Original Article

Available online at
www.bpasjournals.com

Bulletin of Pure and Applied Sciences.
Vol. 38E (Math & Stat.), No.1, 2019. P.245-252
Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2019.00023.7

ON THE HOMOGENEOUS CONE $3x^2 - 8y^2 = 25z^2$

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Received on 07.01.2019 Revised on 20.04.2019 Accepted on 08.05.2019

Abstract:
This paper aims at determining non-zero distinct integer solutions satisfying the homogeneous cone represented by the ternary quadratic equation $3x^2 - 8y^2 = 25z^2$. A few interesting relations among the solutions are presented. A general formula for generating sequence of integer solutions to the given cone based on a given solution is illustrated.

Keywords: Ternary quadratic, homogeneous quadratic, homogeneous cone, integer solutions.

2010 A.M.S. Mathematics Subject Classification: 11D09.

1. INTRODUCTION

The quadratic Diophantine equations with three unknowns offer an unlimited field for research because of their variety [1-3]. For an extensive review of various problems on ternary quadratic Diophantine equations representing specific 3-dimensional surfaces, one may refer to [4-13]. In this communication, we search for non-zero distinct integer solutions satisfying the homogeneous cone represented by the ternary quadratic equation $3x^2 - 8y^2 = 25z^2$. A few interesting relations among the solutions are presented. A general formula for generating sequence of integer solutions to the given cone based on a given solution is illustrated.

2. NOTATIONS

1. $GNO_n = 2n - 1$ - Pentagonal pyramidal number of rank *n*.
2. $PR_n = n(n+1)$ - Pronic number of rank *n*.
3. $t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$ - Polygonal number of rank *n* with size *m*.

4. $S_n = 6n^2 - 6n + 1$ - Star number of rank n.
5. $t_{3,n} = \frac{n(n+1)}{2}$ -triangular number of rank n.

3. METHOD OF ANALYSIS

Consider the homogeneous cone represented by the ternary quadratic equation

$$3x^2 - 8y^2 = 25z^2. \tag{1}$$

We present below different methods of solving (1) and thus, obtain different sets of integer solutions to (1).

Method 1:

Introducing the linear transformations

$$x = X + 8T \quad , \quad y = X + 3T \tag{2}$$

$$X^2 + 5z^2 = 24T^2 \tag{3}$$

Assume

$$T = T(a,b) = a^2 + 5b^2 \tag{4}$$

write the positive integer 24 as

$$24 = \left(2 + i2\sqrt{5}\right)\left(2 - i2\sqrt{5}\right) \tag{5}$$

Using (4), (5) in (3) and employing the method of factorization, define

$$X + i\sqrt{5}z = \left(2 + i2\sqrt{5}\right)\left(a + i\sqrt{5}b\right)^2$$

Equating the real and imaginary parts in the above equation, one obtains

$$X = X(a,b) = 2a^2 - 10b^2 - 20ab \tag{6}$$

$$z = z(a,b) = 2a^2 - 10b^2 + 4ab \tag{7}$$

Substituting (4) and (6) in (2), we have

$$\left.\begin{aligned} x &= x(a,b) = 10a^2 + 30b^2 - 20ab \\ y &= y(a,b) = 5a^2 + 5b^2 - 20ab \end{aligned}\right\} \tag{8}$$

Note that (7) and (8) represent the non-zero distinct integer solutions to (1).

Properties:

- $x(1,b) + 2y(1,b) = 40t_{3,b}$
- $y(a,1) - 10t_{3,n} + 25a \equiv 0 \pmod{5}$
- $z(a,a+1) + 4PR_a + 6GNo_a + 16 = 0$

- $z(1,b) - 5z(1,b) = 40t_{6,b}$
- $x(a,1) - 2y(a,1) - 10GNo_a \equiv 0 \pmod 3$

Method 2:

(3) is written as

$$X^2 + 5z^2 = 24T^2 * 1 \tag{9}$$

Assume $1 = \frac{(2+i\sqrt{5})(2-i\sqrt{5})}{9}$ (10)

Substituting (4), (5) and (11) in (10) and employing the method of factorization, define

$$(X + i\sqrt{5}z) = \frac{(2+i\sqrt{5})}{3}(2+i2\sqrt{5})(a+i\sqrt{5}b)^2 \tag{11}$$

Equating the real and imaginary parts in (11), we have

$$X = X(a,b) = -2a^2 + 10b^2 - 20ab \tag{12}$$

$$z = z(a,b) = 2a^2 - 10b^2 - 4ab \tag{13}$$

Substituting (12) and (4) in (2),

$$\left.\begin{aligned} x = x(a,b) = 6a^2 + 50b^2 - 20ab \\ y = y(a,b) = a^2 + 25b^2 - 20ab \end{aligned}\right\} \tag{14}$$

Thus, (13) and (14) represents the integer solutions to (1).

Properties:

- $x(a,a) - t_{a,14} - 5a = 0$
- $y(a,a) - 5z(a,a) - 11S_a - 33GNo_a \equiv 0 \pmod{11}$
- $x(a,a) - y(a,a) - 30PR_a + 15GNo_a + 15 = 0$
- $y(a,a)$ is a nasty number
- $z(b,b+1) + 24t_{3,n} + 6GNo_b + 16 = 0$

Note: It is to be noted that, in addition to (10), 1 may also be represented as shown below:

Way 1: $1 = \frac{(1+i4\sqrt{5})(1-i4\sqrt{5})}{81}$ (15)

Way 2: $1 = \frac{(2+i3\sqrt{5})(2-i3\sqrt{5})}{49}$ (16)

Following the procedure as above, the corresponding integer solutions to (1) for (15) and (16) are presented below:

Solutions for (15):

$$x = x(A,B) = 306A^2 + 4950B^2 - 900AB$$
$$y = y(A,B) = -99A^2 + 2925B^2 - 900AB$$
$$z = z(A,B) = 90A^2 - 450B^2 - 684AB$$

Solutions for (16):

$$x = x(A,B) = 210A^2 + 2870B^2 - 700AB$$
$$y = y(A,B) = -35A^2 + 1645B^2 - 700AB$$
$$z = z(A,B) = 70A^2 - 350B^2 - 364AB$$

Generation of solutions

Here we obtain the formula for generating sequence of integer solutions to (1) based on its initial solution.

Let (x_0, y_0, z_0) be the initial solution of (1).

Formula 1:

Let (x_1, y_1, z_1) be the second solution of (1),

where $x_1 = 33x_0$, $y_1 = h - 33y_0$, $z_1 = h - 33z_0$ (17)

Substituting (17) in (1) and simplifying, we get

$$h = 16y_0 + 50z_0$$

Thus, the second solution (x_1, y_1, z_1) to (1) is given by

$$x_1 = 33x_0\ ,\ y_1 = -17y_0 + 50z_0\ ,\ z_1 = 16y_0 + 17z_0$$

Express y_1 and z_1 in the form of 2×2 matrix as follows:

$$\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = M \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \quad \text{where } M = \begin{pmatrix} -17 & 50 \\ 16 & 17 \end{pmatrix}$$

Repeating the above process, the general values of y and z are given by

$$\begin{pmatrix} y_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$

If α, β are the eigen values of M, then

$$M^n = \frac{\alpha^n}{(\alpha-\beta)}(M-\beta I) + \frac{\beta^n}{(\beta-\alpha)}(M-\alpha I)$$

$$M^n = \begin{pmatrix} \frac{1}{33}(8\alpha^n + 25\beta^n) & \frac{25}{33}(\alpha^n - \beta^n) \\ \frac{8}{33}(\alpha^n - \beta^n) & \frac{25}{33}(\alpha^n + 8\beta^n) \end{pmatrix}$$

Hence, the general values of x, y, z satisfying (1) are given by

$$x_n = 33^n x_0$$
$$y_n = \frac{1}{33}(8\alpha^n + 25\beta^n)y_0 + \frac{25}{33}(\alpha^n - \beta^n)z_0$$
$$z_n = \frac{8}{33}(\alpha^n - \beta^n)y_0 + \frac{25}{33}(\alpha^n + 8\beta^n)z_0$$

Formula 2:

Let (x_1, y_1, z_1) be the second solution of (1),

where $x_1 = 3h - x_0$, $y_1 = y_0$, $z_1 = h + z_0$ (18)

Substituting (17) in (1) and simplifying, we get

$$h = 9x_0 + 25z_0$$

Thus, the second solution (x_1, y_1, z_1) to (1) is given by

$$x_1 = 26x_0 + 75z_0 \text{ , } y_1 = y_0 \text{ , } z_1 = 9x_0 + 26z_0$$

Express y_1 and z_1 in the form of 2×2 matrix as follows:

$$\begin{pmatrix} x_1 \\ z_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ z_0 \end{pmatrix} \quad \text{where } M = \begin{pmatrix} 26 & 75 \\ 9 & 26 \end{pmatrix}$$

Repeating the above process, the general values of x and z are given by

$$\begin{pmatrix} x_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}$$

If α, β are the eigen values of M , then

$$M^n = \frac{\alpha^n}{(\alpha-\beta)}(M-\beta I) + \frac{\beta^n}{(\beta-\alpha)}(M-\alpha I)$$

$$M^n = \begin{pmatrix} \frac{1}{2}\left(\alpha^n + \beta^n\right) & \frac{5\sqrt{3}}{6}\left(\alpha^n - \beta^n\right) \\ \frac{\sqrt{3}}{16}\left(\alpha^n - \beta^n\right) & \frac{1}{2}\left(\alpha^n + \beta^n\right) \end{pmatrix}$$

Hence, the general values of x, y, z satisfying (1) are given by

$$x_n = \frac{1}{2}\left(\alpha^n + \beta^n\right)x_0 + \frac{5\sqrt{3}}{6}\left(\alpha^n - \beta^n\right)z_0$$

$$y_n = y_0$$

$$z_n = \frac{\sqrt{3}}{16}\left(\alpha^n - \beta^n\right)x_0 + \frac{1}{2}\left(\alpha^n + \beta^n\right)z_0$$

Formula 3:

Let $\left(x_1, y_1, z_1\right)$ be the second solution of (1),

where $x_1 = 5x_0 + h$, $y_1 = h - 5y_0$, $z_1 = 5z_0$ (19)

Substituting (17) in (1) and simplifying, we get

$$h = 6x_0 + 16y_0$$

Thus, the second solution $\left(x_1, y_1, z_1\right)$ to (1) is given by

$$x_1 = 11x_0 + 16y_0 \text{ , } y_1 = 6x_0 + 11y_0 \text{ , } z_1 = 5z_0$$

Express x_1 and y_1 in the form of 2×2 matrix as follows:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad \text{where} \quad M = \begin{pmatrix} 11 & 16 \\ 6 & 11 \end{pmatrix}$$

Repeating the above process, the general values of x and z are given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

If α, β are the eigen values of M , then

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}\left(M - \beta I\right) + \frac{\beta^n}{(\beta - \alpha)}\left(M - \alpha I\right)$$

$$M^n=\begin{pmatrix}\frac{\sqrt{6}(\alpha^n+\beta^n)}{2\sqrt{5}} & \frac{\sqrt{6}(\alpha^n+\beta^n)}{2\sqrt{5}}\\ \frac{3(\alpha^n-\beta^n)}{4\sqrt{5}} & \frac{\sqrt{6}(\alpha^n+\beta^n)}{2\sqrt{5}}\end{pmatrix}$$

Hence, the general values of x, y, z satisfying (1) are given by

$$x_n=\frac{\sqrt{6}(\alpha^n+\beta^n)}{2\sqrt{5}}x_0+\frac{\sqrt{6}(\alpha^n+\beta^n)}{2\sqrt{5}}y_0$$

$$y_n=\frac{3(\alpha^n-\beta^n)}{4\sqrt{5}}x_0+\frac{\sqrt{6}(\alpha^n+\beta^n)}{2\sqrt{5}}y_0$$

$$z_n=5^n z_0$$

4. CONCLUSION

In this paper, an attempt is made to obtain non-zero distinct integer solutions to the cone represented by the ternary quadratic equation $3x^2-8y^2=25z^2$. It is well known that quadratic equations with three unknowns are rich in variety. To conclude, one may search for integer solutions to other choices of cone.

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