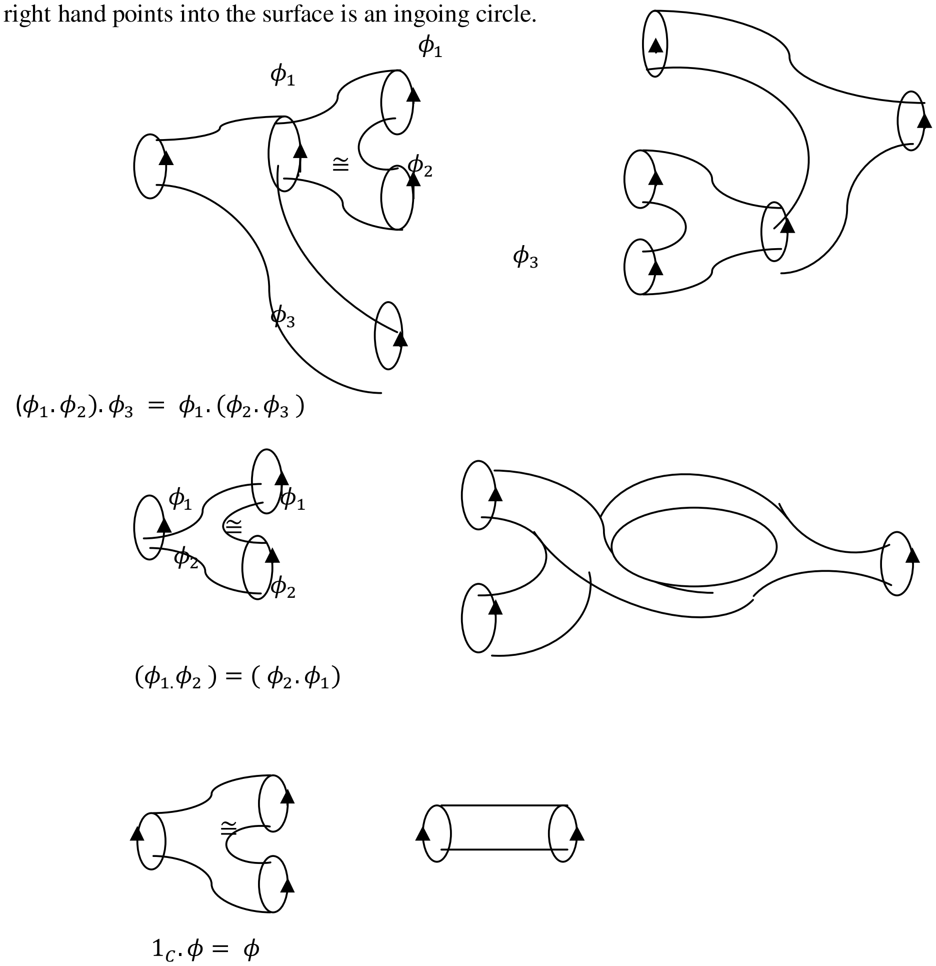



**Figure 3:** Associativity, commutativity, and unit constraints in the closed case. The unit constraint requires the natural assumption that the cylinder correspond to the identity map $C \to C$.

### 2.2 Pictorial representation

Let us explain the pictorial basis for these algebraic conditions. The case of a closed 2d Topological field theory is very well - known. The data of the Frobenius structure is provided by the diagrams in fig.2. The consistency conditions follow from fig.3.

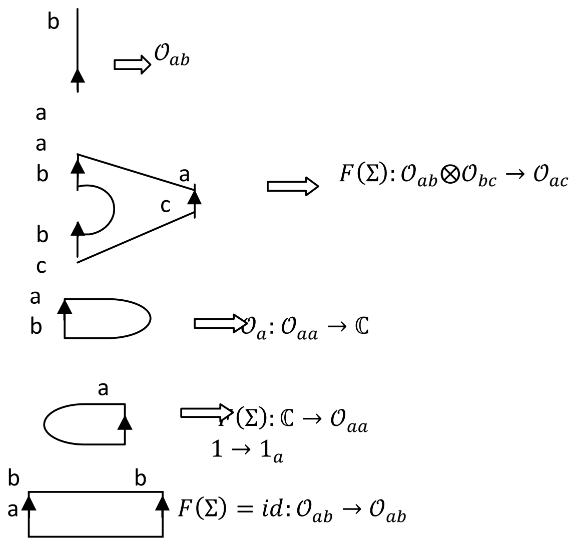



**Figure 4:** Basic data for the open theory. Constrained boundaries are denoted with wiggly lines, and carry a boundary condition a, b, c,…….∈ $B_0$…

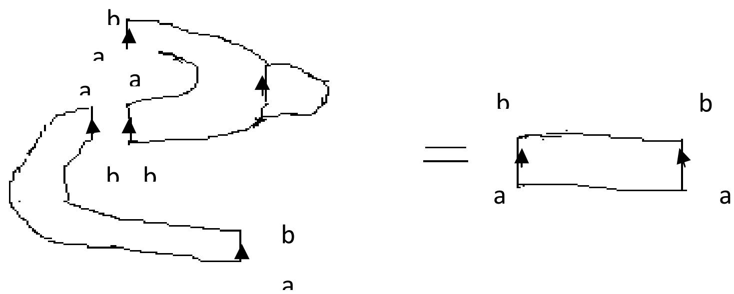



**Figure 5:** Assuming that the strip corresponds to the identity morphism we must have perfect pairings in (2.5)

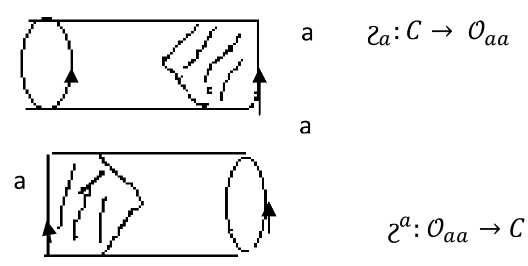



**Figure 6:** Two ways of representing open to closed and closed to open transitions.

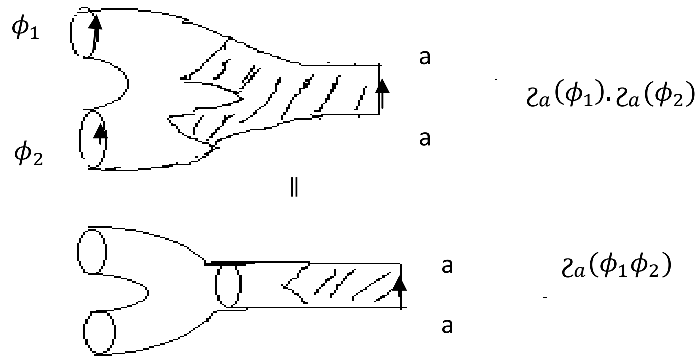
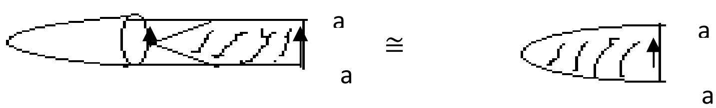
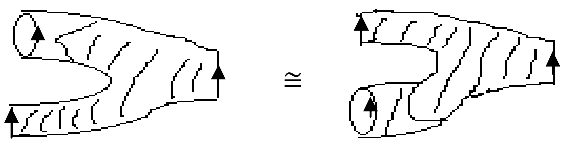



**Figure 7:** $l_a$ is a homomorphism.



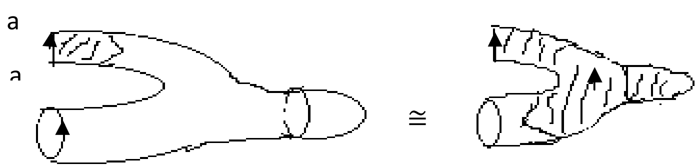

$l_a(1_C) = 1_a$

**Figure 8:** $l_a$ preserves the identity.




$l_a(\phi).\psi = \psi.l_a(\phi)$

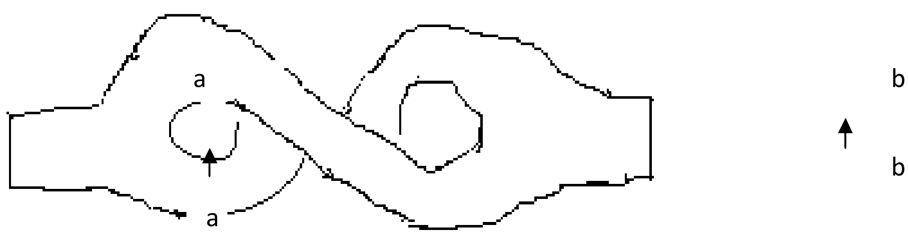
**Figure 9:** $l^a$ maps into the center of$O_{a\,a}$.




$\theta_C(l^a(\psi).\phi) = \theta_a(\psi.l_a(\phi)$

**Figure 10:** $l^a$ is the adjoint of$l_a$.

In the open case, entirely analogous considerations lead to the construction of a non - necessarily commutative Frobenius algebra in the open sector.




$$\pi_b^a : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$$

**Figure 11**: The double - twist diagram defines the map $\pi_b^a : \mathcal{O}_{aa} \to \mathcal{O}_{bb}$

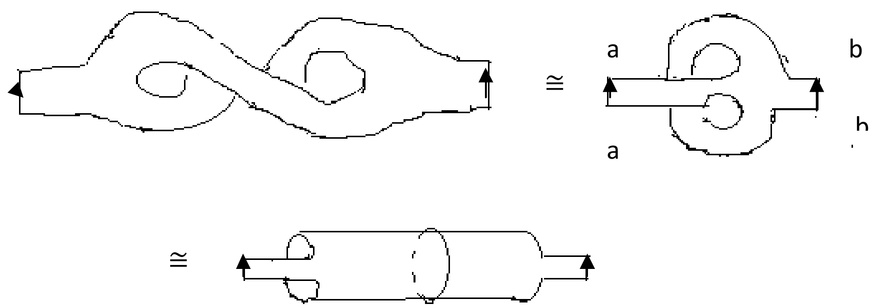



**Figure 12**: the (generalized) Cardy – condition expressing factorization of the double - twist diagram in the closed string channel.

The fact that (2.5) are dual pairings follows from fig.5.The essential new ingredient in the open / closed theories are the open to closed and closed to open transitions. In 2d topological field theory$l^a$, $l_a$.The are represented by fig.6.These are five new consistency conditions associated with the open/ closed transitions. In fig .7 to fig.12.

**2.3 The category of boundary conditions**

The category $B$ of boundary conditions of an open and closed Topological field theory is an additive category. We can always adjoin new objects to it in various ways. For example, we may as well assume that it possesses direct sums, as we can define for any two objects a and b a new object a ⊕ b by

$$\mathcal{O}_{\mathrm{a\oplus b,c}} : \mathcal{O}_{\mathrm{ac}} \otimes \mathcal{O}_{\mathrm{bc}} . \tag{2.14}$$

$$\mathcal{O}_{\mathrm{c,a\oplus b}} : \mathcal{O}_{\mathrm{ca}} \otimes \mathcal{O}_{\mathrm{cb}} . \tag{2.15}$$

and hence

$$\mathcal{O}_{\mathrm{a\oplus b,a\oplus b}} := \begin{pmatrix} \mathcal{O}_{aa} & \mathcal{O}_{ab} \\ \mathcal{O}_{ba} & \mathcal{O}_{bb} \end{pmatrix} \tag{2.16}$$

With the obvious composition laws, and

$$\theta_{\mathrm{a\oplus b}} : \mathcal{O}_{\mathrm{a\oplus b,a\oplus b}} \to \mathbb{C} \tag{2.17}$$

Given by

$$\theta_{\mathrm{a\oplus b}} \begin{pmatrix} \psi_{\mathrm{aa}} & \psi_{\mathrm{ab}} \\ \psi_{\mathrm{ba}} & \psi_{\mathrm{bb}} \end{pmatrix} = \theta_{\mathrm{a}}(\psi_{\mathrm{aa}}) + \theta_{\mathrm{b}}(\psi_{bb}). \tag{2.18}$$

The new object is the direct sum of a and b in the enlarged category of boundary conditions. If there was already a direct sum of a and b in the category B then the new object will be canonically isomorphic to it. In the opposite direction, if we have a boundary condition a and a projection $p \in \mathcal{O}_{aa}$(i.e. an element such that $p^2 = p$) then we may as well assume there is a boundary condition $b = image(p)$ such that for any c we have $\mathcal{O}_{cb} = \{f \in \mathcal{O}_{ab}: pf = f\}$ and $\mathcal{O}_{bc} = \{f \in \mathcal{O}_{ba}: fp = f\}$. Then we shall have $a \cong image(p) \oplus$ Image $(1-p)^4$.

One very special property that the category B possesses is that for any two objects a and b the space $\mathcal{O}_{ab}$of morphisms is canonically dual to $\mathcal{O}_{ba}$, by a pairing which factorizes through the composition in either order. It is natural to call a category with this property a Frobenius category, or perhaps a Calabi-Yau category.[5] It is a strong restriction on the category: for example the category of finitely generated modules over a finite dimensional algebra does not have the property unless the algebra is semisimple.

Example: Probably the simplest example of an open and closed theory of the type we are studying is one associated to a finite group G.The category B is the category of finite dimensional complex representation M of G, and the trace $\theta_M: \mathcal{O}_{MM} = End(M) \to \mathbb{C}$ takes $\psi: M \to M$ to $1/_{|G|}\ trace(\psi)$. The closed algebra $\mathcal{C}$ is the center of the group algebra $\mathbb{C}$ [G], which maps to each End (M) in the obvious way. The trace $\theta_C: \mathcal{C} \to \mathbb{C}$ takes a central element $\sum \lambda_g\, g$ of the group – algebra to $\lambda_1/_{|G|}$.

In this example the partition function of the theory on a surface $\Sigma$ with constrained boundary circles $C_1, C_2, \dots\dots\dots. C_k$ labelled $M_1, M_2, \dots\dots\dots.\ M_k$ is the weighted sum over the isomorphism classes of principal G - bundles P on $\Sigma$ of
$\chi_{M1}(hp(C_1)) \dots\dots\ \chi_{Mk}(hp(C_k))$,

Where $\chi_M$: G→ ℂ is the character of a representation M, and $hp(C)$ denotes the holonomy of P around a boundary circle C.Each bundle P is weighted by the reciprocal of the order of its group of automorphisms.

## 3. COCHAIN LEVEL THEORIES

The most important "generalization" however, of the open and closed topological field theory we have described is the one of which it is intended to be toy model. In closed string theory the central object is the vector space $C = C_{S^1}$ of states of a single parameterized string. This has an integer grading by the "ghost number", and an operator Q: C→ $C$ called the "BRST operator" which raises the ghost number by 1 and satisfies $Q^2 = 0$.In other words, C is a cochain complex. If we think of the string as moving in a space - time M then C is roughly the space of differential forms defined along the orbits of the action of the reparametrization group Diff$^+(s^1)$ on the free loop space
L$\mathcal{M}$. $(More\ precisely, square - integeral\ forms\ of\ semi - infinite\ degree.)$

Similarly, the space C of a topologically -twisted N = 2 supersymmetric theory , as just described, is a cochain complex which models the space of semi - infinite differential forms on the loop space of a Kähler manifold - in this case, all square-integrable differential forms, not just those along the orbits of Diff$^+(s^1)$. In both kinds of example, a cobordisms $\Sigma$ from $\rho$ circles to q circles gives an operator $U_{\Sigma,\mu}: C^{\otimes p} \to C^{\otimes q}$ which depends on a conformal structure $\mu$ on $\Sigma$. This operator is a cochain map, but it's crucial feature is that changing the conformal structure $\mu$ on $\Sigma$ changes the operator $U_{\Sigma,\mu}$ only by a cochain-homotopy. The cohomology $H(C)=\ker(Q)/\mathrm{im}(Q)$ - the "space of physical states" in conventional string theory – is therefore the state space of a topological field theory. (In the usual string theory situation the topological field theory we obtain is not very interesting, for the BRST cohomology is concentrated in one or two degrees, and there is a "grading anomaly" which means that the operator associated to be a cobordisms $\Sigma$ changes the degree by a multiple of the Euler number$\times$ $(\Sigma)$ .In the case of the N = 2 supersymmetric modeles, however, there is no grading anomaly, and the full structure is visible.)

A good way to describe how the operator $U_{\Sigma,\mu}$ varies with $\mu$ is as follows.

If $\mathcal{M}_\Sigma$ is the moduli space of conformal structures on the cobordisms $\Sigma$, modulo diffeomorphisms of $\Sigma$ which are the identity on the boundary circles, then we have a cochain map

$$U_\Sigma: C^{\otimes p} \to \Omega(\mathcal{M}_\Sigma; C^{\otimes q}) \qquad (3.1)$$

where the right-hand side is the de Rham complex of forms on $\mathcal{M}_\Sigma$ with values in $C^{\otimes q}$ . The operator $U_{\Sigma,\mu}$ is obtained from $U_\Sigma$ by restricting from $\mathcal{M}_\Sigma$ to $\{\mu\}$. The composition property when two cobordisms $\sum 1$ and $\sum 2$ are concatenated is that the diagram

$$\begin{array}{ccc} C^{\otimes \mathrm{p}} & \rightarrow & \Omega(\mathcal{M}_{\Sigma 1}; C^{\otimes \mathrm{q}}) \\ \downarrow & & \downarrow \\ \Omega(\mathcal{M}_{\Sigma 2 \circ \Sigma 1}; C^{\otimes \mathrm{r}}) & \rightarrow \Omega(\mathcal{M}_{\Sigma 1} \times \mathcal{M}_{\Sigma 2}; C^{\otimes \mathrm{r}}) = & \Omega(\mathcal{M}_{\Sigma 1}; \Omega(\mathcal{M}_{\Sigma 2}; C^{\otimes \mathrm{r}})) \end{array} \quad (3.2)$$

Commutes, where the lower horizontal arrow in induced by the map $\mathcal{M}_{\Sigma 1} \times \mathcal{M}_{\Sigma 2} \rightarrow \mathcal{M}_{\Sigma 2 \circ \Sigma 1}$ which expresses concatenation of the conformal structures.

Many variants of this formulation are possible. For example; we might to give a cochain map

$$U_{\Sigma}: C.(\mathcal{M}_{\Sigma}) \rightarrow (C^{\otimes \mathrm{p}})^{*} \otimes C^{\otimes \mathrm{q}},$$

Where $C.(\mathcal{M}_{\Sigma})$ is, say, the complex of smooth singular chains of $\mathcal{M}_{\Sigma}$. We may also prefer to use the moduli spaces of Riemannian structures instead of conformal structures.

There is no difficulty in passing from the closed-string picture just presented to an open and closed theory. We shall not discuss these cochain-level theories in any depth in this work, but it is important to realize that they are the real objective. We shall now point out a few basic things about them. A much fuller discussion can be found in Costello [10].

For each pair a, b of boundary conditions we shall still have a vector space - indeed a cochain complex – $\mathcal{O}_{ab}$, but it is no longer the space of morphisms from b to a in a category. Rather, what we have is, in the terminology of Fukaya, Kontsevich, and others, an $A_{\infty} - category$. This means that instead of a composition law $\mathcal{O}_{ab}$ X $\mathcal{O}_{bc}$ $\rightarrow \mathcal{O}_{ac}$ we have a family of ways of composing, parameterized by the contractible space of conformal structures on the surface of fig.1.In particular, any two choices of a composition law from the family are cochain – homotopic. Composition is associative in the sense that we have a contractible family of triple compositions $\mathcal{O}_{ab}$ X $\mathcal{O}_{bc}$ X $\mathcal{O}_{cd} \rightarrow \mathcal{O}_{ad}$, which contains all the maps obtained by choosing a binary composition law from the given family and bracketing the triple in either of the two possible ways.

## 4. CONCLUSION

We conclude that the spacetime and closed strings are fundamental and category of boundary conditions is compatible with that background. We find that the Frobenius category of boundary conditions and derive the closed strings and the spacetime. Thus, our treatment is in harmony with the philosophy of matrix theory. We obtain the closed string algebra from the string algebra by taking the center of the open string algebra Z ($\mathcal{O}$) $\cong C.$ A more sophisticated version of this idea is that the closed string algebra is obtained from the category of boundary conditions.

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