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Convergence theorems in new generalized type of metric spaces and their applications *

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Abstract In the present paper we introduce a new generalized type of metric spaces called the right quasi-metric spaces. We state and prove convergence theorems to a fixed point for any map in these spaces. Finally, we give applications of our results. These results generalize the corresponding results in M. A. Ahmed and F. M. Zeyada (On convergence of a sequence in complete metric spaces and its applications to some iterates of quasi-nonexpansive mappings, J. Math. Anal. Appl., 274(1), 458–465, 2002).

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1 Introduction

In the last few decades, convergence theorems in metric spaces and generalized types of them have been studied (see, e.g., [1]). Recently, in 2012, Ahmed and Zeyada [2] established some convergence theorems of any sequence in complete quasi-metric spaces.

Definition 1.1. Following Waszkiewick [4], let X be a nonempty set. A map $d : X \times X \longrightarrow [0, \infty)$ is called a **distance** on X. A pair (X, d) is said to be a distance space. Now, we list some conditions as follows, for every $x, y, z \in X$,

(i) $d(x, x) = 0$;

(ii) if $d(x, y) = 0$ or $d(y, x) = 0$, then $x = y$;

(iii) $d(x, y) = d(y, x)$;

(iv) $d(x, y) \leq d(x, z) + d(z, y)$.

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If d satisfies the conditions (i)–(iv) $\big($resp. (i), (ii) and (iv)$\big)$, of the Definition 1.1 then it is called a **metric** $\big($resp. **quasi-metric** (abbr. **q-metric**)$\big)$ on X.
It is clear that metric $\Longrightarrow$ q-metric. The reverse of the last implication may not be true.

Definition 1.2. [1] The mapping $T : D \subseteq X \to X$ is said to be quasi-nonexpansive w.r.t. a sequence $\{x_n\}$ if $\{x_n\} \subseteq D$ and for all $n \in \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of all positive integers and for each $p \in F(T)$,

$$d(x_{n+1}, p) \leq d(x_n, p).$$

The present paper is organized as follows. In Section 2 we introduce the concept of right quasi metric spaces. Also, some lemmas and convergence theorems are established. Finally, in Section 3 we give some applications of our main results.

2 Convergence theorem in right quasi metric spaces

In this section, we introduce the concept of right quasi metric spaces (simply rq-metric spaces). We also state and prove some lemmas and convergence theorems in these spaces.
First we define the rq-metric as follows:

Definition 2.1. A distance function d is called rq-metric on X if and only if it satisfies (i), (ii) of the Definition 1.1 and the further condition that

$$d(x, y) \leq d(x, z) + d(y, z) \quad \forall x, y, z \in X.$$

Remark 2.2. One can conclude that q-metric and rq-metric spaces are different generalizations of metric spaces.

Definition 2.3. Let (X, d) be a rq-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is rq-convergent to x if and only if $\lim\limits_{n\to\infty} d(x, x_n) = 0$. In this case, x is called rq-limit of x_n.

Definition 2.4. Let (X, d) be a rq-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is right-Cauchy (simply r-Cauchy) if and only if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $m \geq n > N$.

Now we state and prove some elementary properties of the rq- metric on X.

Lemma 2.5. *Let (X, d) be an rq-metric space. If $x, y \in X$, then*

$$\mid d(x, x_0) - d(y, x_0) \mid \leq d(x, y).$$

Proof. By using the triangle inequality in rq-metric space we get

$$\begin{aligned} d(x, x_0) &\leq d(x, y) + d(x_0, y) \\ &\leq d(x, y) + d(x_0, x_0) + d(y, x_0) \\ &= d(x, y) + 0 + d(y, x_0). \end{aligned}$$

This implies that

$$d(x, x_0) - d(y, x_0) \leq d(x, y). \tag{2.1}$$

Similary, one can deduce that

$$d(y, x_0) \leq d(x, y) + d(x, x_0),$$

which implies that

$$d(x, x_0) - d(y, x_0) \geq -d(x, y). \tag{2.2}$$

From (2.1) and (2.2), we obtain that

$$-d(x,y) \leq d(x,x_0) - d(y,x_0) \leq d(x,y).$$

This is equivalent to

$$\mid d(x,x_0) - d(y,x_0) \mid \leq d(x,y). \tag{2.3}$$

□

Now, we introduce the following lemma which we need in the sequel.

Lemma 2.6. *Let (X,d) be an rq-metric space. For x_0 fixed, the function $x \mapsto d(x,x_0)$ is continuous from X to R.*

Proof. Given $\epsilon > 0$, take $\eta = \epsilon$. From (2.3) we have

$$\mid d(x,x_0) - d(y,x_0) \mid \leq d(x,y) \leq \epsilon.$$

Thus $x \mapsto d(x,x_0)$ is continuous. □

Remark 2.7.

(2.7.1) Lemma 2.5 (resp. Lemma 2.6) generalizes Proposition 4 (resp. Proposition 5) of [3, p. 13].

(2.7.2) For any subset A of a metric space (X,d), $d(x,A) = 0$ if and only if $x \in \bar{A}$ where $\bar{A}$ is the closure of the set A. This fact is valid in q-metric or rq-metric spaces. If A is a closed subset of a metric or a q-metric or a rq-metric space, then $d(x,A) = 0$ if and only if $x \in A$.

Now, we state and prove our main results as follows.

Theorem 2.8. *Let $\{x_n\}$ be a sequence in a subset D of rq-metric space (X,d) and $T : D \longrightarrow X$ be any map such that $F(T) \neq \emptyset$. Assume that $F(T)$ is a closed set. Then*

2.8(a) $\lim_{n\to\infty} d(F(T),x_n) = 0$ *if $\{x_n\}$ rq-converges to a unique point in $F(T)$.*

2.8(b) *$\{x_n\}$ rq-converges to a unique point in $F(T)$ if $\lim_{n\to\infty} d(F(T),x_n) = 0$, and $\lim_{n\to\infty} x_n$ exists.*

Proof.

2.8(a) Let $\{x_n\} \subseteq D$ be rq-convergent to a unique point in $F(T)$. Then $\lim_{n\to\infty} x_n$ exists. From the closedness of $F(T)$, we find that $p := \lim_{n\to\infty} x_n \in F(T) = \overline{F(T)}$.
Hence, we obtain that $d(F(T),p) = 0$. Since d is continuous, we get that

$$\lim_{n\to\infty} d(F(T),x_n) = d(F(T), \lim_{n\to\infty} x_n) = d(F(T),p) = 0.$$

2.8(b) Suppose that $\lim_{n\to\infty} d(F(T),x_n) = 0$. Since d is continuous, then

$$d(F(T), \lim_{n\to\infty} x_n) = \lim_{n\to\infty} d(F(T),x_n) = 0.$$

Therefore, we have that $\lim_{n\to\infty} x_n \in \overline{F(T)}$. The closedness of $F(T)$ leads to $\lim_{n\to\infty} x_n \in F(T)$. Then $\{x_n\}$ rq-converges to a unique point in $F(T)$.

□

Remark 2.9. Theorem 2.8 generalizes and improves Theorem 2.1 in [1].

2.9(I) The quasi-nonexpansiveness of T w.r.t. a sequence $\{x_n\}$ or the weakly quasi metric completeness of X and $\lim_{n\to\infty} d(x_n,F(T)) = 0$ give that $\lim_{n\to\infty} x_n$ exists.

2.9(II) X is rq-metric space instead of being a metric space.

Corollary 2.10. *[1] Let $\{x_n\}$ be sequence in a subset D of a metric space (X,d) and let $T : D \longrightarrow X$ be a map such that $F(T) \neq \emptyset$. Then*

2.10(i) $\lim_{n\to\infty} d(x_n, F(T)) = 0$ *if $\{x_n\}$ converges to a unique point in $F(T)$;*

2.10(ii) *$\{x_n\}$ converges to a point in $F(T)$ if $\lim_{n\to\infty} d(x_n, F(T)) = 0, F(T)$ is closed set, T is quasi-nonexpansive with respect to $\{x_n\}$, and X is complete.*

Corollary 2.11. *[1] Let $\{x_n\}$ be sequence in a subset D of a metric space (X,d) and let $T : D \longrightarrow X$ be a map such that $F(T) \neq \emptyset$. Then*

2.11(i) $\lim_{n\to\infty} d(x_n, F(T)) = 0$ *if $\{x_n\}$ converges to a unique point in $F(T)$;*

2.11(ii) *$\{x_n\}$ converges to a point in $F(T)$ if $\lim_{n\to\infty} d(x_n, F(T)) = 0, F(T)$ is closed set, T is weakly quasi-nonexpansive with respect to $\{x_n\}$, and X is complete.*

As a consequence of Theorem 2.8, we state and prove the following theorem.

Theorem 2.12. *Let $\{x_n\}$ be a sequence in a subset D of rq-metric space (X,d) and $T : D \longrightarrow X$ any map such that $F(T)$ is closed set. Assume that*

2.12(i) *T is quasi-nonexpansive with respect to $\{x_n\}$;*

2.12(ii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$*;*

2.12(iii) *if the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, then*

$$\liminf_{n\to\infty} d(F(T), y_n) = 0 \quad \text{or} \quad \limsup_{n\to\infty} d(F(T), y_n) = 0.$$

Then $\{x_n\}$ rq-converges to a unique point in $F(T)$.

Proof. From **2.12(i)** and the boundedness from below by zero of the sequence $d(F(T), x_n)$, we find that $\lim_{n\to\infty} d(F(T), x_n)$ exists, say, it equals $y \geq 0.$. So, we obtain from the conditions **2.12(ii)** and **2.12(iii)** that

$$\liminf_{n\to\infty} d(F(T), x_n) = 0, \text{ or } \limsup_{n\to\infty} d(F(T), x_n) = 0.$$

From the uniqueness of y, then $\lim_{n\to\infty} d(F(T), x_n) = 0$. Hence, by Theorem **2.8(a)** we conclude that $\{x_n\}$ rq-converges to a unique point in $F(T)$. □

Remark 2.13. Theorem 2.12 generalizes and improves Theorem 2.2 in [2].

2.13(I) X is rq-metric space instead of X is a metric space.

2.13(II) In this theorem we produce rq-convergent, but Theorem 2.2 in [1] was only convergent.

Corollary 2.14. *[2] Let $\{x_n\}$ be a sequence in a subset D of q-metric space (X,d) and $T : D \longrightarrow X$ any map such that $F(T)$ is a closed set. Assume that*

2.14(i) *T is quasi-nonexpansive with respect to $\{x_n\}$;*

2.14(ii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$*;*

2.14(iii) *If the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ and either*

$$\liminf_{n\to\infty} d(y_n, F(T)) = 0 \text{ or } \limsup_{n\to\infty} d(y_n, F(T)) = 0.$$

Then $\{x_n\}$ converges to a unique point in $F(T)$.

Theorem 2.15. *Let $\{x_n\}$ be a sequence in a subset D of rq-metric space (X,d) and $T : D \longrightarrow X$ be a quasi-nonexpansive mapping w.r.t.$\{x_n\}$ such that $F(T)$ is a closed set. Assume that the sequence $\{x_n\}$ contains subsequence $\{x_{n_j}\}$ which is rq-converging to $x^* \in D$ such that there exists a continuous mapping $S : D \longrightarrow D$ satisfying $S(x_{n_j}) = x_{n_{j+1}}$ $\forall n, j \in \mathbb{N}$ and $d(S(x^*), p) < d(x^*, p)$ for some $p \in F(T)$. Then $x^* \in F(T)$ and $\lim_{n\to\infty} x_n = x^*$.*

Proof. The quasi-nonexpansiveness of T w.r.t. (x_n) and the boundedness from below of the sequence $(d(F(T), x_n))$ implies that $\lim_{n\to\infty} d(F(T), x_n)$ exists, say, it equals $r \in [0, \infty)$. Suppose that $x^* \notin F(T)$. So, we have from the assumption made in the statement of this Theorem 2.15, that for some $p \in F(T)$,

$$d(p, x^*) > d(p, S(x^*)) = d(p, S(\lim_{j\to\infty} x_{n_j})) = d(p, \lim_{j\to\infty} S(x_{n_j})) = d(p, \lim_{j\to\infty} x_{n_{j+1}}) = d(p, x^*).$$

This contradiction implies that $x^* \in F(T)$. Then,

$$r = \lim_{j\to\infty} d(F(T), x_n) = \lim_{j\to\infty} d(F(T), x_{n_j}) = d(F(T), \lim_{j\to\infty} x_{n_j}) = d(F(T), x^*) = 0.$$

Thus, we obtain that $\lim_{n\to\infty} x_n = x^*$. □

Corollary 2.16. *[2] Let $\{x_n\}$ be a sequence in a subset D of a complete q-metric space (X, d) and $T : D \longrightarrow X$ be a quasi-nonexpansive mapping w.r.t.$\{x_n\}$ such that $F(T)$ is a closed set. Assume that the sequence $\{x_n\}$ contains the subsequence $\{x_{n_j}\}$ which converges to $x^* \in D$ such that there exists a continuous map $S : D \longrightarrow D$ satisfying $S(x_{n_j}) = x_{n_{j+1}}$ $\forall n, j \in \mathbb{N}$ and $d(S(x^*), p) < d(x^*, p)$ for some $p \in F(T)$. Then $x^* \in F(T)$ and $\lim_{n\to\infty} x_n = x^*$.*

3 Applications

Motivated by the paper of Ahmed and Zeyada [2], we apply Theorem 2.12 for obtaining convergence theorems in rq-metric spaces.
We state the following theorems without proof.

Theorem 3.1. *Let $\{x_n\}$ be a sequence in a subset D of an rq-metric space (X, d) and $T : D \times D \longrightarrow X$ be any map such that $F(T) \neq \emptyset$. Assume that $F(T)$ is a closed set. Then $\{x_n\}$ rq-converges to a unique point in $F(T)$ if and only if $\lim_{n\to\infty} d(F(T), x_n) = 0$.*

Theorem 3.2. *Let $\{x_n\}$ be a sequence in a subset D of an rq-metric space (X, d) and $T : D \times D \longrightarrow X$ be any map such that $F(T)$ is a closed set. Assume that*

3.2(i) *T is quasi-nonexpansive with respect to $\{x_n\}$;*

3.2(ii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$*;*

3.2(iii) *if the sequence $\{y_n\}$ satisfies $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, and either*

$$\liminf_{n\to\infty} d(F(T), y_n) = 0 \text{ or } \limsup_{n\to\infty} d(F(T), y_n) = 0.$$

Then $\{x_n\}$ rq-converges to a unique point in $F(T)$.

Theorem 3.3. *Let $\{x_n\}$ be a sequence in a subset D of an rq-metric space (X, d) and let $T : D \times D \longrightarrow X$ be any map such that $F(T) \neq \emptyset$. Assume that $F(T)$ is a closed set then the sequence $\{x_n\}$ rq-converges to a unique point in $F(T)$ if and only if*

$$\liminf_{n\to\infty} d(F(T), x_n) = 0.$$

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