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APPLICATION OF MIXED QUADRATURE FOR NUMERICAL EVALUATION OF FRACTIONAL INTEGRALS

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Abstract

In this paper, we improve the corrective factor approach using a mixed quadrature rule for numerical integration of fractional integral of order $\alpha,\ 0 < \alpha < 1$.

Keywords: Mixed quadrature rule, fractional integral, corrective factors, semi-fractional integral, Fejer's second quadrature rule, Gaussian rule.

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1. NTRODUCTION

The generalization of ordinary differentiation and integration to arbitrary (non-integer) order is known as fractional calculus. The great mathematicians like Liouville, Riemann, Fourier, Abel, Lebintz and Grunwall have contributed to this theory in a significant manner. Recently, fractional calculus has been applied in various areas of engineering, science, finance, economics, fluid dynamics, bio-engineering etc. In 1977, Leather [9] studied how to overcome singularities in numerical integration. In 1981, Acharya and Das [1] derived the Cauchy principal value integral in an alternative way in the presence of nearby singularity of the integrand. Further, Leather et al. [10] have applied the Gauss Legendre rules of indices and have obtained numerical approximation of the semi integral $D^{-\frac{1}{2}}f(x)$ when $\alpha = \frac{1}{2}$ with respect to the functions $f(t) = e^t$ and $f(t) = \frac{1}{t+1}$. In 2010 Dalir and Bashour [5] introduced some applications fractional calculus while, in 2011 Acharya et. al. [2] introduced a novel approach for evaluation of the semi integral of a function using fractional integrals through corrective factors. The authors have used Gauss-Legendre rule and Radau rule to evaluate the fractional integrals in this process.

In this paper although we use corrective factor approach, the results have been greatly improved introducing a newly designed mixed quadrature rule as shown in the Table-3.1.

The Riemann-Liouville fractional integral operator of order $\alpha, 0 < \alpha < 1$ of a function $f(x)$ is defined as

$$D^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt \tag{1.1}$$

In equation (1.1), the fractional integral $D^{-\alpha}f(x)$ possesses a singularity at $x = 0$ which is the right hand end point of the range of integration $(0,\ x)$. Sometimes we observe that the direct application of any quadrature rule yields results of reasonable accuracy if x is small, but the result is quite inaccurate when x moves farther and farther from the point $x = 0$. For example, the approximate solution of $\left[D^{-\frac{1}{2}}(e^x)\right]_{x=1}$ by using Gauss-Legendre 5-point rule is 1.67245929656961 while the exact value of $\left[D^{-\frac{1}{2}}(e^x)\right]_{x=1}$ is 2.2906982523324. Thus it is essential that some corrective factors should be employed while applying a quadrature rule, if the point where the fractional integral is sought is not in the proximity of $x = 0$.

2. CONSTRUCTION OF CORRECTIVE FACTORS FOR THE MIXED QUADRATURE RULE

For the numerical evaluation of fractional order integrals $D^{-\alpha}f(x)$, open or semi open quadrature rules should be used for x has to be excluded from the set of nodes of the rule. The accuracy of the computed values of the fractional integral $D^{-\alpha}f(x)$ depends upon the degree of precision of the quadrature rule and also on x. Some authors [1,9] have successfully tried to overcome the singularity by using the method of corrective factor. The Taylor series expansion of the generating function $f(t)$ in ascending powers of $(t-x)$ is defined as

$$f(t) = \sum_{i=0}^{\infty} a_i\,(t-x)^i, \quad a_i = \frac{f^i(x)}{i!}. \tag{2.1}$$

Let $h(t) = f(t) - \sum_{i=0}^{r} a_i\,(t-x)^i$. (2.2)

Using equation (1.1), we have

$$\begin{aligned} D^{-\alpha}h(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)-\sum_{i=0}^{r} a_i(t-x)^i}{(x-t)^{1-\alpha}}dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt - \frac{1}{\Gamma(\alpha)}\int_0^x \frac{\sum_{i=0}^{r} a_i(t-x)^i}{(x-t)^{1-\alpha}}dt \\ &= D^{-\alpha}f(x) - \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r} a_i \int_0^x \frac{(t-x)^i}{(x-t)^{1-\alpha}}dt \\ &= D^{-\alpha}f(x) - \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r} a_i\,(-1)^i \int_0^x \frac{(x-t)^i}{(x-t)^{1-\alpha}}dt \\ &= D^{-\alpha}f(x) - \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r} a_i\,(-1)^{i+1}\ \frac{x^{i+\alpha}}{i+\alpha}. \end{aligned}$$

Hence $D^{-\alpha}h(x) = D^{-\alpha}f(x) - C_r(x)$. (2.3)

where $C_r(x) = \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r} a_i\,(-1)^{i+1}\ \frac{x^{i+\alpha}}{i+\alpha}$. (2.4)

is the corrective factor of order r.

From equation (2.3) we have

$$D^{-\alpha}f(x) = D^{-\alpha}h(x) + C_r(x). \tag{2.5}$$

The mixed quadrature rule as designed by Behera et al. in [4] is as follows:

$$R_{2F5GL3}(f) = \frac{1}{2205}\left[896f\left(-\frac{\sqrt{3}}{2}\right) - 375f\left(-\sqrt{\frac{3}{5}}\right) + 1152f\left(-\frac{1}{2}\right) + 1064f(0) + 1152f\left(\frac{1}{2}\right) - 375f\left(\sqrt{\frac{3}{5}}\right) + 896f\left(\frac{\sqrt{3}}{2}\right)\right]. \tag{2.6}$$

The degree of precision of this rule seven.

Using the mixed rule (2.6) for evaluating the fractional integral $D^{-\alpha}h(x)$ over $(0,\ x)$ of equation (2.5) reduces to

$$\begin{aligned} D^{-\alpha}f(x) &\approx R_{\alpha,\ 7}^{2F5GL3}(h;\ x) + C_r(x) \\ &\approx R_{\alpha,\ 7}^{2F5GL3\ C_r}(h;\ x). \end{aligned} \tag{2.7}$$

where $R_{\alpha,\ 7}^{2F5GL3}(f;\ x)$ denotes the approximation of $D^{-\alpha}f(x)$ by the mixed quadrature rule (2.6) and $R_{\alpha,\ 7}^{2F5GL3\ C_r}(h;\ x)$ denotes the corrective approximation of $D^{-\alpha}h(x)$ by the mixed quadrature rule (2.6).

The absolute errors associated to the rules $R_{\alpha,\ 7}^{2F5GL3}(f;\ x)$ and $R_{\alpha,\ 7}^{2F5GL3\ C_r}(h;\ x)$ with respect to $D^{-\alpha}f(x)$ is given by

$$\left|D^{-\alpha}f(x) - R_{\alpha,\ 7}^{2F5GL3}(f;\ x)\right| \text{ and } \left|D^{-\alpha}f(x) - R_{\alpha,\ 7}^{2F5GL3\ C_r}(h;\ x)\right|. \tag{2.8}$$

3. NUMERICAL VERIFICATION

Consider the semi integral $D^{-\frac{1}{2}} f(t)$ (as $\alpha = \frac{1}{2}$) of the function $f(t) = e^t$ for which the exact value is given by

$$D^{-\frac{1}{2}} e^x = \frac{1}{\sqrt{\pi}} \int_0^x \frac{e^t}{(x-t)^{\frac{1}{2}}} dt = e^x \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-t^2} dt \approx e^x \, erf(\sqrt{x}).$$

For applying the mixed quadrature rule (2.6) for finding the semi integral $D^{-\frac{1}{2}} e^x$ by taking $f(t) = e^t$ and $\alpha = \frac{1}{2}$ in equation (2.7) we have

$$D^{-\frac{1}{2}} e^x \approx R_{\alpha,\ 7}^{2F5GL3\ C_r}(e^t;\ x)$$

and the absolute errors associated with the rules $R_{\frac{1}{2},\ 7}^{2F5GL3}(e^t;\ x)$ and $R_{\frac{1}{2},\ 7}^{2F5GL3\ C_r}(e^t;\ x)$ with respect to $D^{-\alpha} f(x)$ is given by

$$\left| D^{-\frac{1}{2}} e^x - R_{\frac{1}{2},\ 7}^{2F5GL3}(e^t;\ x) \right| \text{ and } \left| D^{-\frac{1}{2}} e^x - R_{\frac{1}{2},\ 7}^{2F5GL3\ C_r}(e^t;\ x) \right|.$$

The numerical values have been recorded in the following table (Table-3.1).

Table 3.1:

x	Exact Value	Approximated Value $(Q(f))$ by				
		$R_{\frac{1}{2},\ 7}^{2F5GL3}$	$R_{\frac{1}{2},\ 7}^{2F5GL3C_0}$	$R_{\frac{1}{2},\ 7}^{2F5GL3C_1}$	$R_{\frac{1}{2},\ 7}^{2F5GL3C_2}$	$R_{\frac{1}{2},\ 7}^{2F5GL3C_3}$
		$\lvert Error \rvert$	$\lvert Error \rvert$	$\lvert Error \rvert$	$\lvert Error \rvert$	$\lvert Error \rvert$
		0.5886405838	0.6681864551	0.6683392147	0.6683347957	0.6683350891
0.25	0.6683350725	0.07969448872	1.49×10^{-4}	4.14×10^{-6}	2.77×10^{-7}	1.66×10^{-8}
		1.004941134	1.125038157	1.125592939	1.125519477	1.127522627
0.5	1.123163951	0.1182228167	1.92×10^{-3}	2.43×10^{-3}	2.36×10^{-3}	4.36×10^{-3}
		1.4594827	1.648624142	1.649932829	1.649819426	1.649841531
0.75	1.648656223	0.189173523	3.21×10^{-5}	1.28×10^{-3}	1.16×10^{-3}	1.19×10^{-3}
		2.008642598	2.288344956	2.290932067	2.290633156	2.290710842
1.0	2.290698252	0.282055654	2.35×10^{-3}	2.39×10^{-4}	6.51×10^{-5}	1.26×10^{-5}

4. CONCLUSION

From the above table we observe that the results of semi-fractional integral by using mixed rule with corrective factors is highly encouraging in comparison to those in the paper of Acharya, et al. [2] using the basic rules.

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