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A NOTE ON SEMIDERIVATIONS

Gurninder S. Sandhu*

Author Affiliation:
Department of Mathematics, Punjabi University, Patiala, Punjab-147002; Department of Mathematics, Multani Mal Modi College, Patiala, Punjab-147001, India.

***Corresponding Author:**
Gurninder S. Sandhu, Department of Mathematics, Punjabi University, Patiala, Punjab-147002; Department of Mathematics, Multani Mal Modi College, Patiala, Punjab-147001, India.

E-mail: gurninder_rs@pbi.ac.in

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Abstract

Recently, Filippis et al. introduced the notion of generalized semiderivation [[5], Definition 1.2] in prime rings. Accordingly, let R be a prime ring and $F: R \to R$ be an additive mapping. If there exists a semiderivation d associated with an endomorphism g of R such that $F(xy) = F(x)g(y) + xd(y) = F(x)y + g(x)d(y)$ and $F(g(x)) = g(F(x))$ for all $x,\ y \in R$, then F is called a generalized semiderivation of R. We prove that every generalized semiderivation of a prime ring R is either an ordinary generalized derivation of R or a semiderivation of R.

Keywords: Prime ring; Semiprime ring; Semiderivation; Multiplicative semiderivation.

1. INTRODUCTION

In the beginning of 1980's the notion of semiderivation was introduced by Bergen [1] as follows: let g be an endomorphism of a ring R and d be a mapping of R into itself. Then d is called a semiderivation of R associated with g, if it satisfies

i. $d(x + y) = d(x) + d(y)$
ii. $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$
iii. $d(g(x)) = g(d(x))$

for all $x, y \in R$. If d is a non vanishing semiderivation of a prime ring R, then it is shown by Chang [2] that the associated mapping must necessarily be a ring endomorphism of R. Obviously, a derivation is a semiderivation (when g is identity) but the converse is not true in general; for example, $d = I - g$ is a semiderivation which is not a derivation, where I denotes the identity mapping of R. It is also remarked that a semiderivation which is not necessarily additive is called a multiplicative semiderivation. In [4], Chuang proved that *every semiderivation of a prime ring R is either an ordinary derivation of R or takes the form* $d = \gamma(I - g)$ *for some* $\gamma \in C$, *the extended centroid of* R. At the same time Brešar [3] proved that *every multiplicative semiderivation of a prime ring R is either a multiplicative derivation of R or takes the form* $d = \gamma(I - g)$ *for some* $\gamma \in C$. Recall that, a mapping d (not necessarily additive) of a ring R into itself is called a *multiplicative derivation* of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. And if d is necessarily additive, then it is called a *derivation* of R. An additive mapping $F: R \to R$ is said to be a *generalized derivation* of R if there exists a derivation d of R satisfying $F(xy) =$

$F(x)y + xd(y)$ for all $x, y \in R$. Further, if F is any mapping (not necessarily additive) of R associated with another mapping d (not necessarily additive) such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, then F is called a *multiplicative (generalized)-derivation* of R. For more details of multiplicative (generalized)-derivations of rings, one may see [7]. Moreover, it is known that the mapping d associated with a multiplicative (generalized)-derivation F of a semiprime ring must necessarily be a multiplicative derivation of R. Recently, Filippis et al. [5] introduced the notion of *generalized semiderivations* of prime rings as: let g be an endomorphism of R an additive mapping $F: R \to R$ which is uniquely determined by a semiderivation d of R associated with g is called a *generalized semiderivation* of R if

i. $F(xy) = F(x)g(y) + xd(y) = F(x)y + g(x)d(y)$
ii. $F(g(x)) = g(F(x))$

for all $x, y \in R$. Intuitively, one may think of a multiplicative (generalized)-semiderivation of ring R as follows: let R be a ring and g be an endomorphism of R. A mapping $F: R \to R$ (not necessarily additive) is called *multiplicative (generalized)-semiderivation* of R if there exists a mapping $d: R \to R$(not necessarily additive) such that $F(xy) = F(x)g(y) + xd(y) = F(x)y + g(x)d(y)$ and $F(g(x)) = g(F(x))$ for all $x \in R$. After Chuang [4] and Brešar [3], it is natural to obtain the structure of generalized semiderivations and multiplicative (generalized)-semiderivations of prime rings. In this note, we show that a generalized semiderivation (resp. multiplicative (generalized)-semiderivation) of a prime ring R is either a semiderivation (resp. multiplicative semiderivation) of R or a generalized derivation (resp. multiplicative (generalized)-derivation) of R.

2. **RESULTS**

Proposition 1: *Let R be a semiprime ring and F be a multiplicative (generalized)-semiderivation of R associated with a mapping d and an endomorphism g of R. If g is an epimorphism of R, then d is a multiplicative semiderivation of R.*

Proof: By our hypothesis, we have

$$F(xy) = F(x)g(y) + xd(y) \tag{1}$$

and

$$F(xy) = F(x)y + g(x)d(y) \tag{2}$$

for all $x, y \in R$. We first consider the equation (1). For any $x, y, z \in R$, we have

$$F(xyz) = F((xy)z) = F(xy)g(z) + xyd(z) = F(x)g(yz) + x(d(y)g(z) + yd(z)) \tag{3}$$

On the other hand

$$F(xyz) = F(x(yz)) = F(x)g(yz) + xd(yz) \tag{4}$$

On combining (3) and (4), we obtain $x(d(yz) - d(y)g(z) - yd(z)) = 0$ for all $x, y, z \in R$. Since R is semiprime, we get $d(yz) = d(y)g(z) + yd(z)$ for all $y, z \in R$. Analogously, from (2) we find that $d(yz) = d(y)z + g(y)d(z)$ for all $y, z \in R$. Further, since $F(g(x)) = g(F(x))$ for all $x \in R$. Replacing x by xy, we find that $F(g(x)g(y)) = g(F(xy))$ for all $x, y \in R$. i.e.;

$$F(g(x))g^2(y) + g(x)d(g(y)) = g(F(x))g^2(y) + g(x)g(d(y))$$

for all $x, y \in R$. It implies that $g(x)(d(g(y)) - g(d(y))) = 0$ for all $x, y \in R$. Now, let us suppose that g is an epimorphism of R, we get $R(d(g(y)) - g(d(y))) = (0)$ for all$y \in R$. Hence, semiprimeness of R completes the proof. □

By repeating the same arguments, we can obtain the following result:

Corollary 2: *Let R be a semiprime ring and F be a generalized semiderivation of R. If g is an epimorphism of R, then the associated map d is a semiderivation of R.*

Moreover, we now show that the notion of multiplicative semiderivation (resp. semiderivation) cannot be extended to multiplicative (generalized)-semiderivation (resp. generalized semiderivation) in prime rings. In order to prove this claim, the following lemma is essential.

Lemma 3: *Let R be a prime ring and $a, b, c \in R$ such that $axb = bxc$ for all $x \in R$. Then, either $a = c$ or $b = 0$.*

Proof: Let us replace c by $-c$ in Lemma 1 of [6], we see that the condition $axb = bxc$ for all $x \in R$ implies that $(a - c)xb = 0$ for all $x \in R$. Since R is a prime ring, the last relation yields that either $a = c$ or $b = 0$, as desired.

Theorem 4: *Let R be a prime ring and g be an epimorphism of R. Suppose that $F: R \rightarrow R$ is a multiplicative (generalized)-semiderivation of R associated with a multiplicative semiderivation d of R. Then either $F = d$ or $g = I$.*

Proof: For any $x, y \in R$, our hypothesis yields

$$F(x)g(y) + xd(y) = F(x)y + g(x)d(y)$$

$$F(x)(g(y) - y) = (g(x) - x)d(y)$$

$$F(x)(g - I)(y) = (g - I)(x)d(y) \quad (5)$$

for all $x, y \in R$. Replacing y by yz, we obtain

$$\begin{aligned} F(x)(g - I)(yz) &= F(x)(g(yz) - yz) \\ &= F(x)(g(y) - y)g(z) + y(g(z) - z) \\ &= F(x)(g - I)(y)g(z) + F(x)y(g - I)(z) \end{aligned}$$

Using (5), we find that

$$F(x)(g - I)(yz) = (g - I)(x)d(y)g(z) + F(x)y(g - I)(z) \quad (6)$$

for all $x, y, z \in R$. On the other hand

$$(g - I)(x)d(yz) = (g - I)xd(y)g(z) + (g - I)xyd(z) \quad (7)$$

for all $x, y, z \in R$. In view of (6) and (7), relation (5) implies that $F(x)y(g - I)(z) = (g - I)(x)yd(z)$ for all $x, y, z \in R$. In particular, we have $F(x)y(g - I)(x) = (g - I)(x)yd(x)$ for all $x, y \in R$. With the aid of Lemma 1, we find that either $F(x) = d(x)$ or $(g - I)(x) = 0$ for all $x \in R$. That means, either $F = d$ or $g = I$ as desired. □

We conclude with the following result, which is easy to obtain by repeating the similar arguments as in the Theorem 4.

Theorem 5: *Let R be a prime ring and g be an epimorphism of R. Suppose that $F: R \rightarrow R$ is a generalized semiderivation of R associated with a semiderivation d of R. Then either $F = d$ or $g = I$.*

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