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USE OF MIXED CUBATURE RULE FOR EVALUATION OF INTEGRALS OVER TRIANGULAR REGION IN ADAPTIVE ENVIRONMENT

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Abstract

A new mixed cubature rule is established for 2-simplexes (i.e., triangles). Extending anti-Gauss 3-point rule and Fejer's second 3-point rule in two dimensions and then combining those the mixed cubature rule is formed which is of precision 5. Also an adaptive cubature algorithm is devised to boost up the mixed cubature rule. Some test integrals are also numerically evaluated to show the efficiency of the mixed rule in adaptive environment.

Keywords: Anti-Gauss 3-point rule $\left\{R^2_{aG_3}(f)\right\}$, Fejer's second 3-point rule $\left\{R^2_{2F_3}(f)\right\}$, mixed cubature rule $\left\{R^2_{aG_3F_3}(f)\right\}$, open type cubature rule, adaptive cubature routine, triangular domain.

Mathematics Subject Classification: 65D30, 65D32

1. INTRODUCTION

Many times we encounter functions which cannot have antiderivatives, especially if integration is over regions with curved boundaries. Therefore we look forward to integrate them numerically rather than analytically. Integration formulae for numerical evaluation of integrals over the simplex in n-space have been given inductively by Hammer et al. [2]. In the same paper they derive certain affinely symmetric integration formulae for the triangle and tetrahedron. Using the theory proposed by Hammer and Wymore [3, 4], it is possible to extend the usefulness of the methods developed by transformations of the regions and by the use of Cartesian products. Over the years many authors like Cowper [1], Lyness and Jespersen [12], Lannoy [7], Laurie [9], Laursen and Gellert [10], Lether [11], Hillion [5], Lauge and Baldur [8], were successful in deriving integration techniques over triangles. Many authors gave formula primarily for rectangular regions based on the formulas for the line. Some of the works like Rathod et al. [15], Shivaram [18] on triangular domain have evolved by taking the Generalized Gaussian-quadrature rules.

In recent years, it has been proven that the Finite Element Method (FEM) is a powerful tool for approximate solutions to several phenomena in engineering and it has also found a wide range of applications in Science, Technology etc. For triangular finite elements there will be two dimensional integrals over triangular domain. For such domains there are not so much methods to accomplish that integration with accuracy. Finite Element Method was first appeared in Turner et al. [20] in 1956. Then after some authors Reddy, [16] and Reddy and Shippy [17], applied the Finite Element Method for numerical integration.

Here in this paper we apply the mixed cubature methods to integrate functions over two simplexes in an adaptive environment. So far as known to us, no other authors have used mixed cubature rule(s) for the numerical integration of functions in two dimensions over triangular surface in an adaptive environment.

Let us discuss how we integrate a function numerically over a triangular domain. The basic approach is as follows.

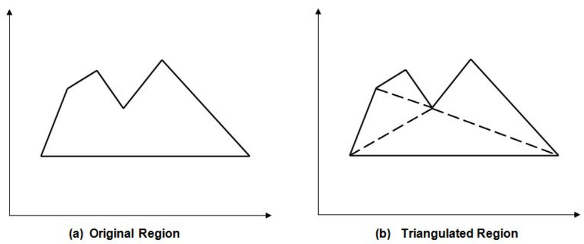

Figure 1: Triangulation of the Domain

Suppose we are given a function in two dimensions over an arbitrary shape as in figure-1. As we can not integrate it directly by applying some cubature rules so our primary approach is to transform each finite element into a standard right triangle element $\Delta st = \{(\mu,\lambda) | 0 \le 1, 0 \le \lambda \le 1, \mu + \lambda \le 1\}$ by setting $\begin{bmatrix} \mu \\ \lambda \end{bmatrix} = = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a' \\ b' \end{bmatrix}$ where $x, y, a_{11,} a_{12}, a_{22} \in \Delta$ and $(a', b') \in \Delta st$. The geometrical interpretation is shown in the figure-2.

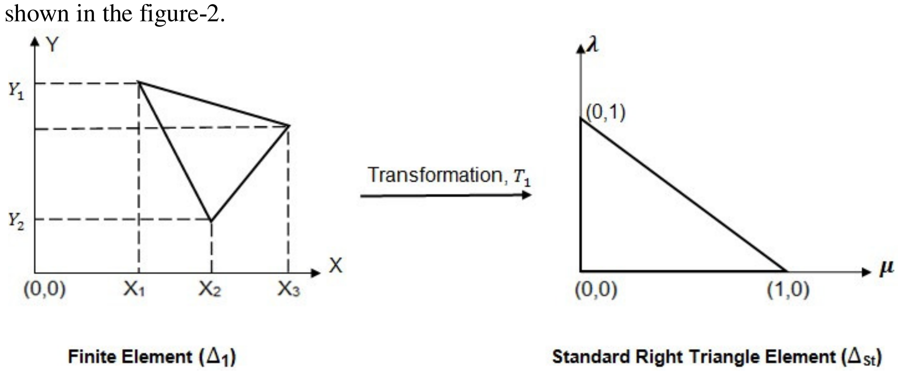

Figure 2: Co-ordinate Transformation of a General Triangular Element into a Standard Right Triangular Element

But the domain of the standard right triangle is not compatible with the domain of our proposed mixed cubature rule. So we should go for another transformation T_2, which transforms a standard right triangle into a standard square i .e ., to say a standard right triangle domain will turn into a standard square domain $\square_{St} = \{(u,v) | 0 \le u \le 1, 0 \le v \le 1\}$ which is shown graphically in figure-3.

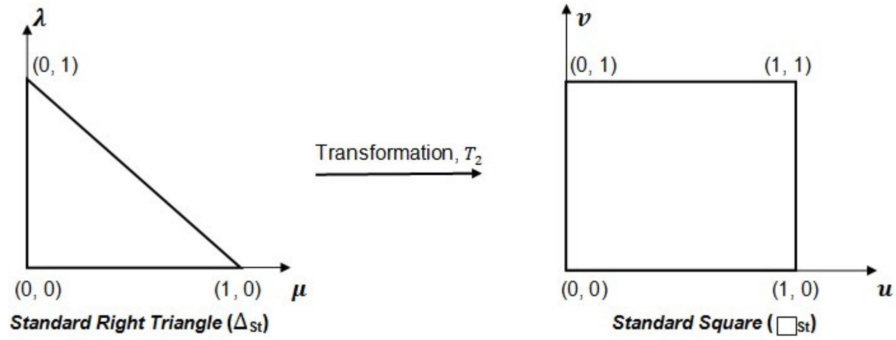

Figure 3: Co-ordinate Transformation of a Standard Right Triangular Element into a Standard Square Domain

But, after this transformation the interval of integrals became $[0,1]$ for both the axes u and v. Again this fact is incompatible with the interval definition for our proposed cubature rules. So we do one last transformation T_3, in order to accomplish our intervals of integration i.e., $[-1,1]$. The standard square $(\square_{St})$ is now will be transformed into a standard 2-square $\square_{St_2} = \{(\xi,\eta) | -1\xi \le 1, -1 \le \eta \le 1\}$ so as to apply our proposed cubature rules i.e., to say we can get the nodes of our cubature rules only if we have a standard 2-square domain. This can be visualized from figure-4.

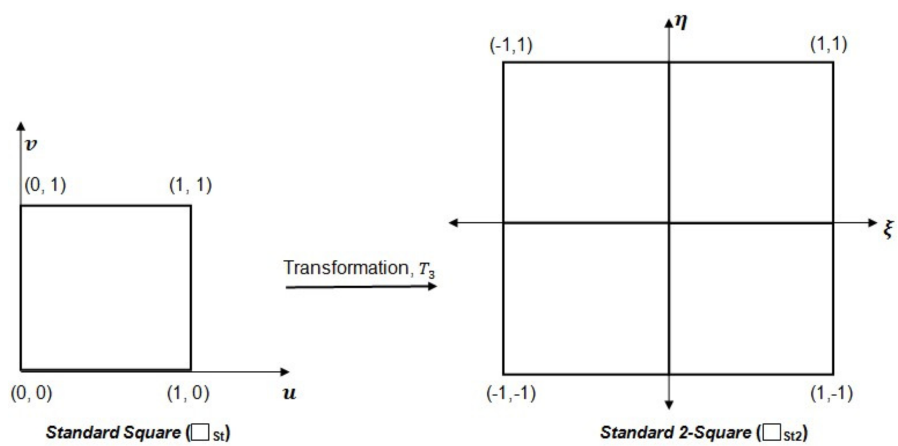

Figure 4: Linear Transformation of a Standard Square Element into a Standard 2-square Element

Some authors like Jena and Dash [6], Patra et al. [13,14] have successfully applied mixed cubature rules to approximate integrals over triangular surfaces in non-adaptive environment.

A comparative study between the constituent rules and the mixed rule is shown only in adaptive cubature routines with the aid of some test integrals in table-1.

2. DOUBLE INTEGRALS OVER TRIANGLE

We consider the integral of an arbitrary function f over the surface of a triangle T as

$$I = \iint_{\Delta_{St}} f(x,y)\,dx\,dy = \int_0^1 dx \int_0^{1-x} f(x,y)\,dy = \int_0^1 dy \int_0^{1-y} f(x,y)\,dx \qquad (2.1)$$

Now we need to approximate (2.1) by a suitable cubature formula

$$I = \sum_{m=1}^{N} W_m f(x_m, y_m) \qquad (2.2)$$

where W_m are the weights associated with specific points (x_m, y_m) and N is the number of pivotal points related to the required precision.

Now we transform the double integral over the triangle in the equation (2.1) to the standard square $\square_{St} = \{(u,v) | 0 \le u \le 1, 0 \le v \le 1\}$ by substituting $x = u$ and $y = (1-u)v$. Thus we have

$$\mathrm{I} = \int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx = \int_0^1 \int_0^1 f\big(x(u,v), y(u,v)\big)|J|\,du\,dv \tag{2.3}$$

where

$$|J(u,v)| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = 1-u$$

From equation (2.3), we have

$$I = \int_0^1 \int_0^1 f\big(u,(1-u)v\big)(1-u)\,du\,dv \tag{2.4}$$

Again in order to get the nodes of our proposed cuabature rules we have to transform the integral I of equation (2.4) into an integral over the standard 2-square as

$$\square_{St_2} = \{(\xi,\eta) | -1 \le \xi \le 1, -1 \le \eta \le 1\} \text{ by substituting } u = \frac{1+\xi}{2}, v = \frac{1+\eta}{2} \tag{2.5}$$

Then clearly the determinant of the Jacobian and the differential area are

$$\left|\frac{\partial(u,v)}{\partial(\xi,n)}\right| = \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n}\frac{\partial v}{\partial \xi} = \frac{1}{2}\left(\frac{1}{2}\right) - 0\times 0 = \frac{1}{4}$$

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(\xi,n)}\right| d\xi\,d\eta = \frac{1}{4}d\xi\,d\eta \tag{2.6}$$

Now on using equations (2.5) and (2.6) in equation (2.4) we have

$$I = \int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx = \int_0^1 \int_0^1 f\big(u,(1-u)v\big)(1-u)\,du\,dv$$

$$= \int_0^1 \int_0^1 f\left(\frac{1+\xi}{2}, \frac{(1-\xi)(1+\eta)}{4}\right)\left(\frac{1-\xi}{8}\right) d\xi\,d\eta \tag{2.7}$$

Equation (2.7) represents an integral over the surface of standard 2-square:

$$\square_{St_2} = \{(\xi,\eta) | -1 \le \xi \le 1, -1 \le \eta \le 1\}$$

From equation (2.7), we can write

$$I = \int_{-1}^1 \int_{-1}^1 f\big(x(\xi,\eta), y(\xi,\eta)\big)\left(\frac{1-\xi}{8}\right) d\xi\,d\eta$$

$$I = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{1-\xi_i}{8}\right)\omega_i\omega_j f\big(x(\xi_i,\eta_j), y(\xi_i,\eta_j)\big) \tag{2.8}$$

where ξ,η are the Gaussian points in the ξ,η directions respectively, and ω_j, ω_i are the corresponding weights.

We can write equation (2.8) as

$$I = \sum_{k=1}^{N=n\times n} w_k f(x_k, y_k) \tag{2.9}$$

where w_k, x_k, y_k are obtained from the relation

$$\left.\begin{aligned} w_k &= \left(\frac{1-\xi_i}{8}\right)\omega_i\omega_j \\ x_k &= \frac{1+\xi_i}{2} \\ y_k &= \frac{(1-\xi_i)(1+\eta_j)}{4} \end{aligned}\right\} \tag{2.10}$$

where
$k = 1, 2, \cdots, n$
$i = 1, 2, \cdots, n$
$j = 1, 2, \cdots, n$

The weighting coefficients w_k and sampling points (x_k, y_k) of various orders can now be easily computed by formulae (2.9) and (2.10). We have shown a numerical verification with some test integrals in table-1.
We now go for our mixed cubature rule to approximate (2.7) in the next section.

3. CONSTRUCTION OF THE MIXED CUBATURE RULE OF PRECISION FIVE

The mixed quadrature rule blending anti-Gauss 3-point rule and Fejer's second- point rule (see Singh and Dash [19]) is given by

$$R_{aG_3 2F_3}(f) = \frac{1}{11}\left[3R_{aG_3}(f) + 8R_{2F_3}(f)\right] \tag{3.1}$$

$$= \frac{1}{11}\left[\frac{15}{13} f\left(-\sqrt{\frac{13}{15}}\right) + \frac{48}{13} f(0) + \frac{15}{13} f\left(\sqrt{\frac{13}{15}}\right)\right.$$

$$\left. + \frac{16}{3} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{16}{3} f(0) + \frac{16}{3} f\left(\frac{1}{\sqrt{2}}\right)\right] \tag{3.2}$$

where
$R_{aG_3}(f) =$ Anti-Gauss 3-point rule in one dimension
$R_{2F_3}(f) =$ Fejer's second 3-point rule in one dimension
Applying mixed quadrature rule (3.2) to the double integral

$$\int_{-1}^{1}\int_{-1}^{1} f\left(x(\xi,\eta), y(\xi,\eta)\right)\left(\frac{1-\xi}{8}\right) d\xi\, d\eta$$

we have

$$I_{mix} \approx \sum_{k=1}^{36} w_k f\left(x_{ij}, y_{ij}\right) = \sum_{k=1}^{36} w_k f\left(x_k, y_k\right) \tag{3.3}$$

where

$$\left.\begin{aligned} w_k &= w_{ij} = \left(\frac{1-\xi_i}{8}\right)\omega_i\omega_j \\ x_k &= x_{ij} \text{ and } y_k = y_{ij} \end{aligned}\right\} \tag{3.4}$$

The weighting coefficient w_k and sampling points (x_k, y_k) of various orders can be easily computed by using the equation (2.10).

4. ERROR ANALYSIS

The error term of the mixed rule given in equation (3.2) in two dimensions is given by Patra et al. [13,14]

$$E^2_{mix}(f) = E^2_{aG_3F_3}(f) = \frac{11}{141750}\left[\frac{\partial^6 f(0,0)}{\partial x^6} + \frac{\partial^6 f(0,0)}{\partial y^6}\right] + \cdots$$

which can be written in a simpler notation as

$$E_{mix}^{2}(f)=E_{aG_3F_3}^{2}(f)=\frac{11}{141750}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\cdots \tag{4.1}$$

Theorem 4.1: Let $f(x,y)$ be a continuously differentiable function in $[-1,1]\times[-1,1]$.Then the error $E_{aG_3F_3}^{2}(f)$ associated with the rule $R_{aG_3F_3}^{2}(f)$ is given by

$$\left|E_{aG_3F_3}^{2}(f)\right|\approx\frac{11}{141750}\left|\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]\right|$$

Proof. Directly follows from equation (4.1).

Theorem 4.2: The bounds for the truncation error $E_{aG_3F_3}^{2}(f)=I(f)-R_{aG_3F_3}^{2}(f)$ is given by

$$\left|E_{aG_3F_3}^{2}(f)\right|\le\frac{2M}{495}\left|\xi_2-\xi_1\right|\times\left|\eta_2-\eta_1\right|$$

where $M=\max\limits_{\substack{-1\le x\le 1\\-1\le y\le 1}}\left|f_{5,0}(x,0)+f_{0,5}(0,y)\right|$

Proof: The error term of anti-Gauss 3-point rule in two dimensions Patra et al. [13,14] is given by

$$E_{aG_3}^{2}(f)\approx-\frac{2}{135}\left[f_{4,0}(\xi_1,\eta_1)+f_{0,4}(\xi_1,\eta_1)\right],\ (\xi_1,\eta_1)\in[-1,1]\times[-1,1]$$

Similarly the error term of Fejer's second 3-point rule in two dimensions Patra et al. [13,14] is given by

$$E_{2_{F_3}}^{2}(f)\approx-\frac{1}{180}\left[f_{4,0}(\xi_2,\eta_2)+f_{0,4}(\xi_2,\eta_2)\right],\ (\xi_2,\eta_2)\in[-1,1]\times[-1,1]$$

We know that $E_{aG_3 2F_3}^{2}(f)=\frac{1}{11}\left[2E_{aG_3}^{2}(f)+8E_{2F_3}^{2}(f)\right]$

$$\approx\frac{1}{11}\left[-\frac{2}{45}\left\{f_{4,0}(\xi_1,0_1)+f_{0,4}(\xi_1,\eta_1)\right\}+\frac{2}{45}\left\{f_{4,0}(\xi_2,0_2)+f_{0,4}(\xi_2,\eta_2)\right\}\right]$$

$$=\frac{2}{495}\left[\left\{f_{4,0}(\xi_2,0)+f_{0,4}(0,\eta_2)\right\}-\left\{f_{4,0}(\xi_1,0)+f_{0,4}(0,\eta_1)\right\}\right]$$

$$=\frac{2}{495}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}\left[f_{5,0}(x,0)+f_0,5(0,y)\right]dxdy \quad \text{(assuming } \xi_1<\xi_2 \text{ and } \eta_1<\eta_2\text{)}$$

Hence $\left|E_{aG_3 2F_3}^{2}(f)\right|\approx\left|\frac{2}{495}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}\left[f_{5,0}(x,0)+f_{0,5}(0,y)\right]dxdy\right|$ where $M=\max\limits_{\substack{-1\le x\le 1\\-1\le y\le 1}}\left|f_{5,0}(x,0)+f_{0,5}(0,y)\right|$

$$\le\frac{2}{495}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}\left|\left[f_{5,0}(x,0)+f_{0,5}(0,y)\right]\right|dxdy$$

$\because f(x,y)$ is defined on a closed and bounded rectangle $[-1,1]\times[-1,1]$ hence compact and $f(x,y)$ attains its maximum over the domain $[-1,1]\times[-1,1]$.

So $\left|E_{aG_3 2F_3}^{2}(f)\right|\le\frac{2}{495}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}dxdy$ where $M=\max\limits_{\substack{-1\le x\le 1\\-1\le y\le 1}}\left|\left[f_{5,0}(x,0)+f_{0,5}(0,y)\right]\right|$

$$=\frac{2M}{495}\left|(\xi_2-\xi_1)\times(\eta_2-\eta_1)\right|$$

which gives only a theoretical error bound as (ξ_1,η_1) and (ξ_2,η_2) are unknown points in $[-1,1]\times[-1,1]$. It shows that the error in the approximation will be less if the points $(\xi_1,\eta_1),(\xi_2,\eta_2)$ get close to each other.

Corollary 4.1: The error bound for the truncation error $E_{aG_3 2F_3}^{2}(f)$ is given by

$$\left|E_{aG_3 2F_3}^{2}(f)\right|\le\frac{8M}{495}$$

Proof: We know from theorem (4.2) that

$$\left|E^2_{aG_3 2F_3}(f)\right| \le \frac{2M}{495}\left|(\xi_2-\xi_1)\times(\eta_2-\eta_1)\right|, (\xi_1,\eta_1),(\xi_2,\eta_2)\in[-1,1]\times[-1,1]$$

where $M = \max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}} \left\|\left[f_{5,0}(x,0)+f_{0,5}(0,y)\right]\right\|$

choosing $\left|(\xi_2-\xi_1)\right|\le 2$ and $\left|(\eta_2-\eta_1)\right|\le 2$ we get $\left|E^2_{aG_3F_3}(f)\right|\le\frac{8M}{495}$.

5. ADAPTIVE CUBATURE ALGORITHM FOR EVALUATION OF DOUBLE INTEGRALS

To evaluate double integrals over any rectangle using adaptive cubature, we adopt the following four steps algorithm.

Input: Function $f:[a,b]\times[c,d]\to\mathbb{R}$ and the prescribed tolerance ε.

Output: An approximation $Q(f)$ to the integral $I(f)=\int_a^b\int_c^d f(x,y)\,dxdy$ such that $\left|Q(f)-I(f)\right|\le\varepsilon$.

Step-1: The mixed cubature rule $R^2_{aG_3 2F_3}(f)$ is applied over the rectangle $[a,b]\times[c,d]$ having corner points $(a,c),(b,c),(b,d)$ and (a,d) to approximate the double integral $I(f)=\int_a^b\int_c^d f(x,y)\,dxdy$. The approximated value is denoted by $R^2_{aG_3 2F_3}\left(f_{[a,b]\times[c,d]}\right)$.

Step-2: The rectangle of integration $[a,b]\times[c,d]$ is split into four equal pieces of rectangles A_1,A_2,A_3,A_4 having the following corner points:

$\{(a,c),(m_1,c),(m_1,m_2),(a,m_2)\}$, $\{(m_1,c),(b,c),(b,m_2),(m_1,m_2)\}$, $\{(m_1,m_2),(b,m_2),(b,d),(m_1,d)\}$ and $\{(a,m_2),(m_1,m_2),(m_1,d),(a,d)\}$ respectively, where $m_1=\frac{a+b}{2}$ and $m_2=\frac{c+d}{2}$. The mixed cubature rule $\left(R^2_{aG_3 2F_3}(f)\right)$ is applied over each small rectangle to approximate the double integrals $I_1(f)=\int_a^{m_1}\int_c^{m_2} f(x,y)\,dxdy$, $I_2(f)=\int_{m_1}^{b}\int_c^{m_2} f(x,y)\,dxdy$, $I_3(f)=\int_{m_1}^{b}\int_{m_2}^{d} f(x,y)\,dxdy$, $I_4(f)=\int_a^{m_1}\int_{m_2}^{d} f(x,y)\,dxdy$ respectively. The approximated values are denoted by $R^2_{aG_3 2F_3}\left(f\,\middle|[a,m_1]\times[c,m_2]\right)$, $R^2_{aG_3 2F_3}\left(f\,\middle|[m_1,b]\times[c,m_2]\right)$, $R^2_{aG_3 2F_3}\left(f\,\middle|[m_1,b]\times[m_2,d]\right)$ and $R^2_{aG_3 2F_3}\left(f\,\middle|[a,m_1]\times[m_2,d]\right)$ respectively.

Step-3: $R^2_{aG_3 2F_3}\left(f\,\middle|[a,m_1]\times[c,m_2]\right)+R^2_{aG_3 2F_3}\left(f\,\middle|[m_1,b]\times[c,m_2]\right)+R^2_{aG_3 2F_3}\left(f\,\middle|[m_1,b]\times[m_2,d]\right)+R^2_{aG_3 2F_3}\left(f\,\middle|[a,m_1]\times[m_2,d]\right)$ is compared with $R^2_{aG_3 2F_3}\left(f\,\middle|[a,b]\times[c,d]\right)$ to estimate the error in $R^2_{aG_3 2F_3}\left(f\,\middle|[a,m_1]\times[c,m_2]\right)+R^2_{aG_3 2F_3}\left(f\,\middle|[m_1,b]\times[c,m_2]\right)+R^2_{aG_3 2F_3}\left(f\,\middle|[m_1,b]\times[m_2,d]\right)+R^2_{aG_3 2F_3}\left(f\,\middle|[a,m_1]\times[m_2,d]\right)$

Step-4: If $\left|\text{estimated error}\right|\le\frac{\varepsilon}{2}(\text{termination criterion})$ then $R^2_{aG_3 2F_3}\left(f\,\middle|[a,m_1]\times[c,m_2]\right)+R^2_{aG_3 2F_3}\left(f\,\middle|[m_1,b]\times[c,m_2]\right)+R^2_{aG_3 2F_3}\left(f\,\middle|[m_1,b]\times[m_2,d]\right)+R^2_{aG_3 2F_3}\left(f\,\middle|[a,m_1]\times[m_2,d]\right)$ is accepted as an approximation to the double integral $I(f)=\int_a^b\int_c^d f(x,y)\,dxdy$. Otherwise the same procedure is applied to each of the four rectangles allowing each piece of rectangles a tolerance $\frac{\varepsilon}{2}$. If the termination criterion is not satisfied on one or more of the rectangles, then those rectangles must be further split into four sub-rectangles and the entire process is repeated. When the process stops, the addition of all accepted values yields the desired approximate value $Q(f)$ to the double integral $I(f)$ such that $|Q(f)-I(f)| \le \varepsilon$.

N.B.: In this algorithm we can use any cubature rule to evaluate real definite integrals in two dimensions in adaptive integration scheme.

6. NUMERICAL VERIFICATION

Table 1: Comparative Study of the Cubature/Mixed Cubature Rule for Approximation of Some Surface Integrals Using Adaptive Cubature Routine

Integrals	Exact Value	Approximate Value $(Q(f))$					
		$R^2_{aG_3}(f)$	#Steps	$R^2_{2F_3}(f)$	#Steps	$R^2_{aG_3 2F_3}(f)$	#Steps
$\int_0^1\int_0^{1-x}(x+y)^{\frac{1}{2}}dxdy$	0.40000…	0.39998886	09	0.4000041725	09	0.39999735	01
$\int_0^1\int_0^{1-x}(x+y)^{-\frac{1}{2}}dxdy$	0.66666666…	0.66670024	25	0.666636553	21	0.666642019	05
$\int_0^1\int_0^{1-x}e^{-y^2}\cos(xy)dxdy$	0.4284998849	0.4285027338	05	0.428498815911	05	0.428499816496	01
$\int_0^1\int_0^{1-x}\frac{\sin x}{x}dxdy$	2	2.000011063	21	1.999995854	21	2.0000001454	05

Note:
Here the prescribed tolerance $\varepsilon = 0.0001$
Steps: Number of steps
$R^2_{aG_3}(f)$ = Anti-Gauss 3-point rule in two dimensions
$R^2_{2F_3}(f)$ = Fejer's second 3-point rule in two dimensions
$R^2_{aG_3 2F_3}(f)$ = Mixed cubature rule blending anti-Gauss 3-point rule and Fejer's second 3-point rule in two dimensions
All the computations are done using 'C' program.

7. CONCLUSION

From the table it is observed that

(i) if the mixed cubature rule $R^2_{aG_3 2F_3}(f)$ is used as the base rule of two dimensional adaptive integration scheme to evaluate the surface integrals, the number of steps are reduced significantly in comparison to its constituent rules $R^2_{aG_3}(f)$ and $R^2_{2F_3}(f)$ and also

(ii) the result came much better than their constituent rules.

(iii) Since the mixed rule is of open type it evaluates the surface integral I2 with singularity (0, 0), successively.

That is why we say that the mixed rule is more efficient than its constituent rules not only theoretically but also practically.

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