

ON INTEGRAL SOLUTIONS OF TERNARY QUADRATIC DIOPHANTINE EQUATION $x^2 + 16y^2 = z^2$

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In this paper, we analyze the ternary quadratic homogeneous Diophantine equation $x^2 + 16y^2 = z^2$ for finding its non – zero integer solutions. The proposed equation represents a homogeneous cone and its distinct integer solutions represent a distinct integer point on it. We obtain four different patterns of integer points on the cone. A few interesting properties of different patterns of solutions are presented with the help of special numbers such as Polygonal number, Pyramidal number, Octahedral number, Pronic number, Stella Octangular number and Oblong number. Finally, the three triples of integers generated from the given solution are exhibited which satisfy the equation of the given cone.

Keywords: ternary quadratic equation, Diophantine equation, integral solution.

Notations Used:

P_n	-	Pronic number of rank n
Obl_n	-	Oblong number of rank n
SO_n	-	Stella Octangular number of rank n
$t_{m,n}$	-	Polygonal number of rank n with size m
Tet_n	-	Tetrahedral number of rank n
PP_n	-	Pentagonal Pyramidal number of rank n

1. INTRODUCTION

Since ancient times mathematicians have tried to solve equation over the integers. Pythagoras, for instance, described all integers as side lengths of rectangular triangles. After Diophantus from Alexandria such equations are called Diophantine equations. He made a study of such equations and was one of the first mathematicians to introduce symbolism into algebra. Since that time, many mathematicians worked on this topic, such as Fermat, Euler, Kummer, Siegel, and Willes. In mathematics, a Diophantine equation is a polynomial equation, usually

in two or more unknowns, such that only the integer solutions are studied. The mathematical study of Diophantine problems that Diophantus initiated is now called Diophantine analysis.

In recent years, many authors have involved themselves in finding such type of integer solutions of diophantine equations of order two with three unknowns. M.A. Gophalan et al. [5-8] have proposed some generalized forms of ternary quadratic Diophantine equations for finding its integral solutions, such equation are $y^2 = Dx^2 + z^2$, $ax^2 + by^2 = c(a+b)z^2$, $ax^2 + by^2 = cz^2$ [6, 7, 8]. Subsequently they have proposed many ternary quadratic Diophantine equations which are particular in nature, among them some equations are mentioned as follows: $43x^2 + y^2 = z^2$, $3x^2 + 5y^2 = 128z^2$, $4x^2 + 3y^2 = 28z^2$, $47x^2 + y^2 = z^2$, $7x^2 + 2y^2 = 135z^2$ [1, 5, 14, 16, 17]. Furthermore, Manju Somanath et al. [13] proposed a generalized form of special type of homogeneous ternary quadratic Diophantine equation $x^2 + (2k+1)y^2 = (k+1)^2 z^2$ to find their solutions which are integral in nature. In this manner, recently P. Jayakumar and others [9, 10, 11, 12] have studied some particular type of ternary quadratic equations for their integral solutions which are $x^2 + 16y^2 = 20z^2$, $x^2 + 15y^2 = 14z^2$, $x^2 + 7y^2 = 16z^2$, $x^2 + 9y^2 = 50z^2$. Moreover, Anbuselvi et al. [2, 3, 4] also proposed some particular type of quadratic Diophantine equation with three unknowns, viz., $15x^2 + y^2 = z^2$, $40x^2 + y^2 = z^2$, $45x^2 + y^2 = z^2$.

2. METHOD OF ANALYSIS

The ternary quadratic equation under consideration is

$$x^2 + 16y^2 = z^2 \quad (1)$$

Different patterns of solutions of (1) are illustrated below.

Pattern I

Consider (1) as

$$x^2 + 16y^2 = z^2 \times 1 \quad (2)$$

Assume

$$z = a^2 + 16b^2 \quad (3)$$

Write (1) as

$$1 = \frac{[(8n^2 + 8n) + i4(2n+1)][(8n^2 + 8n) - i4(2n+1)]}{(8n^2 + 8n + 4)^2} \quad (4)$$

Substituting (3) and (4) in (2) and employing the method of factorization, define

$$x + i4y = \frac{[(8n^2 + 8n) + i4(2n+1)](a + i4b)}{(8n^2 + 8n + 4)}$$

Equating the real and imaginary parts in the above equation, we get

$$x = \frac{(8n^2 + 8n)(a^2 - 16b^2) - 32ab(2n+1)}{(8n^2 + 8n + 4)}$$

$$y = \frac{2ab(8n^2 + 8n) + (a^2 - 16b^2)(2n+1)}{(8n^2 + 8n + 4)}$$

Replacing a by $(8n^2 + 8n + 4)A$, b by $(8n^2 + 8n + 4)B$ in the above equation corresponding integer solution of (1) are given by

$$x = (8n^2 + 8n + 4)[(8n^2 + 8n)(A^2 - 16B^2) - 32AB(2n+1)]$$

$$y = (8n^2 + 8n + 4)[2AB(8n^2 + 8n) + (A^2 - 16B^2)(2n+1)]$$

$$z = (8n^2 + 8n + 4)^2(A^2 + 16B^2)$$

For simplicity and clear understanding, taking $n = 1$ in the above equations, then the corresponding integer solutions of (1) are given by

$$x^2 + 16y^2 = z^2$$

$$x = 320A^2 - 5120B^2 - 1920AB$$

$$y = 60A^2 - 960B^2 + 640AB$$

$$z = 400A^2 + 6400B^2$$

Properties

1. $3x(A, A+1) - 16y(A, A+1) + 16000Pr_A = 0$
2. $5x(A, 2A^2 - 1) + 4z(A, 2A^2 - 1) - 3200Obl_A + 9600So_A \equiv 0 \pmod{3200}$
3. $2x(A, 1) + 9279Pr_A + t_{323,A} \equiv 0 \pmod{2560}$
4. $y(1, B) + 640t_{3,B} + t_{1282,B} \equiv 60 \pmod{321}$
5. $x(B(B+3), B) - 12y(B(B+3), B) + z(B(B+3), B) + 4800So_B + 160t_{202,B} \equiv 0 \pmod{86}$
6. $20x(A, 5A^2 - 3) + 20y(A, 5A^2 - 3) + 19z(A, 5A^2 - 3) - 1520t_{22,A} + 64000So_A \equiv 0 \pmod{662}$

Pattern II

It is worth to note that the numerical 1 in (2) may also be represented as

$$1 = \frac{[(4-n^2) + i4n][(4-n^2) - i4n]}{(4+n^2)^2}$$

Following the analysis presented above, the corresponding integer solution to (1) are then found to be

$$x = (4+n^2)[(4-n^2)(A^2 - 16B^2) - 32ABn]$$

$$y = (4+n^2)[(A^2 - 16B^2)n + 2AB(4-n^2)]$$

$$z = (4+n^2)^2(A^2 + 16B^2)$$

For simplicity and clear understanding, taking $n = 1$ in the above equations, the corresponding integer solutions of (1) are given by

$$x = 15A^2 - 240B^2 - 160AB$$

$$y = 5A^2 - 80B^2 + 30AB$$

$$z = 25A^2 + 400B^2$$

Properties

1. $x(9B^2 - 5, B) + 2y(9B^2 - 5, B) - z(9B^2 - 5, B) + 5400Tet_B - 1900Obl_B = 0 \pmod{400}$
2. $x(A, 3A^2 + 2) + 2y(A, 3A^2 + 2) + z(A, 3A^2 + 2) + 150So_A - 50Pr_A \equiv 0 \pmod{400}$
3. $x((B-3)(B-2), B) - 3y((B-3)(B-2), B) + 636Tet_B - 53t_{10,B} \equiv 0 \pmod{265}$
4. $y(3, B) + 4PP_B - So_B + 80Obl_B \equiv 45 \pmod{173}$

Pattern III

Write (1) as

$$(4y)(4y) = (z+x)(z-x)$$

Which is written in the form of ratio as

$$\frac{(4y)}{(z+x)} = \frac{(z-x)}{(4y)} = \frac{A}{B}, \quad B \neq 0$$

This is equivalent to the following two equations

$$Ax - 4By + Az = 0$$

$$Bx + 4Ay - Bz = 0$$

Applying the method of cross multiplication, we get

$$\left. \begin{aligned} x &= x(A, B) = 4(A^2 - B^2) \\ y &= y(A, B) = -2AB \\ z &= z(A, B) = -4(A^2 + B^2) \end{aligned} \right\} \quad (5)$$

Thus, (5) represents non-zero distinct integral solutions of (1) in two parameters.

Properties

1. $x(A, B^2) - z(A, B^2) - t_{26,A} + 4Obl_A = 0 \pmod{15}$
2. $x((B+3)(B-1), B) + y((B+3)(B-1), B) + z((B+3)(B-1), B) + So_B + 12 Obl_B \equiv 0 \pmod{17}$
3. $x(5, A) + z(A, 5) + 8 Obl_B \equiv 0 \pmod{8}$
4. $x(B, B) + y(B, B) + z(B, B) + So_B - 12 Tet_B + 16 Obl_B \equiv 0 \pmod{11}$

Pattern IV

The general solution of the given equation is

$$x = x(r, s) = 16r^2 - s^2$$

$$y = y(r, s) = 2rs$$

$$z = z(r, s) = 16r^2 + s^2$$

Properties

1. $z(r, 3r) = 25r^2$ *Perfect Square*
2. $y(2s, s) = 4s^2$ *Perfect Square*
3. $x(r, 2) + z(r, 2) + 12 Pr_r - t_{90,r} = 0 \pmod{55}$
4. $x(r, 3) + y(r, 3) + z(r, 3) - 32 Obl_r = 0 \pmod{26}$
5. $x(r, 5) + y(r, 5) + z(r, 5) - 38 Obl_r + t_{14,r} = 0 \pmod{33}$
6. $x(s(s+3), s) + y(s(s+3), s) - z(s(s+3), s) - 4PP_s - 2 Pr_s = 0 \pmod{2}$

Generation of integer solutions

Let (x_0, y_0, z_0) be any given integer solution of (1). Then, each of the following triples of integers satisfies (1):

Triple 1: (x_1, y_1, z_1)

$$x_1 = \frac{1}{2} \left((a^n + b^n)x_0 + (a^n - b^n)z_0 \right)$$

$$y_1 = q^n y_0$$

$$z_1 = \frac{1}{2} \left((a^n - b^n)x_0 + (a^n + b^n)z_0 \right) \quad \text{where } q^2 = ab$$

$$x^2 + 16y^2 = z^2$$

REFERENCES

- [1] Akila, G., Gopalan, M.A., Vidhyalakshmi, S., (2013). Integral Solution $43x^2 + y^2 = z^2$, International journal of Engineering Research, 1(4), 70 – 74.
- [2] Anbuselvi, R., and Ahila, N. (2016). On Homogeneous Ternary Quadratic Diophantine Equation $15x^2 + y^2 = z^2$, *IOSR Journal of Mathematics*, 12(1), 78 – 82.
- [3] Anbuselvi, R., and Ahila, N. (2017). On Ternary Quadratic Diophantine Equation $40x^2 + y^2 = z^2$, *International Journal of Mathematics and its Applications*, 5(4), 2017, 977-983.
- [4] Anbuselvi, R., and Shanmugavadivu, S.A. (2017). On Homogeneous Ternary Quadratic Diophantine Equation $45x^2 + y^2 = z^2$, *International Journal of Engineering Research and Application*, 7(11), 2017, 22 – 25.
- [5] Gopalan, M.A., Vidhyalakshmi, S., Nithya, C., (2014). Integral points on the ternary quadratic Diophantine equation $3x^2 + 5y^2 = 128z^2$, *Bull. Math. & Stat. Res.*, 2(1), 25 – 31.
- [6] Gopalan, M.A. and Kalingarani, J.(2008). Observations on the Diophantine equation $y^2 = Dx^2 + z^2$, *Impact Journal of Sci. Tech.*, 2(2), 91 – 95.
- [7] Gopalan, M.A., Vidhyalakshmi, S., and Krishnamoorthy, A., (2005). Integral Solutions Of Ternary Quadratic $ax^2 + by^2 = c(a+b)z^2$, *Bulletin of Pure and Applied Sciences*, 24E (2), 443 – 446.
- [8] Gopalan, M.A., and Vidhyalakshmi, S. (2007). An Observation on $ax^2 + by^2 = cz^2$, *Acta Ciencia Indica*, XXXIII M(I), 97 – 99.
- [9] Jayakumar, P., and Kanaga Dhurga, C. (2014). On Quadratic Diophantine Equation $x^2 + 16y^2 = 20z^2$, *Galois Journal of Mathematics*, 1(1), 17 – 23.
- [10] Jayakumar, P., and Prabha, S. (2014). On Ternary Quadratic Diophantine Equation, $x^2 + 15y^2 = 14z^2$, *Archimedes Journal of Mathematics*, 4(3), 159 – 164.
- [11] Jayakumar, P., and Meena, J. (2014). Integral Solutions of the Ternary Quadratic Diophantine Equation $x^2 + 7y^2 = 16z^2$, *International Journal of Science and Technology*, 4(4), 1–4.
- [12] Jayakumar, P., and Shankarakalidoss, G. (2015). Lattice Points of Homogenous Cone $x^2 + 9y^2 = 50z^2$, *International Journal of Science and Research*, 4(1), 2053 – 2055.
- [13] Somanath Manju, Sangeetha G., and Gopalan M.A. (2012). Observations on the ternary Quadratic equation $x^2 + (2k+1)y^2 = (k+1)^2 z^2$, *Bessel Math*, 2(2):107-112.
- [14] Meena, K., Vidhyalakshmi, S., Gopalan, M.A., & Aarthy Thangam. S., (2014). Integral solutions on the homogeneous cone $4x^2 + 3y^2 = 28z^2$, *Bull. Math. & Stat. Res.*, 2014, 2(1), 47 – 3.
- [15] Mordell, L.J. (1969). Diophantine Equations, Academic press, New York.
- [16] Nancy, T., Gopalan, M.A., Vidhyalakshmi, S., (2013). On Ternary quadratic Diophantine equation $47x^2 + y^2 = z^2$, International journal of Engineering Research, 1(4), 51 – 55.
- [17] Priya, S., Gopalan, M.A., Vidhyalakshmi, S., (2014). Inegral solutions of ternary Quadratic Diophantine equation $7x^2 + 2y^2 = 135z^2$, *Bull. Math. & Stat. Res.*, 2(1), 32 – 37.