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A FIXED POINT THEOREM IN COMPLEX VALUED $b-$ METRIC SPACE USING CONTRACTIVE MAPPINGS

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Abstract

The aim of the present paper is to establish a fixed point theorem in complex valued $b-$metric space under contractions.

Keywords: Complex valued $b-$metric space, contraction, fixed point

Subject Classification: 47H10, 54H25

1. INTRODUCTION

The study of fixed points of mappings in complex valued metric space is at the center of current research. The concept complex valued b-metric is considered as the generalization of the $b-$metric space. Here it may be mentioned that Azam et al. [2] introduced the concept of complex valued metric space in 2011.

In this paper we prove the existence of fixed point for contractive mapping in complex valued $b-$metric space. This type of contraction is studied earlier by many authors in distinct spaces.

2. PRELIMINARIES

Here we recall the definitions, examples and results which will be used in the sequel.

A $b-$metric space is considered as a generalization of a metric space regarding which the necessary definitions are as stated below:

Definition 2.1: Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to Y$ is called a $b-$metric if the following conditions are satisfied.

(M1) $d(x,y)>0$ and $d(x,y)=0$ if and only if $x=y$.

(M2) $d(x,y)=d(y,x)$ for all $x,y\in X$.

(M3) $d(x,y)\le s\left[d(x,z)+d(z,y)\right]$ for all $x,y\in X$.

Then (X,d) is called a $b-$metric space.

Definition 2.2: Let $\mathbb{C}$ be the set of complex numbers and let $z,w\in\mathbb{C}$. Define a partial order relation on $\mathbb{C}$ such that $\operatorname{Re}(z)\le\operatorname{Re}(w)$ and $\operatorname{Im}(z)\le\operatorname{Im}(w)$.

Definition 2.3: Let X be a non-empty set. Suppose that the mapping $d:X\times X\to\mathbb{C}$ satisfies the following conditions:

(M1) $d(x,y)>0$ and $d(x,y)=0$ if and only if $x=y$.

(M2) $d(x,y)=d(y,x)$ for all $x,y\in X$.

(M3) $d(x,y)\le d(x,z)+d(z,y)$ for all $x,y\in X$.

Then d is called a complex valued metric and (X,d) is called a complex valued metric space.

Example 2.4: Let X be a complex valued $b-$metric space. Consider $d(z,w)=5i|z-w|$, then d is a complex valued metric.

Definition 2.5: Let X be a non-empty set and let $s\ge 1$. The mapping $d:X\times X\to\mathbb{C}$ is called a complex valued metric on X if it satisfies the following conditions.

(M1) $d(x,y)>0$ and $d(x,y)=0$ if and only if $x=y$.

(M2) $d(x,y)=d(y,x)$ for all $x,y\in X$.

(M3) $d(x,y)\le s\left[d(x,z)+d(z,y)\right]$ for all $x,y\in X$.

Then (X,d) is called a complex valued $b-$ metric space. If $b=1$ we get complex valued metric space.

Example 2.6: Let X be a complex valued $b-$metric space. Consider $d(x,y)=(1+i)|x-y|^2$ for $x,y\in X$ with $s=2$, then, d is a complex valued $b-$ metric.

Definition 2.7:

1. If $\{x_n\}$ be a sequence in X and let $x\in X$. Then $\{x_n\}$ is said to be Cauchy sequence, if $\{x_n\}$ converges to $x\in X$. We denote this by writing $\lim_{n\to\infty} x_n = x$.
2. If $c\in X$ with $0<c$ and there exists an $n\in\mathbb{N}$ such that $d(x_{n+m},x_n)<c$ where $m\in\mathbb{N}$, then $\{x_n\}$ is said to be a Cauchy sequence.
3. If every Cauchy sequence in X is convergent then (X,d) is said to be a complete complex valued $b-$metric space.

Definition 2.8: Let (X,d) be a complex valued $b-$metric space and $S:X\to X$. Then

1. S is said to be sequentially convergent if we have for any sequence $\{x_n\}$, if $\{S(x_n)\}$ is convergent, then $\{x_n\}$ also is convergent;
2. S is said to be subsequentially convergent if for every sequence $\{x_n\}$ that $\{S(x_n)\}$ is convergent, $\{x_n\}$ has a convergent subsequence;
3. S is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} S(x_n) = S(x)$ for all $x \in X$.

Definition 2.9: Let f and g be self-maps on a set X, if $w = f(x) = g(x)$ for some $x \in X$, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.

Definition 2.10: Let f and g be self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at their coincidence points.

Definition 2.11: Let X be a non-empty set and $f, g : X \to X$ be mappings. A pair (f, g) is called weakly compatible if $x \in X, f(x) = g(x)$ implies $fg(x) = gf(x)$.

Azam et al. [2] proved the following lemmas for proving their theorems:

Lemma 2.12: Let (X, d) be a complex valued metric space and any sequence $\{x_n\}$ in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$.

Lemma 2.13: Let (X, d) be a complex valued metric space and any sequence $\{x_n\}$ in X. Then $\{x_n\}$ is said to be a Cauchy sequence if and only if $|d(x_n, x)| \to 0$ for $n > m$.

Definition 2.14: (Samet et al. [11]) Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi$ contraction mapping, if two functions $\alpha, \psi : X \to [0,1)$ such that $\alpha(x, y) d(T(x), T(y)) \le \psi(d(x, y))$.

Definition 2.15: (Samet et al. [11]) Let (X, d) be a complete metric space and $T : X \to X$ and $\alpha : X \times X \to [0,1)$. We say that T is $\alpha -$ admissible if for $x, y \in X, d(x, y) \ge 1$ implies that $d(T(x), T(y)) \ge 1$.

3. Main Results

Let $\mathbb{C}^+ = \{z \in \mathbb{C} / \mathrm{Re}(z) \ge 0\}$ and φ be the family of non-decreasing functions $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ such that $\varphi^n(t) < 1$ for each $t > 0$, where φ^n is the n$^{\text{th}}$ iterate of φ.

Definition 3.1: Let (X, d) be a complex valued $b -$ metric space and $f : X \times X \to X$ be a mapping. We say that T is an $\alpha - \psi$ contraction mapping if there exist two functions $\alpha : X \to [0,1)$ and $\psi : X \to [0,1)$ such that

$\alpha(x,y)d(T(x),T(y))\le\psi[M(x,y)]$ for $x,y\in X$, where,

$$M(x,y)=\sup\left\{d(x,T(x)),\frac{d(x,T(x))+d(y,T(y))}{2},1+d(x,T(x))d(y,T(y))\right\}.$$

Theorem 3.2: Let (X,d) be a complex valued $b-$metric space with $s\ge 1$. Let $T:X\to X$ be a contractive mapping satisfying the following conditions:

1. T is admissible,
2. There exists $x_0\in X$ such that $\alpha(x_0,T(x_0))\ge 1$,
3. T is reciprocally continuous,

Then the equation $T(x)=x$ has a solution, i.e., there exists a fixed point.

Proof: Let $x_0\in X$ such that $\alpha(x_0,T(x_0))\ge 1$. Construct a sequence $\{x_n\}$ in X such that $x_{n+1}=T(x_n)$ for all $n\in\mathbb{N}$. If $x_n=x_{n+1}$, then x_n is a fixed point of T. Assume $x_n\ne x_{n+1}$ for all $n\in\mathbb{N}$. Given that T is $\alpha-$admissible. Now, $\alpha(x_0,x_1)=\alpha(x_0,T(x_0))\ge 1$, which implies, that $\alpha(x_1,T(x_1))=\alpha(T(x_0),T(x_1))\ge 0\Rightarrow\alpha(x_1,x_2)\ge 0$. By mathematical induction we get $\alpha(x_k,x_{k+1})\ge 0$ for all $k=0,1,2,3,\ldots$. Since T satisfies

$$\alpha(x,y)\ge 1\Rightarrow d(T(x),T(y))\le\psi[d(M(x),y)]$$

and ψ is non-decreasing we have $d(x_1,x_2)=d(T(x_0),T(x_1))\le\psi[d(x_0,x_1)]$. As ψ is monotonic, so it is continuous at $t=0$, which implies, $x_n=x_{n+1}$. Then x_n is a fixed point of T. Further assuming that $x_{n+1}\ne x_n$ for all $n\in\mathbb{N}$ and using the contractive condition, with the fact that T is $\alpha-$admissible, we have

$$\alpha(x_0,x_1)=\alpha(x_0,T(x_0))\ge 1\Rightarrow\alpha(T(x_0),T(x_1))=\alpha(x_1,x_2)\ge 1,$$
$$\alpha(x_1,x_2)=\alpha(x_1,T(x_1))\ge 1\Rightarrow\alpha(T(x_1),T(x_2))=\alpha(x_2,x_3)\ge 1,$$

So by mathematical induction $\alpha(x_n,x_{n+1})\ge 1$ and by the inequality $\alpha(x,y)d(T(x),T(y))\le\psi[M(x,y)]$ and the triangle inequality

$$\begin{aligned}d(x_n,x_m)&\le s[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+\ldots+d(x_{m-1},x_m)]\\&\le s\psi^n[d(x_0,x_1)]+s\psi^{n+1}[d(x_0,x_1)]+s[\psi^{n+2}d(x_0,x_1)]+\ldots+s\psi^{n+m-1}[d(x_0,x_1)]\end{aligned}$$
$$d(x_n,x_m)\le s\psi^n[d(x_0,x_1)]+s\psi^{n+1}[d(x_0,x_1)]+s\psi^{n+2}[d(x_0,x_1)]+\ldots+s\psi^{n+m}[d(x_0,x_1)]$$

or,

$$d(x_n,x_m)\le\sum\psi^k[d(x_0,x_1)]$$

for all $x,y\in X$ and $\sum\psi^k[d(x_0,x_1)]<\infty$. For each $t\ge 0$, by letting $n,m\to\infty$ we obtain $d(x_n,x_m)\to 0$ for $n>m$. Thus $\{x_n\}$ is a Cauchy sequence in complex valued $b-$metric space X. Since (X,d) is complete, every Cauchy sequence converges, so, $x_n\to u$, $x_n=\lim_{n\to\infty}T(x_{n-1})=T(u)$, which means that as $x_n\to u$ we have, $T(u)=u$. Thus T has a fixed point.

Next we want to prove the uniqueness. There exists $w\in X$ such that $\alpha(u,w)\ge 1$ and $\alpha(v,w)\ge 1$. Using the definition of an $\alpha-$admissible map $\alpha(T(u),T(w))\ge 1$. Consider

$$d\left(v,T^{n}(u)\right)=d\left(T(v),T^{n-1}(u)\right)\leq\psi\left[d\left(v,T^{n-1}(u)\right)\right]\leq\psi^{n}\left[d(v,u)\right]\leq\infty$$

So, finally $d\left(T^{n}(u),u\right)\to 0$, i.e., $T^{n}(u)\to u$.

Next consider

$$d\left(v,T^{n}(u)\right)=d\left(T(v),T^{n+1}(u)\right)\leq\psi\left[d\left(v,T^{n-1}(v)\right)\right]\leq\psi^{n}\left[d(v,v)\right]\leq\infty$$

which gives $d\left(T^{n}(v),v\right)\to 0$, i.e., $T^{n}(v)=v$. By uniqueness of limit, $v=u$. Hence the fixed point is unique.

Example 3.3: Let $X=[0,1)$ with the complex valued metric $d(x,y)=3i|x-y|$ for all $x,y\in X$ with $s=2$. Let $\mathbb{C}^{+}=\{z\in\mathbb{C}/\operatorname{Re}(z)\geq 0\}$ and ψ be the family of non-decreasing functions $\psi:\mathbb{C}^{+}\to\mathbb{C}^{+}$ such that $\psi^{n}(t)<1$ for each $t>0$, where, ψ^{n} is the n$^{\text{th}}$ iterate of ψ. Let (X,d) be a complex valued $b-$metric space and $f:X\times X\to X$ be a mapping. We say that T is an $\alpha-\psi$ contraction mapping if there exist two functions $\alpha:X\to[0,1)$ and $\psi\in\varphi$ such that $\alpha(x,y)d\left(T(x),T(y)\right)\leq\psi\left[M(x,y)\right];x,y\in X$, where, $M(x,y)=\sup\left\{d\left(x,T(x)\right),\dfrac{d\left(x,T(x)\right)+d\left(y,T(y)\right)}{2},1+d\left(x,T(x)\right)d\left(y,T(y)\right)\right\}$.

Here (X,d) is a complex valued $b-$metric space with $s\geq 1$. Define the mapping

$$T(x)=\begin{cases}5x-\dfrac{10}{3}, & \text{if } x>1\\ 0, & \text{if } x\in[0,1)\end{cases}$$

and

$$\alpha(x,y)=\begin{cases}1, & \text{if } x\in[0,1)\\ 0, & \text{otherwise}\end{cases}$$

Here T is continuous and reciprocally continuous. Also T is $\alpha-$admissible, so there exist $x_0\in X$ such that $\alpha\left(x_0,T(x_0)\right)\geq 1$. So 0 is the unique fixed point of T.

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