Original Article

Bulletin of Pure and Applied Sciences.
Vol. 37E (Math & Stat.), No.2, 2018.P.434-445
Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2018.00046.2

MAHLER MEASURE OF CHARGED GRAPHS OVER THE PURE CUBIC FIELD Q ($\sqrt[3]{2}$)

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Received on 21.04.2018, Accepted on 24.09.2018

Abstract

In this paper, the algebraic integer as a edge label from the pure cubic field to the all vertices of the simple graphs thereafter charges to all the vertices and get the edge labeled charged graphs. Further to find the Mahler measure of the edge labeled charged graphs with edge label $\sqrt[3]{2}$. Determine the total number of graphs for particular model graphs and the graphs with maximum and minimum Mahler measure.

Keywords: Mahler measure, transcendental number, pure cubic field.
MSC Code: 11R09, 11C20

1. INTRODUCTION

Mahler [5] introduced measure M (A). Mahler measure is important concept in algebraic number theory. This was applied by Mckee and Smyth [2]. Lehmer's problem is one of finding a monic integer polynomial P with smallest possible Mahler measure M (P) such that M (P) > 1. In [1], Lehmer exhibited the polynomial

$$L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

The Mahler measure M (L) = λ_0 of Lehmer's polynomial is the smallest known as Salem number. In 1970, Smith [4] classified of some graphs having maximum eigenvalue atmost 2. cyclotomic matrices were worked in [3] who classified all cyclotomic matrices over the integer.

Rameshkumar and Nagarajan [6] classified the simple graphs into four variety of graphs namely connected cyclotomic, connected non-cyclotomic , disconnected cyclotomic and disconnected non-cyclotomic . They named the special graphs for the cyclotomic graph whose Perron number is 2. A Pure cubic field is a cubic field can be found by adjoining the real cube root $\sqrt[3]{n}$ of a cubefree positive integer n to the field Q. The complex cubic field Q($\sqrt[3]{2}$) is the pure cubic field. In [7], Cohn found an asymptotic for the number of cyclic cubic fields An extension field Q(ζ_n) of Q generated by $\zeta_n = e^{\frac{2\pi i}{n}}$. The square root of any integer is subset of a cyclotomic field, but other nth roots of any integer are not in cyclotomic extension. . In the section 3 and 4, the classification of edge labeled charged and uncharged graphs of order upto four is determined.

2. PRELIMINARIES

Definition 2.1

Let $P(z) = z^n + a_1 z^{n-1} + ... + a_n = \prod_{i=1}^{n}(z - \alpha_i)$ be a monic non-constant polynomial and $a_1, a_2, ..., a_n \in Q(\sqrt[3]{2})$. The Mahler measure is $M(P) = \prod_{i=1}^{d}\max(1, |\alpha_i|)$.

Definition 2.2

For a vertex v we define its weighted degree as the sum of the weights of the edges incident at v, plus 1 if v has a charge of ± 1. For an edge with label $\sqrt[3]{2}$ we define its weight to be 2(its norm value).

Definition 2.3

Since $\sqrt[3]{2}$ is the root of the cubic equation $x^3 - 2 = 0$ with integer coefficients, $\sqrt[3]{2}$ is the algebraic integer. Consider the simple graph G = (V,E) with the constant map f: E={e_1,e_2,…, }→ $\sqrt[3]{2}$ is called the edge labeled graph.

Definition 2.4

For a monic polynomial $f(x) \in$ Q($\sqrt[3]{2}$)[x] of degree n, the associated reciprocal polynomial is defined to be $z^n f\left(z + \frac{1}{z}\right)$ which is a monic polynomial of degree 2n. For A an symmetric matrix of order n with entries from Q($\sqrt[3]{2}$) . $R_A(z)$ is the associated reciprocal polynomial of its characteristic polynomial $\chi_A(x)$. The Mahler measure of A is the Mahler Measure M($R_A(z)$). All roots of $\chi_A(x)$ and $R_A(z)$ are transcendental numbers.

Definition 2.5

If all the vertices of the graph having charge 0 then the graph is called uncharged graph. If at least one of the vertex having charge either 1 or -1 then the graph is called charged graph.

Definition 2.6

If A is an matrix of order n with non zero diagonal entries as a charge on the corresponding vertex and off diagonal elements from {0, $\sqrt[3]{2}$ }, then If A is the adjacency matrix of the graph with at least one vertex charge either 1 (or) -1 then the graph is called the edge labeled charged graph .

3. CLASSIFICATION OF EDGE LABELED CHARGED GRAPHS

In [8], Taylor found the cyclotomic graphs over the Imaginary Quadratic Field. Consider Simple graphs with all the edges having the labels $\sqrt[3]{2}$. We indicate Vertices with charge 0, 1 and –1 will be drawn as ●, ⊕ and ⊖ respectively. We consider some simple graphs A_1, B_4, B_5, B_6, B_7, C_{26}, C_{30}, C_{55}, C_{57}, C_{62} in [6] as a model.

3.1. Model graph A_1

For the model graph A_1, we give charges 1 (or) -1 to the vertices and edge label to the all the edges for A_1 and get the 6 edge labeled charged graphs with one M value. Here finding the Mahler measure of all the graphs and so find the maximum (minimum) of that named as M_{max} (M_{min}). Let M be the Mahler measure of the graph.

The model graph A_1 is shown below

Fig 1(a), A_1

The six edge labeled charged graphs with $M_{max} = M_{min} = 1$

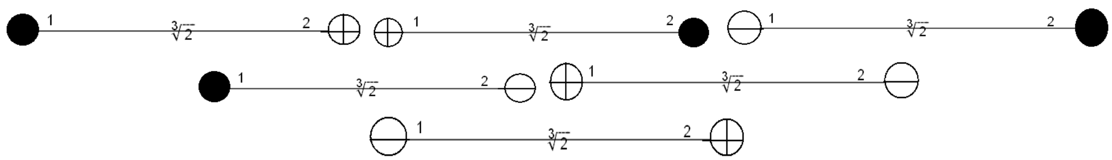


Fig 1(b)

For the first graph in Fig 1(b), the adjacency matrix is A = $\begin{bmatrix} 0 & \sqrt[3]{2} \\ \sqrt[3]{2} & 1 \end{bmatrix}$, the characteristic polynomial $\chi_A(x)$ is $-2^{2/3} - x + x^2$.

The reciprocal polynomial $R_A(z)$ is $z^4 - z^3 + 2z^2 - 2^{2/3}z^2 - z + 1$. The zero $R_A(z)$ are -0.4277 ± 0.9038 i and 0.9277 ± 0.3731 i . The Mahler measure of the $R_A(z)$ is 1 and therefore Mahler measure of the graph is 1 and hence remaining graphs also has M = 1. So the value M_{max} and M_{min} are 1.

3.2. Model Graph B_4

For the model graph B_4, we get the 26 edge labeled charged graphs with 8 different M values. We have two graphs with M_{max} and one with M_{min}.

For the first graph in Fig 2(a), the adjacency matrix is A = $\begin{bmatrix} 1 & \sqrt[3]{2} & \sqrt[3]{2} \\ \sqrt[3]{2} & 1 & 0 \\ \sqrt[3]{2} & 0 & 1 \end{bmatrix}$, the characteristic polynomial $\chi_A(x)$ is $1 - 2\times 2^{2/3} - 3x + 2\times 2^{2/3}x + 3x^2 - x^3$. The reciprocal polynomial $R_A(z)$ is $-1 + 3z - 6z^2 + 2\times 2^{2/3}z^2 + 7z^3 - 2\times 2^{2/3}z^3 - 6z^4 + 2\times 2^{2/3}x^4 + 3x^5 - x^6$. The zero $R_A(z)$ are -0.3908 ± 0.9204 i and 0.5 ± 0.8660 i, 0.4241, 2.3576. The Mahler measure of the $R_A(z)$ is 2.3576 nearly and therefore Mahler measure of the graph is 2.3576 nearly.

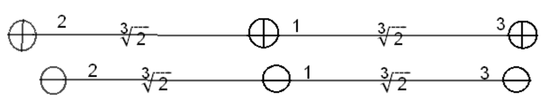


Fig 2(a) , $M_{max} \approx 2.3576$

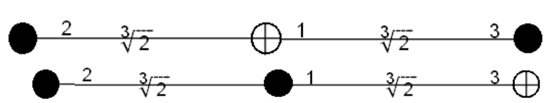


Fig 2(b), $M_{min} = 1.4463$

We may have many number graphs with same Mahler measure in the model graph. So the four graphs with M ≈1.5143

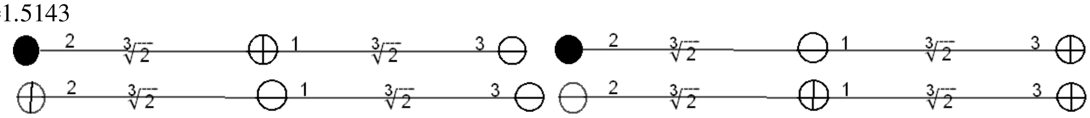


Fig 2(c)

3.3. Model Graph B_5

For the model graph B_5, we get the 26 edge labeled charged graphs with 8 different M values. We have two graphs with M_{max} and one with M_{min} .

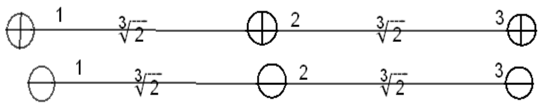


Fig 3(a) , $M_{max} \approx 2.3576$

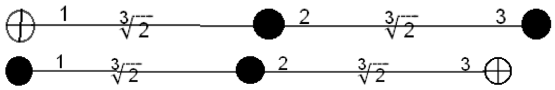


Fig 3(b), $M_{min} = 1.4463$

The four graphs with M ≈1.5143

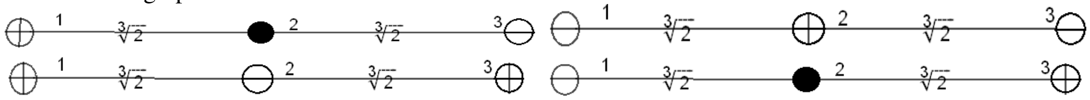


Fig 3(c)

3.4. Model Graph B_6

For B_6, we get the 26 edge labeled charged graphs with 8 different M values. We have two graphs with M_{max} and one graph with M_{min}.

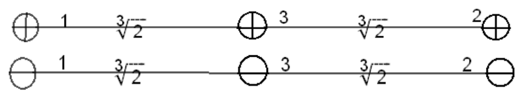


Fig 4(a) , $M_{max} \approx 2.3576$

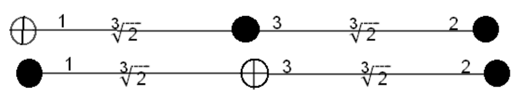


Fig 4(b), $M_{min} = 1.4463$

We have four graphs with M ≈ 1.5143

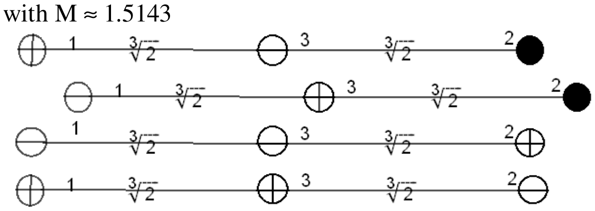


Fig 4(c)

3.5. Model Graph B_7

For B_7, we get the 26 edge labeled charged graphs with 8 different M values. We have one graph with M_{max} and three graphs with M_{min}.

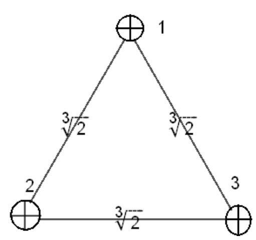


Fig 5(a) , $M_{max} \approx 3.2081$

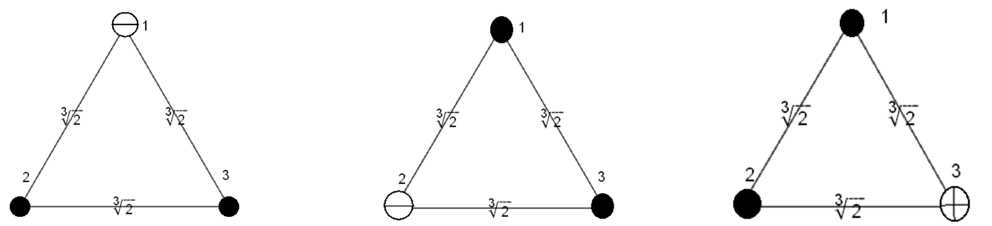


Fig 5(b), $M_{min} \approx$ 1.6241

The six graphs with M ≈ 2.2472

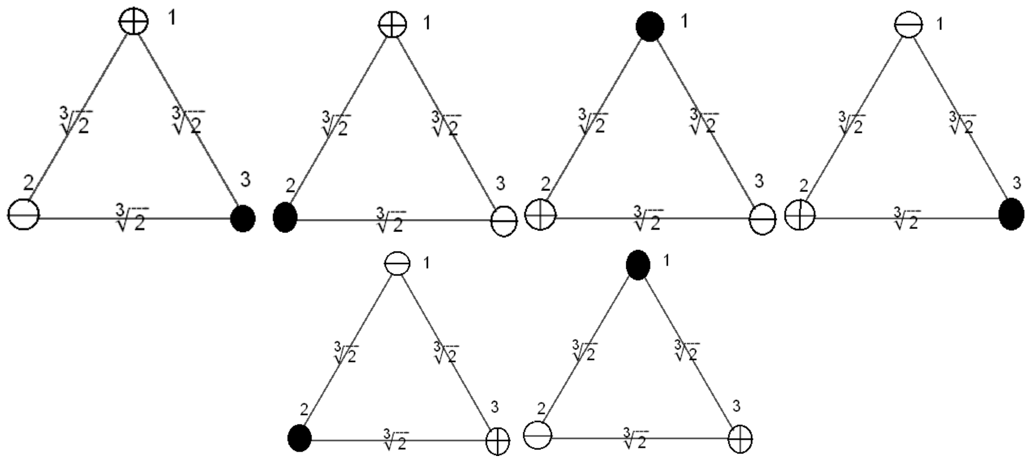


Fig 5(c)

3.6. Model Graph C_{26}

For C26, we get the 79 edge labeled charged graphs with 19 different M values. We have three graphs with M_{max} and one with M_{min} .

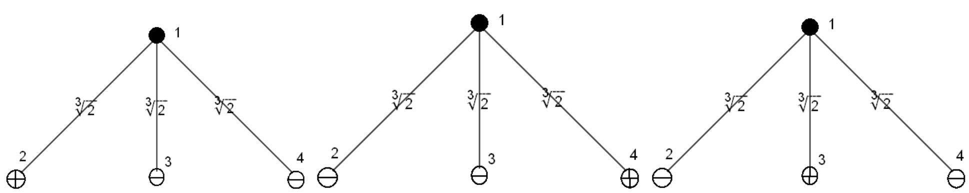


Fig 6(a), $M_{max} \approx$ 3.5563

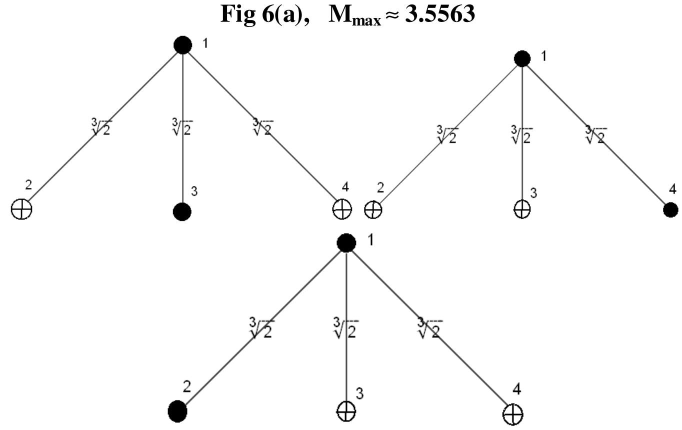


Fig 6(b), $M_{min} \approx$ 2.1279

The four graphs with M ≈ 2.3049

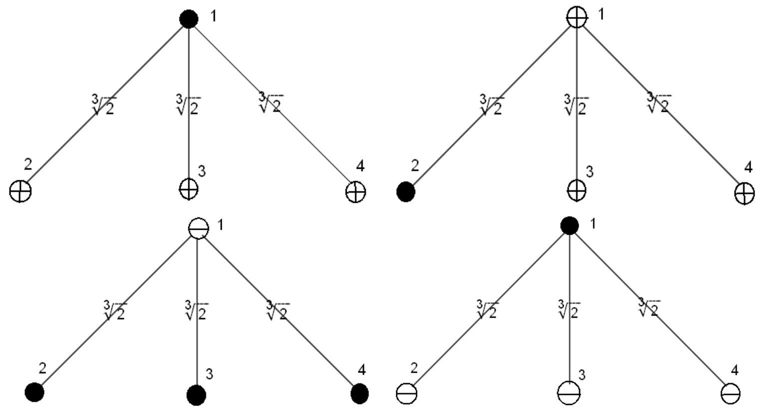


Fig 6(c)

The nine graphs with $M \approx 3.3366$

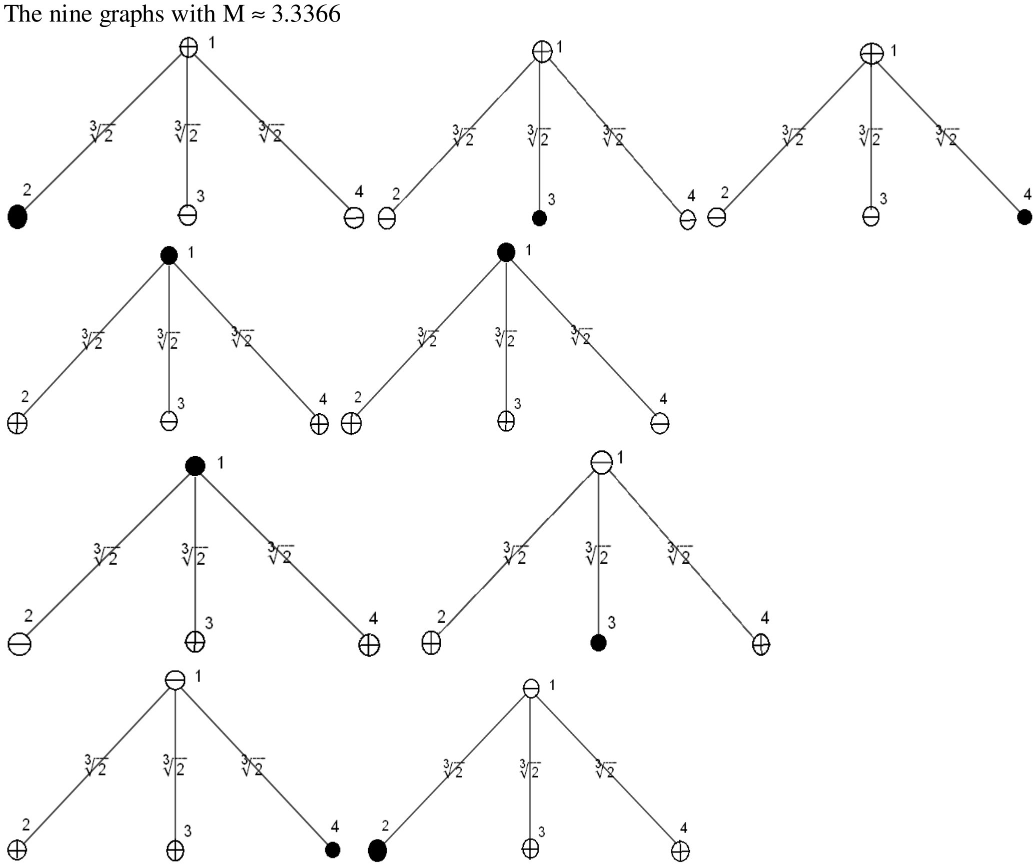


Fig 6(d)

3.7. Model Graph C_{30}

For C_{30}, we get the 65 edge labeled charged graphs with 22 different M values. We have two graphs with M_{max} and one with M_{min} .

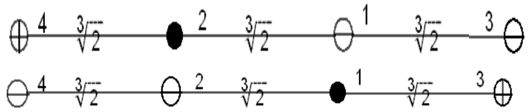


Fig 7(a), $M_{max} \approx$ 1.5276

Fig 7(b), $M_{min} \approx$ 1.6666

The five graphs with M ≈ 1.8477

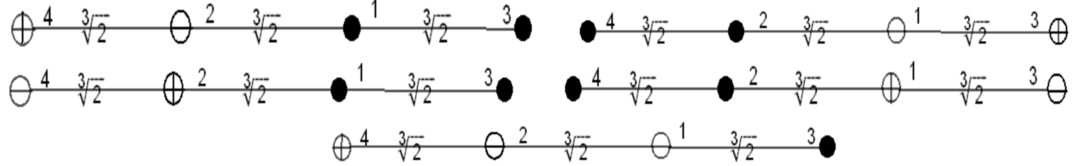


Fig 7(c)

The three graphs with M ≈ 2.4993

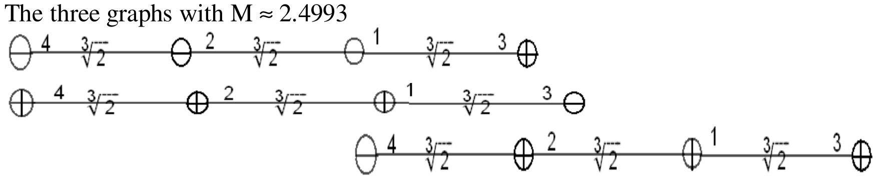


Fig 7(d)

3.8. Model Graph C_{55}

For C_{55}, we get the 80 edge labeled charged graphs with 13 different M values. We have four graphs with M_{max} and four graphs with M_{min}.

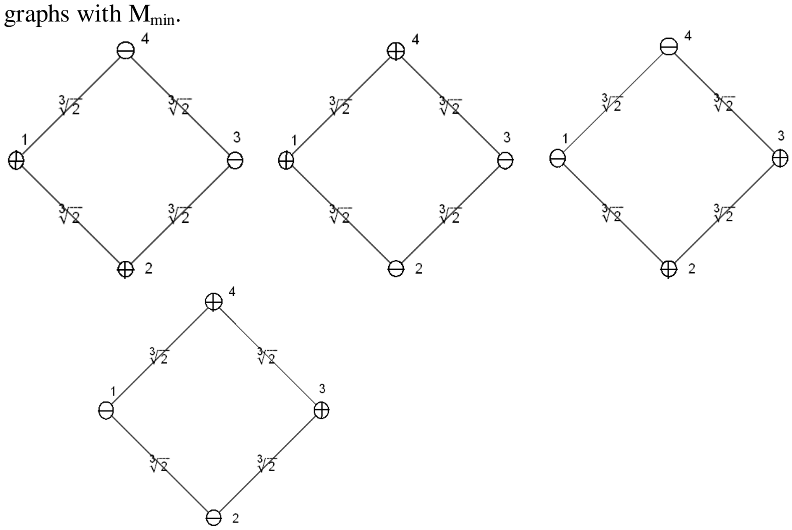


Fig 8(a), $M_{max} \approx$ 6.0629

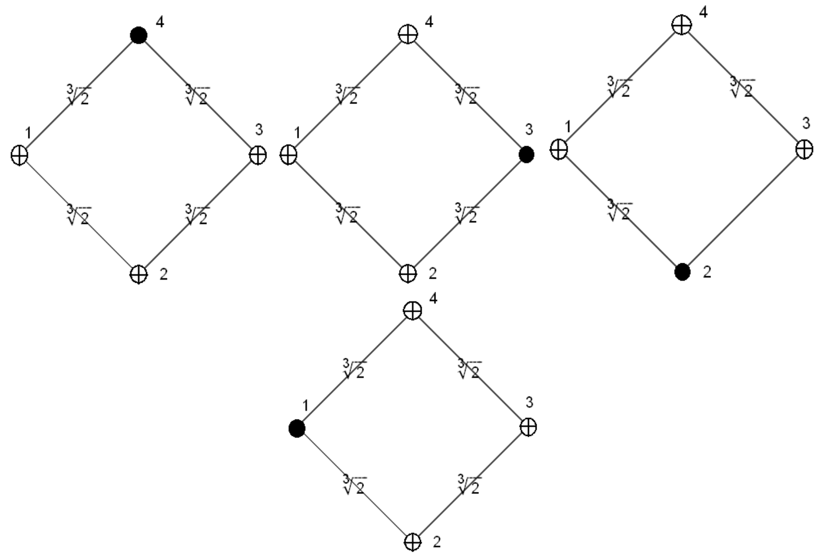


Fig 8(b), **$M_{min} \approx 2.9878$**

The six graphs with $M \approx 3.8591$

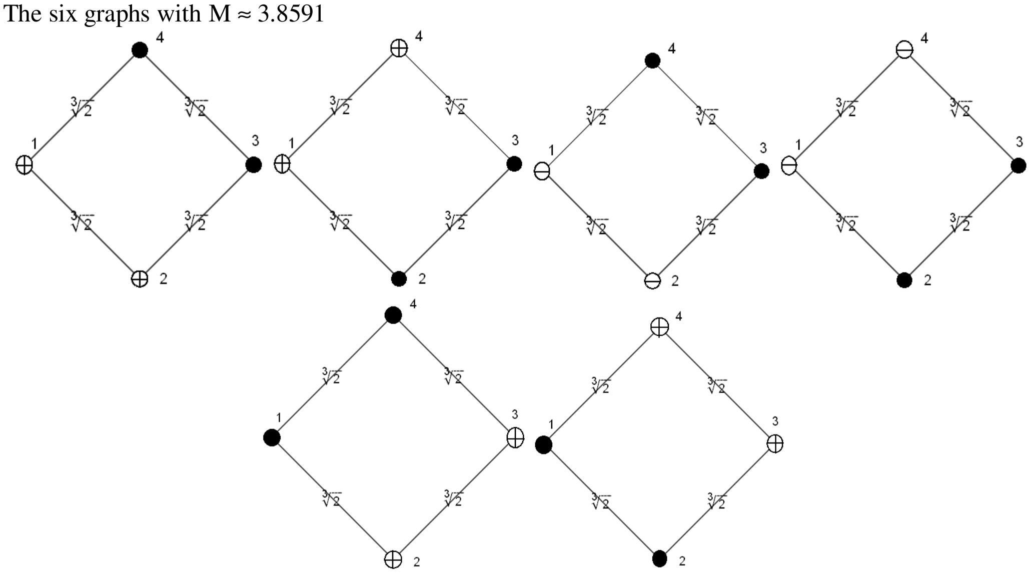


Fig 8(c)

The three graphs with $M \approx 3.5059$

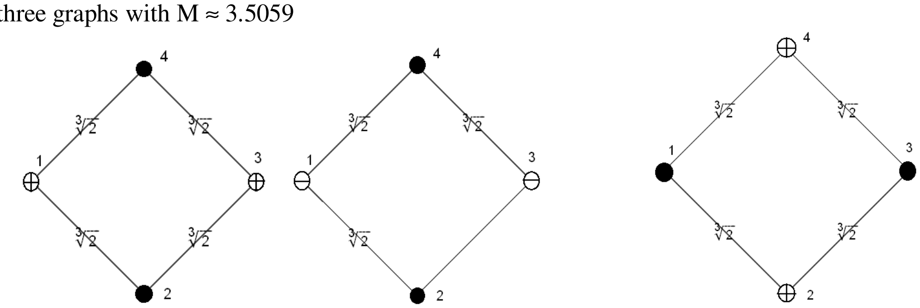


Fig 8(d)

3.9. Model Graph C_{57}

For C_{57}, we get the 80 edge labeled charged graphs with 35 different M values. We have two graphs with M_{max} and two graphs with M_{min} .

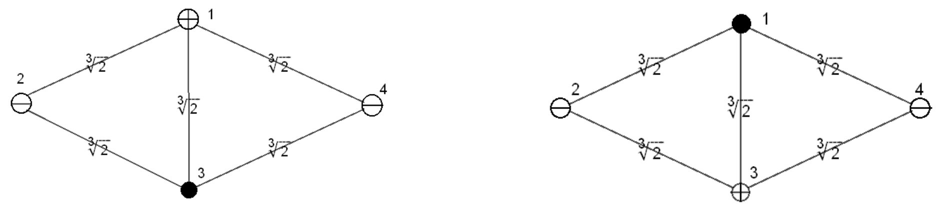


Fig 9(a), $M_{max} \approx$ 9.4279

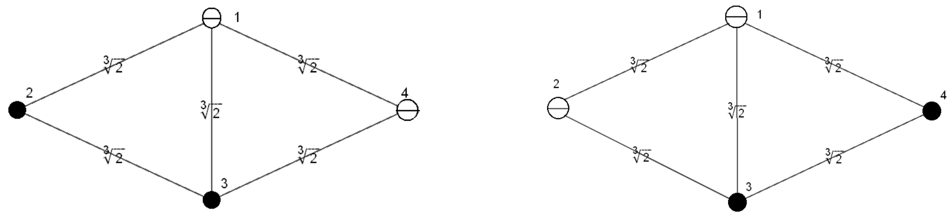


Fig 9(b), $M_{min} \approx$ 2.3679

3.10. Model Graph C_{62}

For C_{62}, we get the 80 edge labeled charged graphs with 14 different M values. We have one graph with M_{max} and one graph with M_{min} .

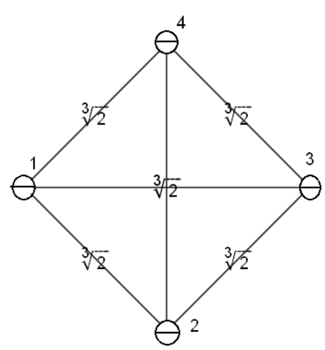


Fig 10(a), $M_{max} \approx$ 10.6952

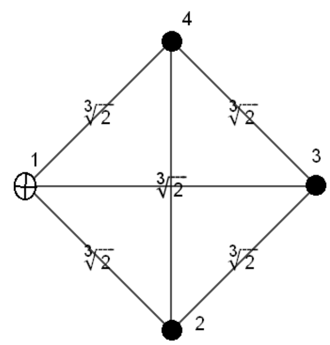


Fig 10(b), $M_{min} \approx$ 3.4935

4. Classification of Edge labeled uncharged graphs

Let n be the number of vertices.

For n = 2, the graph with M =1

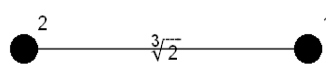


Fig 11(a)

For n = 3, the three graphs with M = 1

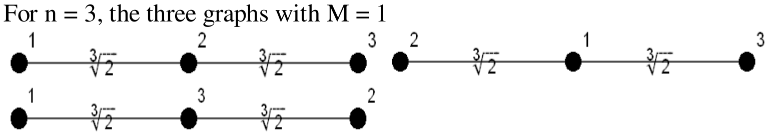
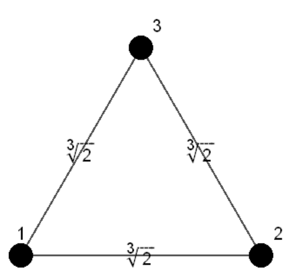


Fig 11(b)



The graph with $M \approx 2.0263$

Fig 11(c)

For n = 4, the four graphs with $M \approx 2.3335$

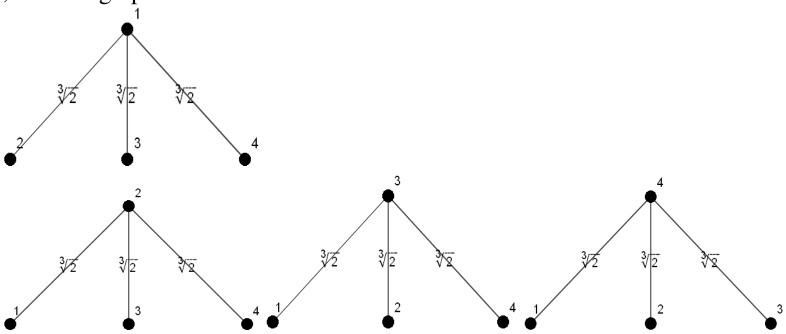


Fig 11(d)

The twelve graphs with $M \approx 1.4803$

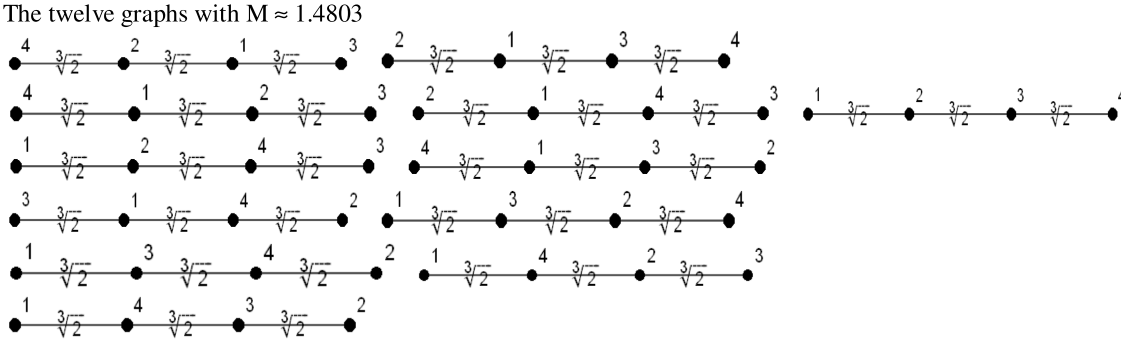
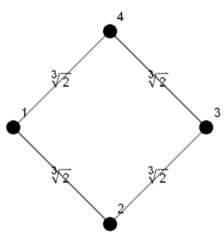


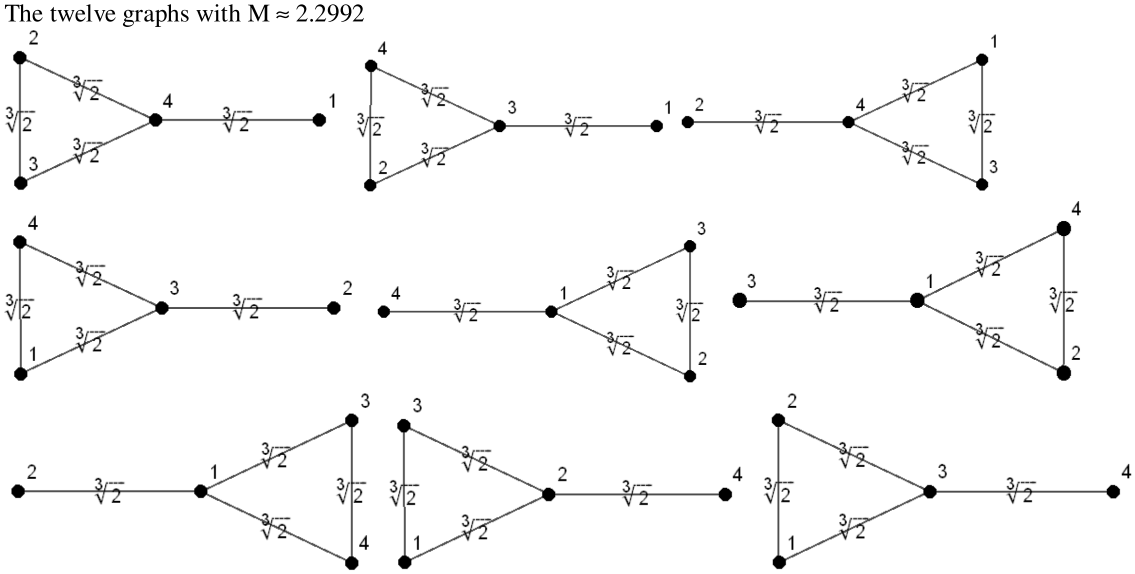
Fig 11(e)



The graph with $M \approx 4.1058$

Fig 11(f)

The twelve graphs with $M \approx 2.2992$



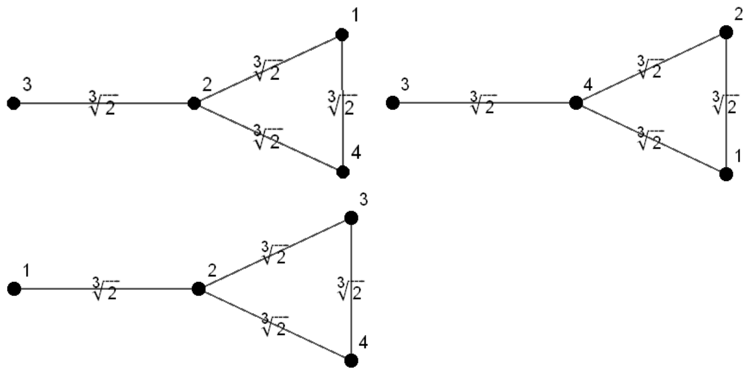


Fig 11(g)

The six graphs with $M \approx 2.8801$

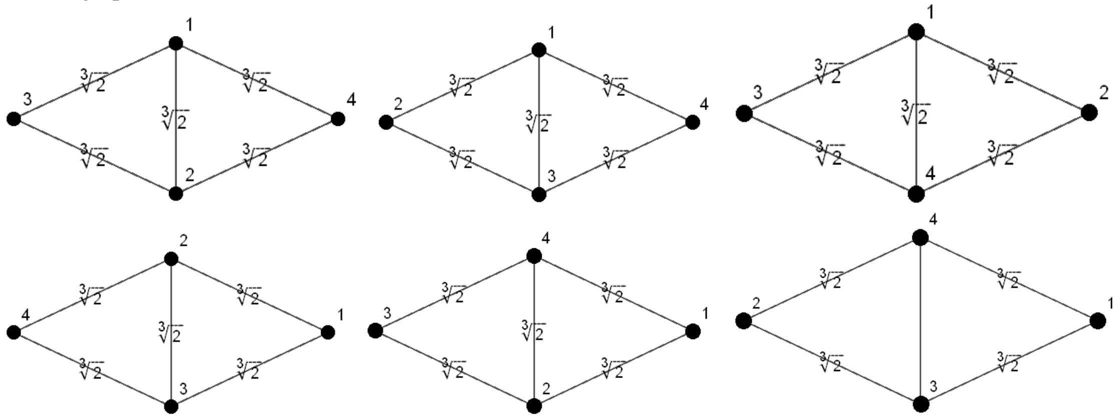
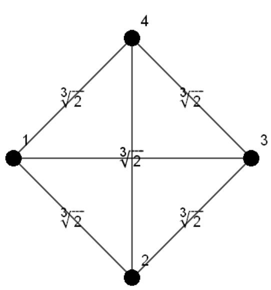


Fig 11(h)



the graph with $M \approx 3.4935$ **Fig 11(i)**

5. CONCLUSION

Hence the Mahler measure of charged graphs over the pure cubic field was found. The total number of graphs for particular model graphs was determined and the graphs with greatest and least Mahler measure were found. The idea discussed can be extended for the graphs over the ring of integers of pure cubic field and quartic field.

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