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SPN-MATRIX COMPLETION PROBLEM FOR STAR BI-DIRECTED GRAPH

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Abstract

In this paper an SPN-Matrix is considered. A Vertex distance path matrix (V_{Dp}) representing theStar bi-directed graph S_n is said to have an SPN-completion if every partial SPN-matrix specifying V_{Dp} can be completed to anSPN-Matrix. It is shown that the SPN-matrices representing all Star bi-directed graphs $S_n$of n vertices and 2(n-1) edges have an SPN-matrix completion. In addition a study on another distance matrix related to the q distance matrix andthe exponential distance matrix of a tree thathave an SPN–matrix completion is also characterized.

Keywords: Bi-directed Star graph, vertex distance path matrix, distance matrix,SPN-completion.

AMS Mathematics Subject Classification:15A18, 15A57

1. INTRODUCTION

In this paper, we consider a partial matrix is a matrix in which some entries are specified and other are not (all entries specified is also allowed). A completion of a partial matrix is a matrix obtained by choosing values for the unspecified entries. A pattern for n X n matrix, is a list of position of an n X n matrix, that is a subset of N X N, where N = {1, 2, 3,.... n}. The pattern of specified and unspecified entries is sufficient to assure that any partial vertex distance path matrix with this pattern can be completed to a SPN- Matrix. For a star bi-directed graph, we obtain a formula for the characteristic polynomial for a star graph, q-distance matrix and the exponential distance matrix of a tree. They are defined as follows:

Definition 1.1:
A matrix is an array of elements in a rectangular form which is arranged in rows and columns. The elements of an array are also known as entries. It can be manipulated in various ways such as addition, multiplication or

decomposition, to make them a keyconcept in linear algebra and matrix theory. In a matrix rows and columns are termed as m and n which is represented as an m x n matrix.

Definition 1.2:
A Sub matrix of an n x n matrix A is obtained by deleting some rows and columns of that matrix. For α a subset of {1,2,3,…,n}, a principal sub matrix $A(\alpha)$ of A is obtained from A by deleting all rows and columns that are not in α .The determinant of such a principal sub matrix $A(\alpha)$ is called a principal minor of $A(\alpha)$.

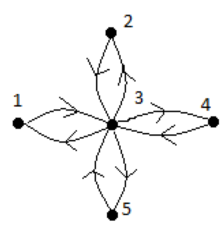
Definition 1.3:
The matrix completion problem [1] is concerned with determining whether or not a completion of a partial matrix that consist of a certain class of matrices. In such matrices, a description of positions is found in which choices for the unspecified entries may be made from a set so that the resulting matrix is known as a completion of the matrix, and is of the preferred type.

Definition 1.4:
A graph G = (V, E),where V represents the vertices and E denotes the edges,that is strongly joined, weighted and bi-directed is called a bi-directed graph. A bi-directed graph has twice the number of edges that the equivalent undirected graph has. Bi-directed graphs [2] are distinctive cases of Eulerian graphs (graphs for which the number of incoming edges equals the number of outgoing edges for each vertex).

Definition 1.5:
A tree (T) on n vertices and n-1 edges. The distance matrix D of a tree is an n×n matrix with D_{ij} = k, if the path from the vertex i to the vertex j is of length k; and D_{ii} = 0.

Definition 1.6:
A Star graph is a simple connected graph without any circuit. Let’s consider a star graph in which each edge is replaced by two arcs in whichever direction. In this paper, such star graph are called Star bi-directed graph.

A Star bi-directed graph $\mathbf{S_n}$, is a connected graph on n vertices where one vertex has degree 2(n-1) and the other n-1 vertices have degree 2. A star graph is a distinct case of a complete bipartite graph in which one set has one vertex and the other set has n-1 vertices.
$\mathbf{S_n=K_{1,2(n-1)}}$
Below is the graph $\mathbf{S_5=K_{1,8}}$



Definition 1.7:
The standard distance matrix or the vertex-distance matrix (or the minimum path matrix) of a vertex-labeled connected graph *G*, denoted by $\mathbf{V_D}$, is a real symmetric V x Vmatrix whose elements are defined as:
$\mathbf{V_{Dij}}$= {l_{ij} if $i \neq j$, 0 otherwise}
where$l(i,j)$ is the length of the shortest path, *i.e.*, the minimumnumber of edges, between the vertices *i* and *j* in *G*. The length $l(ij)$ is also called the distancebetween the vertices *i* and *j* in *G*, hence the term distance matrix.

Definition 1.8:
The entries in the vertex-distance-path matrix [3,4], denoted by $\mathbf{V_{Dp}}$are based on the elements of the vertex-distance matrix:

$$[V_{D_P}]_{ij} = \begin{pmatrix} [V_D]_{ij} + 1 \\ 2 \end{pmatrix} \text{if } i = j \text{, 0 otherwise.} \quad \text{---------------------- (1)}$$

It should be noted that the elements $[\mathbf{V_{Dp}}]_{ij}$count all the internal paths included in the shortest paths between the vertices *i* and *j* in a graph.

Definition 1.9:
A matrix is aP-matrix [5] if every principal minor is positive. A matrix is an N-matrix if every principal minor is negative.

Definition 1.10:
Ina matrix,if every principal minor of odd orderis positive and every principal minor of even order is negative then it is a (semi-) PN-matrix [6,7]. We call a real matrix an SPN-matrix if all the sums of its principal minors of odd order are positivein which$S_1(A) = 0$ always. Here,$S_1(A)$ is indicated as trace of A and all the sums of its principal minors of even order are negative.

Definition 1.11:
The partial SPN-matrices are from the star bi-directed graphs as follows: A specified entry a_{ij} will be used to signify an arc in the directed graph (digraph), an unspecified entry x_{ij} will be used to signify a missing arc in the digraph while d_i will specify the diagonal entries.

A Vertex distance path graph $[\mathbf{V}_{\mathbf{D}p}]_{ij}$ representing the star bi-directed graph S_n has SPN-completion if every partial SPN-matrix specifying $[\mathbf{V}_{\mathbf{D}p}]$ can be completed to a SPN-matrix for Star graph.

Definition 1.12:
Let $n \geq 3$ and A be an n x n SPN-matrix. The characteristic polynomial of A is given by $P(\lambda) = (-\lambda)^n + S_1(-\lambda)^{n-1} + S_2(-\lambda)^{n-2} + ... - S_{n-1}\lambda + S_n$,whereS_k denotes the sum of the principal minors of order k ofA. By definition, we have sign $S_k = (-1)^{k+1}, k = 1,2,...,n$.

Definition 1.13:
The well known formula to find the number of different combinations of n distinct objects taken r at a time is

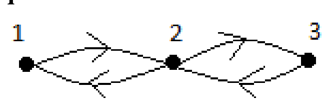
$$C(n,r)=\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

2. GENERALIZED N X N MATRICES SPECIFYING VERTEX DISTANCE PATH MATRIX FOR BI-DIRECTED STAR GRAPH

In this section, we extend certain theorems and results on vertex distance path matrix for star graph obtained by n x n matrices.A formula for the determinant is obtained, which generalizes the existing formulae principal minors for star graph. We also give a characteristic polynomial of the concerned matrix.

Theorem 2.1:
Suppose S_n is a star graph [8] with vertex set $V(S_n)=\{v_1,v_2,\ldots,v_n\}$.Let $D = (d_{i,j})_{n \times n}$ be the vertex distance path matrix of S_n , where $d_{i,j}$is the distance between the vertices v_i and v_j. Then the determinant of the matrix is $\det(D) = (-1)^{n-1} (n-1) 3^{n-2}$.

2.1. Analysis of the 3 x 3 matrices which specify the bi-directed star graph with 3 vertices and 4 edges. Deliberate the digraph below: p = 3, q = 4.



Let $A = \begin{bmatrix} d_{11} & a_{12} & x_{13} \\ a_{21} & d_{22} & a_{23} \\ x_{31} & a_{32} & d_{33} \end{bmatrix}$ be the partial vertex distance path matrix representing the bidirected star graph S_3.By the definition of $[\mathbf{V}_{\mathbf{D}p}]_{ij}$ completion,

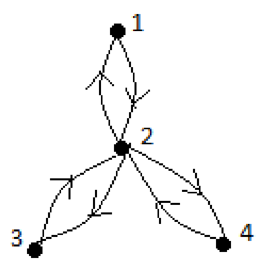
$$A[V_{D_p}]_{ij} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$, considering the principal minors;

Sub matrix	Determinant
A(1,2)	-1
A(1,3)	-9
A(2,3)	-1
A(1,2,3)	6

By the definition of the completion of $[\mathbf{V}_{\mathbf{D}p}]_{ij}$, $S_1(A)$ = trace of A = 0, $S_2(A) = -11$, $S_3(A) = 6$.
Then $S_1(A) = 0$, $S_2(A) < 0$ and $S_3(A) > 0$. The characteristic polynomial of A is P(A) = $-\lambda^3 + S_1\lambda^2 - S_2\lambda + S_3 = -x^3 + 11x + 6 = x^3 - 11x - 6$ .Hence A has an SPN-matrix completion.

2.2. Analysis of the 4 x 4 matrices which specifythe bi-directed star graph with 4 vertices and 6 edges.Consider the digraph below: p = 4, q = 6

Let $B = \begin{bmatrix} d_{11} & a_{12} & x_{13} & x_{14} \\ a_{21} & d_{22} & a_{23} & a_{24} \\ x_{31} & a_{32} & d_{33} & x_{34} \\ x_{41} & a_{42} & x_{43} & d_{44} \end{bmatrix}$ be the partial vertex distance path matrix representing the bidirected star graph S_4.



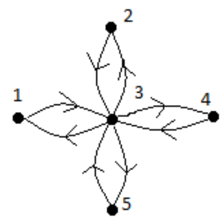
By the definition of $[\mathbf{V}_{\mathbf{D}p}]_{ij}$ completion, $B[V_{D_p}]_{ij} = \begin{bmatrix} 0 & 1 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 3 \\ 3 & 1 & 3 & 0 \end{bmatrix}$, considering the principal minors;

Sub matrix	Determinant	Sub matrix	Determinant
B(1,2)	-1	B(1,2,3)	6
B(1,3)	-9	B(1,2,4)	6
B(1,4)	-9	B(1,3,4)	54
B(2,3)	-1	B(2,3,4)	6
B(2,4)	-1	B(1,2,3,4)	-27
B(3,4)	-9		

By the definition of the completion of $[\mathbf{V}_{\mathbf{D}p}]_{ij}$, $S_1(B)$ = trace of B = 0, $S_2(B) = -30$, $S_3(B) = 72$, $S_4(B) = -27$.Then $S_1(B) = 0$, $S_3(B) > 0$ and $S_2(B), S_4(B) < 0$. The characteristic polynomial of A is P(B) = $\lambda^4 - S_1\lambda^3 + S_2\lambda^2 - S_3\lambda + S_4 = x^4 - 30x^2 - 72x - 27$ .Hence B has an SPN-matrix completion.

2.3 Analysis of the 5 x 5 matrices which specifythe bi-directed star graph with 5 vertices and 8 edges. Consider the digraph below: p = 5, q = 8

Let $C = \begin{bmatrix} d_{11} & x_{12} & a_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ a_{31} & a_{32} & d_{33} & a_{34} & a_{35} \\ x_{41} & x_{42} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & a_{53} & x_{54} & d_{55} \end{bmatrix}$ be the partial vertex distance path matrix representing the bidirected star graph S_5.



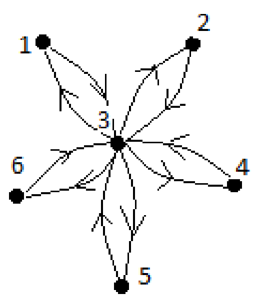
By the definition of $[V_{D_p}]_{ij}$ completion,

$$C[V_{D_p}]_{ij} = \begin{bmatrix} 0 & 3 & 1 & 3 & 3 \\ 3 & 0 & 1 & 3 & 3 \\ 1 & 1 & 0 & 1 & 1 \\ 3 & 3 & 1 & 0 & 3 \\ 3 & 3 & 1 & 3 & 0 \end{bmatrix}$$, considering the sum of all principal minors;

$S_1(C) = 0, S_2(C) = -58, S_3(C) = 252, S_4(C) = -351, S_5(C) = 108,$

Then $S_1(C) = 0, S_3(C), S_5(C) > 0$ and $S_2(A), S_4(C) < 0$. The characteristic polynomial of C is P(C) = $-\lambda^5 + S_1\lambda^4 - S_2\lambda^3 + S_3\lambda^2 - S_4\lambda + S_5 = x^5 - 58x^3 - 252x^2 - 351x - 108$.Hence C has an SPN-matrix completion.

2.4. Analysis of the 6 x 6 matrices which specify the bi-directed star graph with 6 vertices and 10 edges. Consider the digraph below: p = 6, q = 10



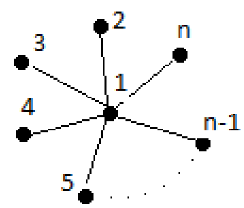
Let $D = \begin{bmatrix} d_{11} & x_{12} & a_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} & x_{26} \\ a_{31} & a_{32} & d_{33} & a_{34} & a_{35} & a_{36} \\ x_{41} & x_{42} & a_{43} & d_{44} & x_{45} & x_{46} \\ x_{51} & x_{52} & a_{53} & x_{54} & d_{55} & x_{56} \\ x_{61} & x_{62} & a_{63} & x_{64} & x_{65} & d_{66} \end{bmatrix}$ be the partial vertex distance path matrix representing the bidirected star graph S_6.By the definition of $[V_{D_p}]_{ij}$ completion,

$$D[V_{D_p}]_{ij} = \begin{bmatrix} 0 & 3 & 1 & 3 & 3 & 3 \\ 3 & 0 & 1 & 3 & 3 & 3 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 3 & 3 & 1 & 0 & 3 & 3 \\ 3 & 3 & 1 & 3 & 0 & 3 \\ 3 & 3 & 1 & 3 & 3 & 0 \end{bmatrix}$$, considering the sum of all principal minors;

By the definition of the completion of $[\mathbf{V}_{\mathbf{D}p}]_{ij}$,
$S_1(D) = 0, S_2(D) = -95, S_3(D) = 600, S_4(D)=1485, S_5(D) = 512, S_6(D) = -405$
Then $S_1(D) = 0, S_3(D), S_5(D) > 0$ and $S_2(D), S_4(D)$, $S_6(D) < 0$. The characteristic polynomial of C is P(D) = $x^6 - 95x^4 - 600x^3 - 1485x^2 - 1512x - 405$ .Hence D has an SPN-matrix completion.

2.5. Analysis of the n x n matrices which specify the bi-directed star graph with n vertices and 2(n-1) edges.

Consider the graph below: p = n, q = 2(n-1)



Let $G = \begin{bmatrix} d_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & d_{22} & x_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & \dots & d_{nn} \end{bmatrix}$ be the partial vertex distance path matrix representing the bidirected star graph S_n.By the definition of $[\mathbf{V}_{\mathbf{D}p}]_{ij}$ completion,

$$G[V_{D_p}]_{ij} = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 3 & \dots & 3 \\ 1 & 3 & 0 & \dots & 3 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 3 & 3 & \dots & 0 \end{bmatrix}$$, considering the sum of all n x n principal minors;

$$S_n(G) = (-1)^{n-1}(n-1)3^{n-2}$$
$$S_{n-1}(G) = (-1)^{n-2}(n-2)[(n-1)3^{n-3} + 3^{n-1}]$$
$$S_{n-2}(G) = (-1)^{n-3}(n-3)\left[\binom{n-1}{2}3^{n-4} + (n-1)3^{n-2}\right]$$
$$S_{n-3}(G) = (-1)^{n-4}(n-4)\left[\binom{n-1}{3}3^{n-5} + \binom{n-1}{2}3^{n-3}\right]$$
$$S_{n-4}(G) = (-1)^{n-5}(n-5)\left[\binom{n-1}{4}3^{n-6} + \binom{n-1}{3}3^{n-4}\right] \text{and so on.}$$
$$S_{n-5}(G) = (-1)^{n-6}(n-6)\left[\binom{n-1}{5}3^{n-7} + \binom{n-1}{4}3^{n-5}\right]$$
$$S_{n-6}(G) = (-1)^{n-7}(n-7)\left[\binom{n-1}{6}3^{n-8} + \binom{n-1}{5}3^{n-6}\right]$$

....

....

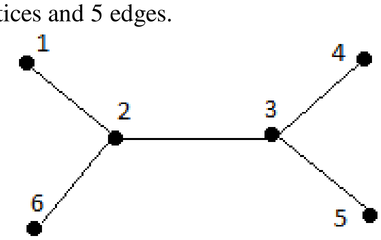
Hence G has an SPN-matrix completion.

3. q-DISTANCE MATRIX OF A TREE

In this section, we extend the result of Graham and Lovasz [9,10] by considering a new distance matrix which is termed as the q-distance matrix,denoted D = (D_{ij}) and defined as follows:
Let T be a tree on n vertices and let D = (D_{ij}) be its classical distance matrix. For an indeterminate q,we define

$$D_{ij} = \begin{cases} 1+q+q^2+\ldots+q^{k-1}, \text{ if } & D_{ij} = k \\ 0, \text{ if } & i = j \end{cases}$$

Analysis of the 6 x 6 matrices using the exponential distance matrix of a tree:
Consider the graph below with 6 vertices and 5 edges.



Let $E = \begin{bmatrix} 0 & 1 & 1+q & 1+q+q^2 & 1+q+q^2 & 1+q \\ 1 & 0 & 1 & 1+q & 1+q & 1 \\ 1+q & 1 & 0 & 1 & 1 & 1+q \\ 1+q+q^2 & 1+q & 1 & 0 & 1+q & 1+q+q^2 \\ 1+q+q^2 & 1+q & 1 & 1+q & 0 & 1+q+q^2 \\ 1+q & 1 & 1+q & 1+q+q^2 & 1+q+q^2 & 0 \end{bmatrix}$ be the distance matrix representing the q-distance matrix of T.

$S_1(F) = 0$
$S_2(F) = -q^4+f_1(t)$ where $\deg f_1(t) \leq 3$
$S_3(F) = 8q^5+f_2(t)$ where $\deg f_2(t) \leq 4$
$S_4(F) = -4q^6+f_3(t)$ where $\deg f_3(t) \leq 5$
$S_5(F) = 6q^5+f_4(t)$ where $\deg f_4(t) \leq 4$
$S_6(F) = -5q^4+f_5(t)$ where $\deg f_5(t) \leq 3$
Then$S_3(F)$,$S_5(F)$> 0 and $S_2(F)$,$S_4(F)$,$S_6(F)$ < 0 and $S_1(F)$ = 0.Hence E has an SPN-matrix completion.

4. THE EXPONENTIAL DISTANCE MATRIX OF A TREE

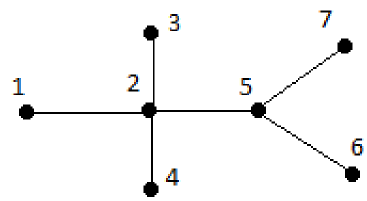
We extend another result of Graham and Lovasz [10] in this section by using a new distance matrix which is termed as the exponential distance matrix, denoted $D = (D_{ij})$ and is defined as follows:

Let T be a tree on n vertices and let $D = (D_{ij})$ be its distance matrix. We now consider an n x n matrix $F = (F_{ij})$, called the exponential distance matrix, with

$F_{ij} = \{0, \text{ if } i = j;\ q^{D_{ij}} \text{ if } i \neq j\}$.

Analysis of the 7 x 7 matrices using the exponential distance matrix of a tree:

Consider the graph belowwith 7vertices and 6 edges.



Let $F = \begin{bmatrix} 0 & q & q^2 & q^2 & q^2 & q^3 & q^3 \\ q & 0 & q & q & q & q^2 & q^2 \\ q^2 & q & 0 & q^2 & q^2 & q^3 & q^3 \\ q^2 & q & q^2 & 0 & q^2 & q^3 & q^3 \\ q^2 & q & q^2 & q^2 & 0 & q & q \\ q^3 & q^2 & q^3 & q^3 & q & 0 & q^2 \\ q^3 & q^2 & q^3 & q^3 & q & q^2 & 0 \end{bmatrix}$ be the distance matrix representing the exponential distance matrix of T.

$S_1(F) = 0$
$S_2(F) = -6q^6+f_1(t)$ where $\deg f_1(t) \leq 4$
$S_3(F) = 18q^8+f_2(t)$ where $\deg f_2(t) \leq 6$
$S_4(F) = -10q^{10}+f_3(t)$ where $\deg f_3(t) \leq 8$
$S_5(F) = 6q^{12}+f_4(t)$ where $\deg f_4(t) \leq 10$
$S_6(F) = -5q^{12}+f_5(t)$ where $\deg f_5(t) \leq 10$
$S_7(F) = q^{12}+f_6(t)$ where $\deg f_6(t) \leq 10$

Then $S_3(F), S_5(F), S_7(F) > 0$ with $S_1(F) = 0$ and $S_2(F), S_4(F), S_6(F) < 0$.
Hence F has an SPN-matrix completion.

5. CONCLUSION

This work presents the result of star bidirected graph by means of vertex distance path matrix. Here a condition for S_n to have an SPN-completion is obtained. In the case of q-distance matrix and the exponential distance matrix of a tree it is necessary to ensure that T must also have an SPN-completion. Furthermore, this work can be extended to other different classes of matrices.

REFERENCES

[1] HogbenL. (2001). Graph Theoretic methods for matrix completion problems, in Handbook of Linear Algebra, (L. Hogben, Editor), Chapman & Hall/CRC Press, Boca Raton, 328:161-202.

[2] WestD. (2001). Introduction to Graph theory, Second Edition, Prentice Hall.

[3] Diudea MV. (1996). Walk numbers $^{e}W_M$: Wiener-type numbers of higher rank, *Journal of chemical information and computer sciences,* .May 23; 36(3):535-40.

[4] DiudeaMV(1996). Wiener and hyper-Wiener numbers in a single matrix, *Journal of chemical information and computer sciences*. 1996 Jul 24; 36(4):833-836.

[5] DeAlbaL.M., HogbenL. (2000). Completions of P-matrix Patterns, *Linear Algebra and its Applications*, 319;83-102.

[6] Garloff Jurgen, Hattenbach Volker. (1987)., The Spectra of Matrices having Sums of Principal Minors with Alternating Sign,*SIAM Journal on Algebraic Discrete Methods,* Vol.8, No.1,January 1987.

[7] John.S. Maybee (1976), Some Aspects of the theory of PN-Matrices, *SIAM Journal on Applied Mathematics*, Vol.31, and No.2:PP.397-410.

[8] TilliaValerie (2010).M.S. in Mathematical Sciences, Portland State University, Project: Determinants of Distance Matrices of Trees, *Linear Algebra and its Applications*,

[9] YanWeigen, Yeh Yeong-Nan (2006). A Simple Proof of Graham and Pollak's Theorem, *Journal of Combinatorial Theory Series*, 892-893, July 2006.

[10] Graham R.L., Lovasz L. (2001). Distance matrix Polynomials of trees, *Adv. Math.*, 29, 60-88.