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TRANSIENT ANALYSIS OF $M^{[X_1]}, M^{[X_2]}/G_1, G_2^{(a,b)}/1$ QUEUEING SYSTEM WITH PRIORITY SERVICES

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Abstract

In this paper, we consider a single server queuing system with two types of batch arrivals and services under non preemptive priority rule. Arrivals follow a compound Poisson process. The server provides single service to the high priority customers and the general bulk service rule for the low priority customers on a FCFS discipline. The server starts service to the low priority customers only if the high priority queue is empty and the number of customers in the low priority queue is greater than or equal to $'a'$. If there are no customer in the high priority queue and the number of customers in the the low priority queue are less than $'a'$ then the server becomes idle. The service time for each service follows a general(arbitrary) distribution. Using the supplementary variable technique, the time dependent probability generating functions of the distributions $P_{m,n}$, $q_{m,n}$ and $P^{+}_{m,n}$, $q^{+}_{m,n}$ under equilibrium, have been derived in terms of their Laplace transforms and the corresponding steady state results are also derived. The average number of customers in the queues and the average waiting time are derived. Numerical case has been worked out on the assumption that the service time follows a specified exponential and Erlang-2 distributions.

Key words: Batch arrival; Bulk service; Non-preemptive priority service; Average queue size

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1. INTRODUCTION

In our day to day life queueing situations often arise in which service is rendered in bulk. Many researchers have put their efforts in this area by considering various aspects, like, Madan,K.C.et.al. (2003) have studied the steady state analysis of $M^{[X]}/M^{a,b}/1$ queueing model with random breakdown, Holman,D.F. et.al. (1981) have studied some results for the general bulk service queueing system, Medhi,J. (1994) have studied batch arrival and bulk service, Neuts,M.F. (1967) have studied a general class of bulk queues with Poisson input, Chaudhry, M.L.et. al. (1981) have studied the queueing system $M/G^{B}/1$ and its ramifications. Some of them have discussed the single and batch arrival with priority services, for example, Jain, M.et.al (2008) have studied a bulk arrival retrial queue with unreliable server and priority subscribers, Chesoong Kim.et. al. (2016) have studied priority tandem queueing system with retrial and reservation of channels as a model of call center, Atencia, I. et.al. (2005) have studied a single-server retrial queue with general retrial times and Bernoulli schedule. Further, Ayyappan, G.et.al. (2009) have studied the single server retrial queueing system with non-pre-emptive priority service and single vacation-exhaustive service type. Deepak, C.et.al. (2014) have studied a delay analysis of a discrete-time non-preemptive priority queue with priority jumps, Jinbiao Wu.et.al. (2013) have studied a single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule. Madan,K.C. (2011) have studied a non-preemptive priority queueing system with a single server serving two queues $M/G/1$ and $M/D/1$ with optional server vacations based on exhaustive service of the priority units, Rajadurai,P. et.al. (2014) have studied an analysis of an $M^{[X]}/(G_1,G_2)/1$ retrial queueing system with balking, optional re-service under modified vacation policy and service interruption. Gautam Choudhury.et.al. (2012) have studied a batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair.

Now, we consider a single server queuing system with two types of batch arrivals and services under non preemptive priority rule. Arrivals follow a compound Poisson process. The server provides single service to the high priority customers and general bulk service rule to the low priority customers on a FCFS discipline. The service rule for low priority customers is as follows: The server starts service only when a minimum number of customers $'a'$ is present in the queue and the maximum service capacity is $'b'$. If the number of customers in the queue is less than $'a'$ the server becomes idle if more than $'b'$ customers waiting in the queue the server serves first $'b'$ customers in the queue and the remaining customers wait in the queue. The service time follows a general(arbitrary) distribution. Using the supplementary variable technique, the time dependent probability generating functions of the distributions $P_{m,n}, q_{m,n}$ and $P^{+}_{m,n}, q^{+}_{m,n}$ under equilibrium, have been derived in terms of their Laplace transforms and the corresponding steady state results are also derived. The average number of customers in the queues and the average waiting time are derived. Numerical case has been worked out on the assumption that the service time follows a specified exponential and Erlang-2 distributions.

The rest of the paper is organized as follows: Mathematical description of our model isdetailed in section (2). The definitions and the equations governing of our model and the time dependent solution are obtained in sections (3) and (4). The corresponding steady state results have been derived explicitly in section (5).Relation to the imbedded Morkov chain is given in section (6). The calculation of basic $P^{+}_{0,k}, q^{+}_{0,k}$ and $Q_{0,k}$ probabilities is given in section (7). Average queue size and the average waiting time are computed in sections (8) and (9). Some particular cases have been discussed in section (10). The numerical results are discussed in section (11) and the references to the paper are stated in section (12).

2 MATHEMATICAL DESCRIPTION OF OUR MODEL

We assume the following to describe the queueing model of our study.

1. High priority and low-priority units arrive at the system in batches of variable sizes in a compound Poisson process and they are provided one by one on a FCFS basis. Let $\lambda_1 c_i\, dt$ and $\lambda_2 c_i\, dt(i = 1,2,3,..$ be the first order probability that a batch of i customers arrive at the system during a short interval of time $(t, t+dt)$, where $0 \le c_i \le 1$, $\sum_{i=1}^{\infty} c_i = 1$, and $\lambda_1 > 0$, $\lambda_2 > 0$ are the

average arrival rates for high-priority and low-priority customers and high-priority and low-priority customers forms seperate queues, if the server is busy. The server must serve all the high-priority units present in the system before taking up low-priority unit for service. In other words, there is no high-priority unit present in the system at the time of starting the bulk service of a low-priority unit. Further, we assume that the server follows a non-preemptive priority rule, which means that if one or more high-priority units arrive during the service time of a low-priority unit, the current service of a low-priority unit is not stopped and a high-priority unit will be taken up for service only after the current service of a low-priority unit is completed.

2. Each customer under high-priority and low-priority service provided by a single server. The service time for both high-priority and low-priority units follows general(arbitrary) distribution with distribution functions $B(v)$ and the density functions $b(v)$.

3. Let $\mu(x)dx$ be the conditional probability of completion of the high-priority and low-priority unit service during the interval $(x, x+dx]$, given that the elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)}$$

and therefore,

$$b(v) = \mu(v) e^{-\int_0^s \mu(x)dx}.$$

$P^{+}_{m,n}$ =Prob {The number of customers in the high priority queue is m and low priority queue is n just after a depature epoch}

$q^{+}_{m,n}$ = Prob {The number of customers in the high priority queue is m and low priority queue is n just after a depature epoch}

$P_{m,n}$ =Prob {The number of customers in the high priority queue is m and low priority queue is n just after a random epoch}

$q_{m,n}$ =Prob {The number of customers in the high priority queue is m and low priority queue is n just after a random epoch}

where $m, n = 0,1,2,..$.

4. Let $M_q(t)$ and $N_q(t)$ be the high-priority and low-priority queue size(defined here to denote the number of customers waiting for service, not including those in service respectively) and $X(t)$ be the elapsed service time of the customers undergoing service at time t. The supplementary variable $X(t)$ is introduced in order to obtain a bivariate Markov Process $(M_q(t), N_q(t), X(t))$. Let we define the joint probabilities:

$$Q_{0,n}(t) = \text{Prob}\{N_q(t) = n \text{ and the server is idle}\},\ 0 \le n < a,$$

$P_{m,n}$=Prob {Mq(t)=m, N_q(t)=n and the server is busy with high priority customers}, x,t>0, m,n≥0,
$q_{m,n}$= Prob {Mq(t)=m, N_q(t)=n and the server is busy with low-priority customers}, x,t>0, m,n≥0,

5. Various stochastic process involved in the system are assumed to be independent of each other.

3 DEFINITIONS AND NOTATIONS

1. $P_{m,n}(x,t) =$ Probability that at time t, the server is active providing service and there are $m\ (m \ge 0)$ priority units and $n\ (n \ge 0)$ non-priority units in the queue excluding the one priority unit in service with elapsed service time for this customer is x. Accordingly, $P_{m,n}(t) = \int_0^\infty P_{m,n}(x,t)dx$ denotes the probability that at time t there are $m\ (m \ge 0)$ priority units and $n\ (n \ge 0)$

non-priority units in the queue excluding one priority unit in service without regard to the elapsed service time x of a priority unit.

2. $q_{m,n}(x,t) =$ Probability that at time t, the server is active providing service and there are $m\ (m \geq 0)$ priority units and $n\ (n \geq 0)$ non-priority units in the queue excluding one batch of low-priority customers being served and the elapsed service time for this batch of customers is x. Accordingly, $q_{m,n}(t) = \int_0^\infty q_{m,n}(x,t)dx$ denotes the probability that at time t there are $m\ (m \geq 0)$ priority units and $n\ (n \geq 0)$ non-priority units in the queue excluding one batch of low-priority unit in service without regard to the elapsed service time x of a low-priority unit.

3. $Q_{0,n}(t) =$ Probability that at time t, there are $(a-1)$ customers in the system and the server is idle but available in the system.

4. EQUATIONS GOVERNING THE SYSTEM

The system is then governed by the following set of differential-difference equations:

$$\frac{d}{dt}Q_{0,0}(t) = -(\lambda_1+\lambda_1)Q_{0,0}(t) + \int_0^\infty P_{0,0}(x,t)\mu(x)dx + \int_0^\infty q_{0,0}(x,t)\mu(x)dx, \tag{1}$$

$$\frac{d}{dt}Q_{0,n}(t) = -(\lambda_1+\lambda_1)Q_{0,n}(t) + \sum_{k=1}^{n}\lambda_2 C_k Q_{0,n-k}(t) + \int_0^\infty P_{0,n}(x,t)\mu(x)dx$$
$$+\int_0^\infty q_{0,n}(x,t)\mu(x)dx,\ n=1,2,...a-1. \tag{2}$$

$$\frac{\partial}{\partial x}P_{m,n}(x,t) + \frac{\partial}{\partial t}P_{m,n}(x,t) = -(\lambda_1+\lambda_2+\mu(x))P_{m,n}(x,t) + \sum_{i=1}^{n}\lambda_1 C_i P_{m-i,n}(x,t)$$
$$+\sum_{i=1}^{n}\lambda_2 C_i P_{m,n-i}(x,t),\ m,n \geq 1 \tag{3}$$

$$\frac{\partial}{\partial x}P_{m,0}(x,t) + \frac{\partial}{\partial t}P_{m,0}(x,t) = -(\lambda_1+\lambda_2+\mu(x))P_{m,0}(x,t) + \sum_{i=1}^{n}\lambda_1 C_i P_{m-i,0}(x,t), \qquad m \geq 1 \tag{4}$$

$$\frac{\partial}{\partial x}P_{0,n}(x,t) + \frac{\partial}{\partial t}P_{0,n}(x,t) = -(\lambda_1+\lambda_2+\mu(x))P_{0,n}(x,t) + \sum_{i=1}^{n}\lambda_2 C_i P_{0,n-i}(x,t), \qquad n \geq 1 \tag{5}$$

$$\frac{\partial}{\partial x}P_{0,0}(x,t) + \frac{\partial}{\partial t}P_{0,0}(x,t) = -(\lambda_1+\lambda_2+\mu(x))P_{0,0}(x,t), \tag{6}$$

$$\frac{\partial}{\partial x}q_{m,n}(x,t) + \frac{\partial}{\partial t}q_{m,n}(x,t) = -(\lambda_1+\lambda_2+\mu(x))q_{m,n}(x,t) + \sum_{i=1}^{n}\lambda_1 C_i q_{m-i,n}(x,t)$$
$$+\sum_{i=1}^{n}\lambda_2 C_i q_{m,n-i}(x,t),\ m,n \geq 1 \tag{7}$$

$$\frac{\partial}{\partial x} q_{m,0}(x,t) + \frac{\partial}{\partial t} q_{m,0}(x,t) = -(\lambda_1 + \lambda_2 + \mu(x)) q_{m,0}(x,t) + \sum_{i=1}^{n} \lambda_1 C_i q_{m-i,0}(x,t), \qquad m \geq 1 \quad (8)$$

$$\frac{\partial}{\partial x} q_{0,n}(x,t) + \frac{\partial}{\partial t} q_{0,n}(x,t) = -(\lambda_1 + \lambda_2 + \mu(x)) q_{0,n}(x,t) + \sum_{i=1}^{n} \lambda_2 C_i q_{0,n-i}(x,t), n \geq 1 \qquad (9)$$

$$\frac{\partial}{\partial x} q_{0,0}(x,t) + \frac{\partial}{\partial t} q_{0,0}(x,t) = -(\lambda_1 + \lambda_2 + \mu(x)) q_{0,0}(x,t). \qquad (10)$$

Equations $(1)-(10)$ are to be solved subject to the following boundary conditions:

$$P_{0,0}(0,t) = \lambda_1 C_1 Q(t) + \int_0^\infty P_{1,0}(x,t)\mu(x)dx + \int_0^\infty q_{1,0}(x,t)\mu(x)dx, \qquad (11)$$

$$P_{m,0}(0,t) = \lambda_1 C_{m+1} Q(t) + \int_0^\infty P_{m+1,0}(x,t)\mu(x)dx + \int_0^\infty q_{m+1,0}(x,t)\mu(x)dx, \qquad m \geq 1 \qquad (12)$$

$$P_{0,n}(0,t) = \int_0^\infty P_{1,n}(x,t)\mu(x)dx + \int_0^\infty q_{1,n}(x,t)\mu(x)dx, \ n \geq 1 \qquad (13)$$

$$P_{m,n}(0,t) = \int_0^\infty P_{m+1,n}(x,t)\mu(x)dx + \int_0^\infty q_{m+1,n}(x,t)\mu(x)dx, \ m,n \geq 1 \qquad (14)$$

$$q_{0,0}(0,t) = \sum_{m=a}^{b}\sum_{k=0}^{a-1} \lambda_2 C_{m-k} Q_{0,k}(t) + \int_0^\infty P_{0,n}(x,t)\mu(x)dx + \int_0^\infty q_{0,n}(x,t)\mu(x)dx, \qquad a \leq n \leq b \quad (15)$$

$$q_{0,n}(0,t) = \sum_{k=0}^{a-1} \lambda_2 C_{b+n-k} Q_{0,k}(t) + \int_0^\infty P_{0,n+b}(x,t)\mu(x)dx + \int_0^\infty q_{0,n+b}(x,t)\mu(x)dx, \qquad n \geq 1. \quad (16)$$

We assume that initially there are no sufficient customers in the system, that is the server is idle. So that the initial conditions are

$$Q_{0,0}(0) = 1, P_{m,n}(0) = 0, q_{m,n}(0) = 0 \text{ for } m,n = 0,1,2,.. \qquad (17)$$

5. PROBABILITY GENERATING FUNCTIONS OF THE QUEUE LENGTH: THE TIME DEPENDENT SOLUTION

In this section we obtain the transient solution for the above set of differential-difference equations.

THEOREM:

The system of differential difference equations to describe an $M^{[X_1]}, M^{[X_2]}/G_1, G^{(a,b)}/1$ queueing system with priority servics are given by equations $(1)-(16)$ with the initial conditions (17) gives the probability generating functions of the transient solution.

PROOF:

The probability generating functions are

$$P(x,z_1,z_2,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n P_{m,n}(x,t) q(x,z_1,z_2,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n q_{m,n}(x,t)$$

$$C(z_1)=\sum_{n=1}^{\infty} z_1^n C_n \text{ and } C(z_2)=\sum_{n=1}^{\infty} z_2^n C_n \tag{18}$$

We also define the $W_q(z_1,z_2)$ to be the probability generating function of the number of customers in the queue, regardless of whether the server is busy or not. Then,

$$W_q(z_1,z_2)=P(z_1,z_2)+q(z_1,z_2)+\sum_{n=0}^{a-1} z_2^n Q_{0,n} \tag{19}$$

which are convergent inside the circle is given by $|z_1|\le 1, |z_2|\le 1$ and define the Laplace transform of a functions

$$\bar{f}(s)=\int_0^\infty e^{-st} f(t)dt,\ \ \Re(s)>0$$

Now taking the Laplace transform of equations $(1)-(16)$ and using (17) we get

$$(s+\lambda_1+\lambda_2)\bar{Q}_{0,0}(s)-1=\int_0^\infty \bar{P}_{0,0}(x,t)\mu(x)dx+\int_0^\infty \bar{q}_{0,0}(x,t)\mu(x)dx, \tag{20}$$

$$(s+\lambda_1+\lambda_2)\bar{Q}_{0,n}(s)=\sum_{k=1}^{n}\lambda_2 C_k \bar{Q}_{0,n-k}(s)+\int_0^\infty \bar{P}_{0,n}(x,s)\mu(x)dx$$

$$+\int_0^\infty \bar{q}_{0,n}(x,s)\mu(x)dx,\ n=1,2,...a-1. \tag{21}$$

$$\frac{\partial}{\partial x}\bar{P}_{m,n}(x,s)+(s+\lambda_1+\lambda_2+\mu(x))\bar{P}_{m,n}(x,s)=\sum_{i=1}^{m}\lambda_1 C_i \bar{P}_{m-i,n}(x,s)$$

$$+\sum_{i=1}^{n}\lambda_2 C_i \bar{P}_{m,n-i}(x,s),\ m,n\ge 1 \tag{22}$$

$$\frac{\partial}{\partial x}\bar{P}_{m,0}(x,s)+(s+\lambda_1+\lambda_2+\mu(x))P_{m,0}(x,s)=\sum_{i=1}^{m}\lambda_1 C_i P_{m-i,0}(x,s),\ m\ge 1 \tag{23}$$

$$\frac{\partial}{\partial x}\bar{P}_{0,n}(x,s)+(s+\lambda_1+\lambda_2+\mu(x))\bar{P}_{0,n}(x,s)=\sum_{i=1}^{n}\lambda_2 C_i \bar{P}_{0,n-i}(x,s),\ n\ge 1 \tag{24}$$

$$\frac{\partial}{\partial x}\bar{P}_{0,0}(x,s)+(s+\lambda_1+\lambda_2+\mu(x))\bar{P}_{0,0}(x,s)=0, \tag{25}$$

$$\frac{\partial}{\partial x}\bar{q}_{m,n}(x,s)+(s+\lambda_1+\lambda_2+\mu(x))\bar{q}_{m,n}(x,s)=\sum_{i=1}^{m}\lambda_1 C_i \bar{q}_{m-i,n}(x,s)$$

$$+\sum_{i=1}^{n}\lambda_2 C_i \bar{q}_{m,n-i}(x,s),\ m,n\ge 1 \tag{26}$$

$$\frac{\partial}{\partial x}\bar{q}_{m,0}(x,s)+(s+\lambda_1+\lambda_2+\mu(x))q_{m,0}(x,s)=\sum_{i=1}^{m}\lambda_1 C_i q_{m-i,0}(x,s),\ m\ge 1 \tag{27}$$

$$\frac{\partial}{\partial x}\bar{q}_{0,n}(x,s)+(s+\lambda_1+\lambda_2+\mu(x))\bar{q}_{0,n}(x,s)=\sum_{i=1}^{n}\lambda_2 C_i \bar{q}_{0,n-i}(x,s),\ n\ge 1 \tag{28}$$

$$\frac{\partial}{\partial x}\bar{q}_{0,0}(x,s)+(s+\lambda_1+\lambda_2+\mu(x))\bar{q}_{0,0}(x,s)=0, \tag{29}$$

$$\bar{P}_{0,0}(0,s) = \lambda_1 C_1 \bar{Q}(s) + \int_0^\infty \bar{P}_{1,0}(x,s)\mu(x)dx + \int_0^\infty \bar{q}_{1,0}(x,s)\mu(x)dx, \tag{30}$$

$$\bar{P}_{m,0}(0,s) = \lambda_1 C_{m+1} \bar{Q}(s) + \int_0^\infty \bar{P}_{m+1,0}(x,s)\mu(x)dx + \int_0^\infty \bar{q}_{m+1,0}(x,s)\mu(x)dx, \quad m \ge 1 \tag{31}$$

$$\bar{P}_{0,n}(0,s) = \int_0^\infty \bar{P}_{1,n}(x,s)\mu(x)dx + \int_0^\infty \bar{q}_{1,n}(x,s)\mu(x)dx, \; n \ge 1 \tag{32}$$

$$\bar{P}_{m,n}(0,s) = \int_0^\infty \bar{P}_{m+1,n}(x,s)\mu(x)dx + \int_0^\infty \bar{q}_{m+1,n}(x,s)\mu(x)dx, \; m,n \ge 1 \tag{33}$$

$$\bar{q}_{0,0}(0,s) = \sum_{m=a}^{b}\sum_{k=0}^{a-1} \lambda_2 C_{m-k} \bar{Q}_{0,k}(s) + \int_0^\infty \bar{P}_{0,n}(x,s)\mu(x)dx + \int_0^\infty \bar{q}_{0,n}(x,s)\mu(x)dx, \quad a \le n \le b \tag{34}$$

$$\bar{q}_{0,n}(0,s) = \sum_{k=0}^{a-1} \lambda_2 C_{b+n-k} \bar{Q}_{0,k}(s) + \int_0^\infty \bar{P}_{0,n+b}(x,s)\mu(x)dx + \int_0^\infty \bar{q}_{0,n+b}(x,s)\mu(x)dx, \quad n \ge 1. \tag{35}$$

Now multiply equation (21) by z_2^n summing over n from 0 to $(a-1)$ and then add to equation (20), we get

$$\sum_{n=0}^{a-1} z_2^n (s+\lambda_1+\lambda_2)\bar{Q}_{0,n}(s) - 1 = \sum_{n=1}^{a-1} z_2^n \sum_{k=1}^{n} \lambda_2 C_k \bar{Q}_{0,n-k}(s)$$
$$+ \int_0^\infty \sum_{n=0}^{a-1} z_2^n \bar{P}_{0,n}(x,s)\mu(x)dx + \int_0^\infty \sum_{n=0}^{a-1} z_2^n \bar{q}_{0,n}(x,s)\mu(x)dx, \; n = 1,2,\ldots,a-1. \tag{36}$$

Next, we multiply equations $(22), (23), (26)$ and (27) by z_1^m summing over m from1 to ∞, and add the resulting equations to the equations $(24), (25), (28)$ and (29), we get

$$\frac{\partial}{\partial x}\bar{P}_n(x,z_1,s) + (s+\lambda_1+\lambda_2+\mu(x))\bar{P}_n(x,z_1,s) = \lambda_1 C(z_1)\bar{P}_n(x,z_1,s)$$
$$+ \sum_{i=1}^{n} \lambda_2 C_i \bar{p}_{n-i}(x,z_1,s), \; \ge 1 \tag{37}$$

$$\frac{\partial}{\partial x}\bar{P}_0(x,z_1,s) + (s+\lambda_1+\lambda_2+\mu(x))\bar{P}_0(x,z_1,s) = \lambda_1 C(z_1)\bar{P}_0(x,z_1,s) \tag{38}$$

$$\frac{\partial}{\partial x}\bar{q}_n(x,z_1,s) + (s+\lambda_1+\lambda_2+\mu(x))\bar{q}_n(x,z_1,s) = \lambda_1 C(z_1)\bar{q}_n(x,z_1,s)$$
$$+ \sum_{i=1}^{n} \lambda_2 C_i \bar{q}_{n-i}(x,z_1,s), \; \ge 1 \tag{39}$$

$$\frac{\partial}{\partial x}\bar{q}_0(x,z_1,s) + (s+\lambda_1+\lambda_2+\mu(x))\bar{q}_0(x,z_1,s) = \lambda_1 C(z_1)\bar{q}_0(x,z_1,s). \tag{40}$$

Next, we multiply equations (37) and (39) by z_2^n summing over n from 1 to ∞, and add the resulting equations to the equations (38) and (40), we get

$$\frac{\partial}{\partial x}\overline{P}(x,z_1,z_2,s)+(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]+\mu(x))\overline{P}(x,z_1,z_2,s)=0 \quad (41)$$

$$\frac{\partial}{\partial x}\overline{q}(x,z_1,z_2,s)+(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]+\mu(x))\overline{q}(x,z_1,z_2,s)=0. \quad (42)$$

Integrating equations (41) and (42) between 0 to x, we get

$$\overline{P}(x,z_1,z_2,s)=\overline{P}(0,z_1,z_2,s)e^{-(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])x-\int_0^x \mu(t)dt} \quad (43)$$

$$\overline{q}(x,z_1,z_2,s)=\overline{q}(0,z_1,z_2,s)e^{-(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])x-\int_0^x \mu(t)dt}. \quad (44)$$

Again integrating equation (43) and (44) by parts with respect to x, yields

$$\overline{P}(z_1,z_2,s)=\overline{P}(0,z_1,z_2,s)\left[\frac{1-\overline{B}_1(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}{(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}\right] \quad (45)$$

$$\overline{q}(z_1,z_2,s)=\overline{q}(0,z_1,z_2,s)\left[\frac{1-\overline{B}_2(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}{(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}\right] \quad (46)$$

now multiply equations (43) and (44) by $\mu(x)$ and then integrate with respect to x, we get

$$\int_0^\infty \overline{P}(x,z_1,z_2,s)\mu(x)dx=\overline{P}(0,z_1,z_2,s)\overline{B}_1(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]) \quad (47)$$

$$\int_0^\infty \overline{q}(x,z_1,z_2,s)\mu(x)dx=\overline{q}(0,z_1,z_2,s)\overline{B}_2(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]). \quad (48)$$

At $z_1=0$, equations (47) and (48) gives,

$$\int_0^\infty \overline{P}_0(x,z_2,s)\mu(x)dx=\overline{P}_0(0,z_2,s)\overline{B}_1(s+\lambda_1+\lambda_2[1-C(z_2)]) \quad (49)$$

$$\int_0^\infty \overline{q}_0(x,z_2,s)\mu(x)dx=\overline{q}_0(0,z_2,s)\overline{B}_2(s+\lambda_1+\lambda_2[1-C(z_2)]). \quad (50)$$

However by its definition, $\overline{q}(0,z_1,z_2,s)=\overline{q}_0(0,z_2,s)$ the equation (48) becomes

$$\int_0^\infty \overline{q}(x,z_1,z_2,s)\mu(x)dx=\overline{q}_0(0,z_2,s)\overline{B}_2(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]). \quad (51)$$

Next, we multiply the boundary conditions (31) and (33) by z_1^m summing over m from 1 to ∞ and then add the resulting equations to $z_1\times(30)$ and $z_1\times(32)$

we get,

$$z_1\overline{P}_0(0,z_1,s)=\lambda_1C(z_1)\overline{Q}(s)+\int_0^\infty \overline{P}_0(x,z_1,s)\mu(x)dx-\int_0^\infty \overline{P}_{0,0}(x,s)\mu(x)dx$$
$$+\int_0^\infty \overline{q}_0(x,z_1,s)\mu(x)dx-\int_0^\infty \overline{q}_{0,0}(x,s)\mu(x)dx \quad (52)$$

$$z_1\overline{P}_n(0,z_2,s)=\int_0^\infty \overline{P}_n(x,z_1,s)\mu(x)dx-\int_0^\infty \overline{P}_{0,n}(x,s)\mu(x)dx$$
$$+\int_0^\infty \overline{q}_n(x,z_1,s)\mu(x)dx-\int_0^\infty \overline{q}_{0,n}(x,s)\mu(x)dx \quad (53)$$

now multiply equation (53) by z_2^n summing over m from 1 to ∞ and then add the resulting equation to (52), we get.

$$z_1\overline{P}(0,z_1,z_2,s)=\lambda_1C(z_1)\overline{Q}(s)+\int_0^\infty \overline{P}(x,z_1,z_2,s)\mu(x)dx-\int_0^\infty \overline{P}_0(x,z_2,s)\mu(x)dx$$

$$+\int_0^\infty \overline{q}(x,z_1,z_2,s)\mu(x)dx - \int_0^\infty \overline{q}_0(x,z_2,s)\mu(x)dx \tag{54}$$

now multiply equation (35) by z_2^n summing over m from 1 to ∞ and then add the resulting equation (34), we get

$$\begin{aligned}\overline{q}_0(0,z_2,s) &= \lambda_2 z_2^{-b}\sum_{n=1}^{b}\sum_{k=0}^{a-1}C_n\overline{Q}_{0,k}(s)(z_2^b - z_2^{n+k}) - \lambda_2 C_b z_2^{-b}\sum_{k=1}^{a-1}(z_2^b - z_2^{k+b})\overline{Q}_{0,k}(s)\\
&\quad -\lambda_2 C_{b-1}z_2^{-b}\sum_{k=2}^{a-1}\overline{Q}_{0,k}(s)(z_2^b - z_2^{k+b-1}) - \ldots - \lambda_2 C_{b-a}z_2^{-b}\overline{Q}_{0,a-1}(s)(z_2^b - z_2^{b-a})\\
&\quad +\lambda_2 z_2^{-b}C(z_2)\sum_{k=0}^{a-1}\overline{Q}_{0,k}(s)z_2^k - \lambda_2\sum_{n=1}^{a-1}\sum_{k=0}^{n}C_k\overline{Q}_{0,n-k}(s) + z_2^{-b}\sum_{k=0}^{b}(z_2^b - z_2^k)\\
&\quad \int_0^\infty \overline{P}_{0,k}(x,s)\mu(x)dx + z_2^{-b}\sum_{k=0}^{b}(z_2^b - z_2^k)\int_0^\infty \overline{q}_{0,k}(x,s)\mu(x)dx\\
&\quad + z_2^{-b}\int_0^\infty \overline{P}_0(x,z_2,s)\mu(x)dx + z_2^{-b}\int_0^\infty \overline{q}_0(x,z_2,s)\mu(x)dx\\
&\quad -\int_0^\infty \overline{P}_{0,0} + \overline{P}_{0,1} + \ldots + \overline{P}_{0,a-1}(x,s)\mu(x)dx\\
&\quad -\int_0^\infty \overline{q}_{0,0} + \overline{q}_{0,1} + \ldots + \overline{q}_{0,a-1}(x,s)\mu(x)dx\end{aligned} \tag{55}$$

using equations (20) and (21) into equation (55), we get

$$\begin{aligned}z_2^b\overline{q}_0(0,z_2,s) &= \lambda_2\sum_{n=1}^{b}\sum_{k=0}^{a-1}C_n\overline{Q}_{0,k}(s)(z_2^b - z_2^{n+k}) - \lambda_2 C_b\sum_{k=1}^{a-1}(z_2^b - z_2^{k+b})\overline{Q}_{0,k}(s)\\
&\quad -\lambda_2 C_{b-1}\sum_{k=2}^{a-1}\overline{Q}_{0,k}(s)(z_2^b - z_2^{k+b-1}) - \ldots - \lambda_2 C_{b-a}\overline{Q}_{0,a-1}(s)(z_2^b - z_2^{b-a})\\
&\quad +\lambda_2 C(z_2)\sum_{k=0}^{a-1}\overline{Q}_{0,k}(s)z_2^k + \sum_{k=0}^{b}(z_2^b - z_2^k)\int_0^\infty \overline{P}_{0,k}(x,s)\mu(x)dx\\
&\quad +\sum_{k=0}^{b}(z_2^b - z_2^k)\int_0^\infty \overline{q}_{0,k}(x,s)\mu(x)dx + \int_0^\infty \overline{P}_0(x,z_2,s)\mu(x)dx\\
&\quad +\int_0^\infty \overline{q}_0(x,z_2,s)\mu(x)dx - z_2^b\{\sum_{k=0}^{a-1}(s+\lambda_1+\lambda_2)\overline{Q}_{0,k}(s) - 1\}.\end{aligned} \tag{56}$$

Now substitute equations (47), (50), (51) into (54) and (56), we get

$$\begin{aligned}\{z_1 - \overline{B}_1(\phi_1(z,s)\}\overline{P}(0,z_1,z_2,s) &= \lambda_1 C(z_1)\overline{Q}(s) + \overline{q}_0(0,z_2,s)\{\overline{B}_2(\phi_1(z,s))\\
&\quad -\overline{B}_2(\psi_1(z,s))\} - \overline{P}_0(0,z_2,s)\overline{B}_1(\psi_1(z,s))\end{aligned} \tag{57}$$

$$\begin{aligned}z_2^b\overline{q}_0(0,z_2,s) &= \lambda_2\sum_{n=1}^{b}\sum_{k=0}^{a-1}C_n\overline{Q}_{0,k}(s)(z_2^b - z_2^{n+k}) - \lambda_2 C_b\sum_{k=1}^{a-1}(z_2^b - z_2^{k+b})\overline{Q}_{0,k}(s)\\
&\quad -\lambda_2 C_{b-1}\sum_{k=2}^{a-1}\overline{Q}_{0,k}(s)(z_2^b - z_2^{k+b-1}) --- \lambda_2 C_{b-a}\overline{Q}_{0,a-1}(s)(z_2^b - z_2^{b-a})\\
&\quad +\lambda_2 C(z_2)\sum_{k=0}^{a-1}\overline{Q}_{0,k}(s)z_2^k + \sum_{k=0}^{b}(z_2^b - z_2^k)\int_0^\infty \overline{P}_{0,k}(x,s)\mu(x)dx\end{aligned}$$

$$+\sum_{k=0}^{b}(z_2^b - z_2^k)\int_0^\infty \bar{q}_{0,k}(x,s)\mu(x)dx + \bar{P}_0(0,z_2,s)\bar{B}_1(\psi_1(z,s))$$
$$+\bar{q}_0(0,z_2,s)\bar{B}_2(\psi_1(z,s)) - z_2^b\{\sum_{k=0}^{a-1}(s+\lambda_1+\lambda_2)\bar{Q}_{0,k}(s)-1\} \quad (58)$$

by using Rouche's theorem, let $z_1 = g(z_2)$ into equation (57), we get

$$\bar{P}_0(0,z_2,s)\bar{B}_1(\psi_1(z,s)) = \lambda_1 C(g(z_2))\bar{Q}(s) + \bar{q}_0(0,z_2,s)\{\bar{B}_2(\phi_2(z,s)) - \bar{B}_2(\psi_1(z,s))\}. \quad (59)$$

Now substitute equation (59) into equation (58), we get

$$\bar{q}_0(0,z_2,s) = \frac{N_1}{z_2^b - \bar{B}_2(\phi_1(z,s)} \quad (60)$$

where

$$N_1 = \lambda_1 C(g(z_2))\bar{Q}(s) + \lambda_2\sum_{n=1}^{b}\sum_{k=0}^{a-1}C_n\bar{Q}_{0,k}(s)(z_2^b - z_2^{n+k}) - \lambda_2 C_b\sum_{k=1}^{a-1}(z_2^b - z_2^{k+b})\bar{Q}_{0,k}(s)$$
$$-\lambda_2 C_{b-1}\sum_{k=2}^{a-1}\bar{Q}_{0,k}(s)(z_2^b - z_2^{k+b-1}) --- \lambda_2 C_{b-a}\bar{Q}_{0,a-1}(s)(z_2^b - z_2^{b-a})$$
$$+\lambda_2 C(z_2)\sum_{k=0}^{a-1}\bar{Q}_{0,k}(s)z_2^k + \sum_{k=0}^{b}(z_2^b - z_2^k)\int_0^\infty \bar{P}_{0,k}(x,s)\mu(x)dx$$
$$+\sum_{k=0}^{b}(z_2^b - z_2^k)\int_0^\infty \bar{q}_{0,k}(x,s)\mu(x)dx - z_2^b\{\sum_{k=0}^{a-1}(s+\lambda_1+\lambda_2)\bar{Q}_{0,k}(s)-1\}$$

Next, we substitute equation (59) and (60) into equation (57), we get

$$\bar{P}(0,z_1,z_2,s) = \frac{\left\{\begin{array}{l}\{1-(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])\bar{Q}(s)\} \\ +\bar{q}_0(0,z_2,s)\{\bar{B}_2(\phi_2(z,s)-\bar{B}_2(\psi_1(z,s)\}\end{array}\right\}}{z_1 - \bar{B}_1(\phi_1(z,s)} \quad (61)$$

Again, we substitute equations (61) and (60) into equations (45) and (46) thus we get the complete solution for the following states, $\bar{P}(z_1,z_2,s)$ and $\bar{q}(z_1,z_2,s)$ are

$$\bar{P}(z_1,z_2,s) = \bar{P}(0,z_1,z_2,s)[\frac{1-\bar{B}_1(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}{(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}] \quad (62)$$

$$\bar{q}(z_1,z_2,s) = \bar{q}(0,z_1,z_2,s)[\frac{1-\bar{B}_2(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}{(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}] \quad (63)$$

6. STEADY STATE RESULTS

In this section, we derive the steady state probability distribution for our queueing model.
By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t).$$

to the equations (62) and (63). In order to determine Q we use the normalizing condition

$$Q + P(1,1) + q(1,1) = 1. \quad (64)$$

The steady state probability generating function for the no.of customers in the queue when the server is busy with high priority and low priority customers are

$$P(z_1,z_2) = P(0,z_1,z_2)[\frac{1-\overline{B_1}(\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}{(\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}] \quad (65)$$

and

$$q(z_1,z_2) = q_0(0,z_2)[\frac{1-\overline{B_2}(\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}{(\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])}] \quad (66)$$

where

$$P(0,\ z_1,z_2) = \frac{-(\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)])Q+q_0(0,z_2)\{\overline{B_2}(\phi_2(z)-\overline{B_2}(\psi_1(z)\}}{z_1-\overline{B_1}(\phi_1(z)}, \quad (67)$$

$$q_0(0,z_2) = \frac{N_2}{z_2^b-\overline{B_2}(\phi_1(z)}. \quad (68)$$

$$N_2 = \lambda_1 C(g(z_2))Q + \lambda_2 \sum_{n=1}^{b}\sum_{k=0}^{a-1} C_n Q_{0,k}(z_2^b - z_2^{n+k}) - \lambda_2 C_b \sum_{k=1}^{a-1}(z_2^b - z_2^{k+b})Q_{0,k}$$
$$-\lambda_2 C_{b-1}\sum_{k=2}^{a-1} Q_{0,k}(z_2^b - z_2^{k+b-1}) - \ldots - \lambda_2 C_{b-a} Q_{0,a-1}(z_2^b - z_2^{b-a})$$
$$+\lambda_2 C(z_2)\sum_{k=0}^{a-1} Q_{0,k} z_2^k + \sum_{k=0}^{b}(z_2^b - z_2^k)\int_0^\infty P_{0,k}(x)\mu(x)dx$$
$$+\sum_{k=0}^{b}(z_2^b - z_2^k)\int_0^\infty q_{0,k}(x)\mu(x)dx - z_2^b\{\sum_{k=0}^{a-1}(\lambda_1+\lambda_2)Q_{0,k}\}$$

The steady state equations corresponding to equations (20) and (21) becomes,

$$0 = -(\lambda_1+\lambda_2)Q + \int_0^\infty P_{0,0}(x)\mu(x)dx + \int_0^\infty q_{0,0}(x)\mu(x)dx \quad (69)$$

$$(\lambda_1+\lambda_2)Q_{0,n} = \sum_{k=1}^{n}\lambda_2 C_k Q_{0,n-k} + \int_0^\infty P_{0,n}(x)\mu(x)dx + \int_0^\infty q_{0,n}(x)\mu(x)dx, \qquad n = 1,2,\ldots,\ a-1. \quad (70)$$

Further discussion of this will be taken up in a later section. It is of considerable interest to note that the equations (65) and (66) expression has no dependence on a , the quorum. It agrees exactly with the similar expression found by Chaudhry and Templeton (1981) for the system $M/G^B/1$.

7. RELATION TO THE IMBEDED MORKOV CHAIN

Next we wish to relate the probability generating function in equations (65) and (66) to $P^+(z)$ and $q^+(z)$, are the probability generating function of the number in the system immediately after departure epochs of high and low priority customers or equivalently just before commencement of service epochs: $P^+(z) = \sum_{i=0}^{\infty} P_i^+ z^i, q^+(z) = \sum_{i=0}^{\infty} q_i^+ z^i$, where P_i^+ and q_i^+ are probabilities that the number is i, P_n^+ and q_n^+ are related to $P_n(x), q_n(x)$ are as follows. This procedure is new and has been used for the first time by Chaudhry and Templeton (1981) for analyzing bulk service queues. For computational results, this allows us to use the steady-states imbedded Markov chain probabilities, which can be calculated as shown below. We note from Chaudhry and Templeton (1981) that $P^+(z)$ and $q^+(z)$ are related to P(z) and q(z) as follows:

$$P^{+}_{0,n} = d\int_0^\infty P_{0,n}(x)\mu(x)dx \tag{71}$$

$$q^{+}_{0,n} = d\int_0^\infty q_{0,n}(x)\mu(x)dx \tag{72}$$

where d is a normalizing constant. Then equation (71) and (72) is based on the definitions of

$P_n(x), q_n(x)$ and $\mu(x)$.

The probability generating function $P^{+}(z)$ and $q^{+}(z)$ can be seen to be

$$P^{+}_{0,n}(z) = d\int_0^\infty P_{0,n}(x,z)\mu(x)dx \tag{73}$$

$$q^{+}_{0,n}(z) = d\int_0^\infty q_{0,n}(x,z)\mu(x)dx \tag{74}$$

using equations (43) and (44), we get

$$P^{+}(z) = dP(0,z_1,z_2)\overline{B}_1(\phi_1(z)) \tag{75}$$

and

$$q^{+}(z) = dq_0(0,z_2)\overline{B}_2(\phi_1(z)) \tag{76}$$

where we have used the definition of $\mu(x)$. Now taking $P(0, z_1, z_2)$ and $q_0(0,z_2)$ from equations (67) and (68), we get

$$P^{+}(z) = [\frac{dq_0(0,z_2)\{\overline{B_2}(\phi_1(z)) - \overline{B_2}(\phi_2(z))\} - d\{\lambda_1[1-C(z_1)] + \lambda_2[1-C(z_2)]\}}{z_1 - \overline{B}_1(\phi_1(z))}]\overline{B}_1(\phi_1(z)) \tag{77}$$

$$q^{+}(z) = dq_0(0,z_2)\overline{B}_2(\phi_1(z) = \{\frac{N_3\overline{B}_2(\phi_1(z))}{z_2^b - \overline{B}_2(\phi_1(z))}\} \tag{78}$$

where

$$\begin{aligned} N_3 = d\{&\lambda_1 C(g(z_2))Q + \lambda_2\sum_{n=1}^{b}\sum_{k=0}^{a-1} C_n Q_{0,k}(z_2^b - z_2^{n+k}) - \lambda_2 C_b \sum_{k=1}^{a-1}(z_2^b - z_2^{k+b})Q_{0,k} \\ &- \lambda_2 C_{b-1}\sum_{k=2}^{a-1} Q_{0,k}(z_2^b - z_2^{k+b-1}) - \ldots - \lambda_2 C_{b-a} Q_{0,a-1}(z_2^b - z_2^{b-a}) \\ &+ \lambda_2 C(z_2)\sum_{k=0}^{a-1} Q_{0,k} z_2^k - z_2^b \sum_{k=0}^{a-1}(\lambda_1 + \lambda_2)Q_{0,k}\} + \sum_{k=0}^{b}(z_2^b - z_2^k)P^{+}_{0,k} \\ &+ \sum_{k=0}^{b}(z_2^b - z_2^k)q^{+}_{0,k} \end{aligned}$$

Now, by using equation (68), we, get

$$q^{+}_0(z) = dq_0(0,z_2)\frac{N_3}{z_2^b - \overline{B}_2(\phi_1(z))}.$$

It needs to be pointed out here that equation (77) and (78) may be compared with equation (67) and (68) of Neuts with $H_i(.) = H(.)$, $i = 0,1,2,\ldots, B$. Next we can evaluate the constant 'd' by applying the normalizing condition, $P^{+}(1) + q^{+}(1) = 1$, to equation (75) and (76). We get

$$d = (P(0,1,1) + q_0(0,1))^{-1}. \tag{79}$$

Now, we evaluate $(P(0,1,1) + q_0(0,1))^{-1}$ from equations (64) and (65), by applying the normalizing condition to this equation, recalling from (19)

$$P(1,1)+q(1,1)=1-\sum_{n=0}^{a-1}Q_{0,n} \tag{80}$$

this is the total probability that the server is busy.Finally, we get

$$P(0,1,1) = P(1,1)\mu \tag{81}$$

and

$$q_0(0,1)=q(1,1)\mu \tag{82}$$

a result which is intuitively appealing and may be obtained by the conservation principle. This gives

$$d=[\mu\{1-\sum_{n=0}^{a-1}Q_{0,n}\}]^{-1} \tag{83}$$

Now compare equations (65) and (66) to (77) and (78) and using (83) we get the relation

$$P(z_1,z_2)+q(z_1,z_2)=[\{\frac{1-\overline{B}_1(\phi_1(z))}{\phi_1(z))\overline{B}_1(\phi_1(z))}\}P^+(z_1,z_2)+\{\frac{1-\overline{B}_2(\phi_1(z))}{\phi_1(z))\overline{B}_2(\phi_1(z))}\}q^+(z_1,z_2)]$$
$$\times\mu(1-\sum_{k=0}^{a-1}Q_{0,k}) \tag{84}$$

8. CALCULATION OF BASIC $P_{0,k}^+$, $q_{0,k}^+$ AND $Q_{0,k}$ PROBABILITIES

We next discuss the calculation of the unknown probabilities $P_{0,k}^+$ and $q_{0,k}^+$, $k = 0,1,2,...b$ and then these are related to the server's idle probability, $Q_{0,k}, k=0,1..a-1$.

The characteristic equation of this queueing system is, from the denominator of (78),

$$z_2^b-\overline{B}_2(\phi_1(z))=0 \tag{85}$$

The equation (85) has b zeros inside and on C. There is a simple zero at $z_2 = 1$, and therefore, there are $b-1$ other zeros inside the unit circle. Let us call them $z_i, i=0,1,..b-1$. Consider again the equation (75). Since it must converge inside and on the unit circle, the numerator must vanish at all the zeros of the denominator which are inside and on the unit circle. The zero at $z_2 = 1$ clearly cancels from both numerator and denominator, leaving $b-1$ other equations,

$$d\{\lambda_1 C(g(z_2))Q+\lambda_2\sum_{n=1}^{b}\sum_{k=0}^{a-1}C_nQ_{0,k}(z_2^b-z_2^{n+k})-\lambda_2C_b\sum_{k=1}^{a-1}(z_2^b-z_2^{k+b})Q_{0,k}$$
$$-\lambda_2C_{b-1}\sum_{k=2}^{a-1}Q_{0,k}(z_2^b-z_2^{k+b-1})-...-\lambda_2C_{b-a}Q_{0,a-1}(z_2^b-z_2^{b-a})$$
$$+\lambda_2C(z_2)\sum_{k=0}^{a-1}Q_{0,k}z_2^k-z_2^b\sum_{k=0}^{a-1}(\lambda_1+\lambda_2)Q_{0,k}\}+\sum_{k=0}^{b}(z_2^b-z_2^k)P_{0,k}^+ +\sum_{k=0}^{b}(z_2^b-z_2^k)q_{0,k}^+=0. \tag{86}$$

To this we add the normalization condition, $q^+(1,1)=1$, which gives

$$\lambda_1E(I)E(I_1)\sum_{k=0}^{a-1}Q_{0,k}+\lambda_2\sum_{n=1}^{b}\sum_{k=0}^{a-1}C_nQ_{0,k}(b-n-k)+\lambda_2C_b\sum_{k=1}^{a-1}Q_{0,k}k$$
$$-\lambda_2C_{b-1}\sum_{k=2}^{a-1}Q_{0,k}(1-k)-...-\lambda_2C_{b+1}Q_{0,a-1}+\lambda_2E(I)\sum_{k=0}^{a-1}Q_{0,k}$$
$$+\lambda_2\sum_{k=0}^{a-1}Q_{0,k}k-\lambda_2\sum_{k=0}^{a-1}Q_{0,k}b+\mu(1-\sum_{k=0}^{a-1}Q_{0,k})\sum_{k=0}^{b}(b-k)(P_{0,k}^+ +q_{0,k}^+)$$

$$= \mu(1-\sum_{n=0}^{a-1}Q_{0,n})(b-\rho_2). \tag{87}$$

Together, the system of equations (86) and (87) form a set of b linear independent equations in the b unknowns $P_{0,k}^{+}$ and $q_{0,k}^{+}$. Therefore, the $P_{0,k}^{+}$ and $q_{0,k}^{+}$ can be found from the zeros of the characteristic equation inside and on the unit circle. Thus the probability generating function $P_{0,k}^{+}$ and $q_{0,k}^{+}$ is completely determined. We now wish to relate the $P_{0,k}^{+}$ and $q_{0,k}^{+}$ to the $Q_{0,k}$

Recall equations (71) and (72) and using the value of d from (83), we have,

$$\int_0^\infty P_{0,n}(x)dx = \mu(1-\sum_{n=0}^{a-1}Q_{0,n})P_{0,n}^{+}$$

and

$$\int_0^\infty q_{0,n}(x)dx = \mu(1-\sum_{n=0}^{a-1}Q_{0,n})q_{0,n}^{+}$$

This relationship is substituted into equations (69) and (70), gives

$$(\lambda_1+\lambda_2)Q_{0,0} = \mu(1-\sum_{i=0}^{a-1}Q_{0,i})P_{0,0}^{+} + \mu(1-\sum_{i=0}^{a-1}Q_{0,i})q_{0,0}^{+}, \tag{88}$$

$$(\lambda_1+\lambda_2)Q_{0,n} = \mu(1-\sum_{i=0}^{a-1}Q_{0,i})P_{0,n}^{+} + \mu(1-\sum_{i=0}^{a-1}Q_{0,i})q_{0,n}^{+},\ 1\le n\le a-1. \tag{89}$$

Multiplying equation (88) by a and adding it to the equation (89) multiplied by $(a-r)$ and summed over $n=1$ to $a-1$, we get an equation which gives

$$(1-\sum_{i=0}^{a-1}Q_{0,i}) = \frac{\left\{\frac{(\lambda_1+\lambda_2)}{\mu} - \frac{(\lambda_1+\lambda_2)}{\mu}\sum_{i=0}^{a-1}Q_{0,i} + \frac{(\lambda_1+\lambda_2)}{\mu}\sum_{n=0}^{a-1}Q_{0,n}(a-n) - \sum_{n=1}^{a-1}(a-n)\sum_{k=1}^{n}Q_{0,n-k}\frac{\lambda_2 C_k}{\mu}\right\}}{\frac{(\lambda_1+\lambda_2)}{\mu} + \sum_{n=0}^{a-1}(a-n)(P_{0,n}^{+}+q_{0,n}^{+})}, \tag{90}$$

since $P^{+}(z_1,z_2)$ and $q^{+}(z_1,z_2)$ is already fully determined, (90) together with the equation (84) is sufficient to give $W_q(z_1,z_2)$ completely. However, it is still necessary to find the values of individual $Q_{0,i}$ to use in (19). First substituting (90) into (88)

$$Q_{0,0} = \frac{\left\{\frac{(\lambda_1+\lambda_2)}{\mu} - \frac{(\lambda_1+\lambda_2)}{\mu}\sum_{i=0}^{a-1}Q_{0,i} + \frac{(\lambda_1+\lambda_2)}{\mu}\sum_{n=0}^{a-1}Q_{0,n}(a-n) - \sum_{n=1}^{a-1}(a-n)\sum_{k=1}^{n}Q_{0,n-k}\frac{\lambda_2 C_k}{\mu}\right\}\left[\frac{(P_{0,0}^{+}+q_{0,0}^{+})\mu}{\lambda_1+\lambda_2}\right]}{\frac{(\lambda_1+\lambda_2)}{\mu} + \sum_{n=0}^{a-1}(a-n)(P_{0,n}^{+}+q_{0,n}^{+})}. \tag{91}$$

Rewrite the equations (88) and (89) for n=r, we get

$$(\lambda_1+\lambda_2)Q_{0,0} = \mu(1-\sum_{i=0}^{a-1}Q_{0,i})\{P_{0,0}^{+}+q_{0,0}^{+}\},$$

$$(\lambda_1+\lambda_2)Q_{0,r} = \sum_{k=1}^{r}\lambda_2 C_k Q_{0,r-k} + \mu(1-\sum_{i=0}^{a-1}Q_{0,i})\{P_{0,r}^{+}+q_{0,r}^{+}\},\ 1 \le r \le a-1,$$

summing these two equations for r=1 to n gives an equation which, with the use of equation (90) gives,

$$Q_{0,n} = -\{Q_{0,0}+Q_{0,1}+Q_{0,2}+\ldots+Q_{0,n-1}\}+[\frac{1}{\lambda_1+\lambda_2}]\{\lambda_2 C_1\sum_{r=0}^{n-1}Q_{0,r}$$
$$+\lambda_2 C_2\sum_{r=0}^{n-2}Q_{0,r}+\lambda_2 C_3\sum_{r=0}^{n-3}Q_{0,r}+\ldots+\lambda_2 C_n Q_{0,r}\}$$
$$+[\frac{\mu}{\lambda_1+\lambda_2}](1-\sum_{i=0}^{a-1}Q_{0,i})\sum_{r=0}^{n}\{P_{0,r}^{+}+q_{0,r}^{+}\}. \quad (92)$$

We now wish to evaluate the term $\sum_{n=0}^{a-1} z_2^n Q_{0,n}$ to complete the determination of $W_q(z_1,z_2)$ in equation (19). From (92), we get

$$\sum_{n=0}^{a-1}z_2^n Q_{0,n} = -\sum_{n=1}^{a-1}(\frac{1-z_2^a}{1-z_2})Q_{0,n-1}+(\frac{1-z_2^a}{1-z_2})[\frac{1}{\lambda_1+\lambda_2}]\{\lambda_2 C_1\sum_{n=0}^{a-2}Q_{0,n}$$
$$+\lambda_2 C_2\sum_{n=0}^{a-3}Q_{0,n}+\lambda_2 C_3\sum_{n=0}^{a-4}Q_{0,n}+\ldots+\lambda_2 C_{a-1}Q_{0,0}\}$$
$$+[\frac{\mu}{\lambda_1+\lambda_2}](1-\sum_{i=0}^{a-1}Q_{0,i})\sum_{n=0}^{a-1}(\frac{z_2^n-z_2^a}{1-z_2})\{P_{0,n}^{+}+q_{0,n}^{+}\}. \quad (93)$$

Finally, this gives the expression for $W_q(z_1,z_2)$:

$$W_q(z_1,z_2) = P(z_1,z_2)+q(z_1,z_2)+\sum_{n=0}^{a-1}z_2^n Q_{0,n}. \quad (94)$$

9. AVERAGE NUMBER OF CUSTOMERS IN THE QUEUES

The average number of customers in the queues, L_{q_1} and L_{q_2} is found as usual by taking the first derivative of equation (94) at $z_1 = 1$ and $z_2 = 1$ respectively.

$$L_{q_1} = \frac{d}{dz_1}W_q(z_1,1)|_{z_1=1}, \quad (95)$$

$$L_{q_2} = \frac{d}{dz_2}W_q(1,z_2)|_{z_2=1}. \quad (96)$$

Now

$$L_{q_1} = \frac{1}{2}\mu(1-\sum_{i=0}^{a-1}Q_{0,i})\{\lambda_1 E(I)P^{+}(1,1)(E[B_1^2]-2(E[B_1])^2)+2E[B_1]P^{+'}(1,1)$$
$$+\lambda_1 E(I)q_0^{+}(1)(E[B_2^2]-2(E[B_2])^2)\}+a\{-\sum_{n=1}^{a-1}Q_{0,n-1}$$
$$+[\frac{1}{\lambda_1+\lambda_2}]\{\lambda_2 C_1\sum_{n=0}^{a-2}Q_{0,n}+\lambda_2 C_2\sum_{n=0}^{a-3}Q_{0,n}+\lambda_2 C_3\sum_{n=0}^{a-4}Q_{0,n}$$
$$+\ldots+\lambda_2 C_{a-1}Q_{0,0}\}\}+[\frac{\mu}{\lambda_1+\lambda_2}](1-\sum_{i=0}^{a-1}Q_{0,i})\sum_{n=0}^{a-1}\{P_{0,n}^{+}+q_{0,n}^{+}\}, \quad (97)$$

where

$$P^{+}(1,1)=\{\frac{d\{\lambda_1 E(I)+q_0(0,1)E[B_2]\}}{1-\lambda_1 E(I)E[B_1]}\},\quad P^{+'}(1,1)=\{\frac{N_4}{\{2-\lambda_1 E(I)E[B_1]\}^2}\},$$

$$\begin{aligned}N_4 &= dq_0(0,1)\{(\lambda_1 E(I))^2 E[B_2^2]+E[B_2]\lambda_1 E(I[I-1])+2E[B_1]E[B_2](\lambda_1 E(I))^2\\ &-(\lambda_1 E(I))^3 E[B_1]E[B_2^2]-2E[B_2]\lambda_1 E(I[I-1])(\lambda_1 E(I)E[B_1])^2+E[B_2]\\ &(\lambda_1 E(I))^3 E[B_1^2]\}+d\{\lambda_1 E(I[I-1])+2(\lambda_1 E(I))^2 E[B_1]-2(\lambda_1 E(I))^3\\ &(E[B_1])^2+(\lambda_1 E(I))^3 E[B_1])^2\},\end{aligned}$$

$$q_0^{+}(1)=dq_0(0,1)=\frac{N_5}{b-E[B_2]E(I)\lambda_1},$$

$$\begin{aligned}N_5 &= d\{\lambda_1 E(I)E(I_1)\sum_{k=0}^{a-1}Q_{0,k}+\lambda_2\sum_{n=1}^{b}\sum_{k=0}^{a-1}C_n Q_{0,k}(b-n-k)+\lambda_2 C_b\sum_{k=1}^{a-1}Q_{0,k}k\\ &-\lambda_2 C_{b-1}\sum_{k=2}^{a-1}Q_{0,k}(1-k)-\ldots-\lambda_2 C_{b+1}Q_{0,a-1}+\lambda_2 E(I)\sum_{k=0}^{a-1}Q_{0,k}\\ &+\lambda_2\sum_{k=0}^{a-1}Q_{0,k}k-\lambda_2\sum_{k=0}^{a-1}Q_{0,k}b\}+\mu(1-\sum_{k=0}^{a-1}Q_{0,k})\sum_{k=0}^{b}(b-k)(P_{0,k}^{+}+q_{0,k}^{+}),\end{aligned}$$

and

$$\begin{aligned}L_{q_2} &= \frac{\mu(1-\sum_{i=0}^{a-1}Q_{0,i})}{2\lambda_2 E(I)}\{\{(\lambda_2 E(I))^2 E[B_1^2]+E[B_1]\lambda_2 E(I[I-1])\}P^{+}(1,1)\\ &+2E[B_1]\lambda_2 E(I)P^{+'}(1,1)-E[B_1]P^{+}(1,1)\{\lambda_2 E(I[I-1])\\ &+2(\lambda_2 E(I))^2 E[B_1]\}+\{(\lambda_2 E(I))^2 E[B_2^2]+E[B_2]\lambda_2 E(I[I-1])\}q_0^{+}(1)\\ &+2E[B_2]\lambda_2 E(I)q_0^{+'}(1)-E[B_2]q_0^{+}(1)\{\lambda_2 E(I[I-1])+2(\lambda_2 E(I))^2 E[B_2]\}\}\\ &+\frac{a(a-1)}{2}\{-\sum_{n=1}^{a-1}Q_{0,n-1}+[\frac{1}{\lambda_1+\lambda_2}]\{\lambda_2 C_1\sum_{n=0}^{a-2}Q_{0,n}+\lambda_2 C_2\sum_{n=0}^{a-3}Q_{0,n}+\lambda_2 C_3\sum_{n=0}^{a-4}Q_{0,n}+\ldots+\lambda_2 C_{a-1}Q_{0,0}\}\}\\ &+[\frac{\mu}{\lambda_1+\lambda_2}](1-\sum_{i=0}^{a-1}Q_{0,i})\sum_{n=0}^{a-1}(\frac{a(a-1)-n(n-1)}{2})\{P_{0,n}^{+}+q_{0,n}^{+}\},\end{aligned} \tag{98}$$

where

$$P^{+}(1,1)=\{\frac{d\{\lambda_1 E(I)E(I_1)q_0(0,1)E[B_2]-\lambda_2 E(I)\}}{\lambda_2 E(I)E[B_1]}\},P^{+'}(1,1)=\{\frac{N_6}{2\{\lambda_2 E(I)E[B_1]\}^2}\},$$

$$\begin{aligned}N_6 &= d\{\lambda_2 E(I)-\lambda_1 E(I)E(I_1)q_0(0,1)E[B_2]\}\{E[B_1]\lambda_2 E(I[I-1])+(\lambda_2 E(I))^2 E[B_1^2]\}\\ &-E[B_1]\lambda_2 E(I)\{dq_0(0,1)\{(\lambda_2 E(I))^2 E[B_2^2]+E[B_2]\lambda_2 E(I[I-1])-E[B_2^2]\\ &\{\lambda_1 E(I)E(I_1)+\lambda_2 E(I)\}^2+E[B_2]\{\lambda_1[E(I[I-1])E(I_1)+E(I)E(I_1[I_1-1])\\ &+\lambda_2 E(I[I-1])\}\}+d\lambda_2 E(I[I-1])+2E[B_1]\lambda_2 E(I)\{dq_0(0,1)E[B_2]\lambda_1 E(I)E(I_1)\end{aligned}$$

$$+d\lambda_2 E(I)\} - 2dq_0^{'}(0,1)E[B_2]\lambda_1 E(I)E(I_1)\},$$

$$q_0^{+}(1) = dq_0(0,1) = \frac{N_7}{b - E[B_2]E(I)\lambda_2},$$

$$N_7 = d\{\lambda_1 E(I)E(I_1)\sum_{k=0}^{a-1} Q_{0,k} + \lambda_2 \sum_{n=1}^{b}\sum_{k=0}^{a-1} C_n Q_{0,k}(b-n-k) + \lambda_2 C_b \sum_{k=1}^{a-1} Q_{0,k} k$$
$$-\lambda_2 C_{b-1}\sum_{k=2}^{a-1} Q_{0,k}(1-k) - \ldots - \lambda_2 C_{b+1} Q_{0,a-1} + \lambda_2 E(I)\sum_{k=0}^{a-1} Q_{0,k}$$
$$+\lambda_2 \sum_{k=0}^{a-1} Q_{0,k} k - \lambda_2 \sum_{k=0}^{a-1} Q_{0,k} b\} + \sum_{k=0}^{b}(b-k)(P_{0,k}^{+} + q_{0,k}^{+}),$$

$$dq_0^{'}(0,1) = \frac{N_8}{2\{b - E[B_2]E(I)\lambda_2\}^2},$$

$$N_8 = \{b - E[B_2]E(I)\lambda_2\}\{d\{-\sum_{k=0}^{a-1} Q_{0,k}\lambda_1[E(I[I-1])(E(I_1))^2$$
$$+E(I)E(I_1[I_1-1]) + 2E(I)E(I_1)b] + \lambda_2 \sum_{n=1}^{b}\sum_{k=0}^{a-1} C_n Q_{0,k}(b(b-1)$$
$$-(n+k)(n+k-1)) - \lambda_2 C_b \sum_{k=1}^{a-1} Q_{0,k}(b(b-1) - (k+b)(k+b-1))$$
$$-\lambda_2 C_{b-1}\sum_{k=2}^{a-1} Q_{0,k}(b(b-1) - (k+b-1)(k+b-2)) - \ldots - \lambda_2 C_{b+1} Q_{0,a-1}$$
$$-b(b-1)\lambda_2 \sum_{k=0}^{a-1} Q_{0,k}\} + \sum_{k=0}^{b}(b(b-1) - k(k-1))(P_{0,k}^{+} + q_{0,k}^{+})\}$$
$$-\{b(b-1) - (\lambda_2 E(I))^2 E[B_2^2] - E[B_2]\lambda_2 E(I[I-1])\}\{d\{-\lambda_1 E(I)$$
$$E(I_1)\sum_{k=0}^{a-1} Q_{0,k} + \lambda_2 \sum_{n=1}^{b}\sum_{k=0}^{a-1} C_n Q_{0,k}(b-n-k) + \lambda_2 C_b \sum_{k=1}^{a-1} Q_{0,k} k$$
$$-\lambda_2 C_{b-1}\sum_{k=2}^{a-1} Q_{0,k}(1-k) - \ldots - \lambda_2 C_{b+1} Q_{0,a-1} + \lambda_2 E(I)\sum_{k=0}^{a-1} Q_{0,k}$$
$$+\lambda_2 \sum_{k=0}^{a-1} Q_{0,k} k - \lambda_2 \sum_{k=0}^{a-1} Q_{0,k} b\} + \sum_{k=0}^{b}(b-k)(P_{0,k}^{+} + q_{0,k}^{+})\},$$

$$q_0^{+'}(1) = \frac{N_9}{2\{b - E[B_2]E(I)\lambda_2\}^2},$$

$$N_9 = \{b - E[B_2]E(I)\lambda_2\}\{d\{-\sum_{k=0}^{a-1} Q_{0,k}\lambda_1[E(I[I-1])(E(I_1))^2$$

$$+E(I)E(I_1[I_1-1])+2E(I)E(I_1)b]+\lambda_2\sum_{n=1}^{b}\sum_{k=0}^{a-1}C_nQ_{0,k}(b(b-1)$$

$$-\lambda_2C_{b-1}\sum_{k=2}^{a-1}Q_{0,k}(b(b-1)-(k+b-1)(k+b-2))-...-\lambda_2C_{b+1}Q_{0,a-1}$$

$$(b(b-1)-(b-1))+\lambda_2[E(I[I-1])+2E(I)k+k(k-1)]\sum_{k=0}^{a-1}Q_{0,k}$$

$$-b(b-1)\lambda_2\sum_{k=0}^{a-1}Q_{0,k}\}+\sum_{k=0}^{b}(b(b-1)-k(k-1))(P_{0,k}^{+}+q_{0,k}^{+})+2\{d\{-\lambda_1E(I)$$

$$E(I_1)\sum_{k=0}^{a-1}Q_{0,k}+\lambda_2\sum_{n=1}^{b}\sum_{k=0}^{a-1}C_nQ_{0,k}(b-n-k)+\lambda_2C_b\sum_{k=1}^{a-1}Q_{0,k}k$$

$$-\lambda_2C_{b-1}\sum_{k=2}^{a-1}Q_{0,k}(1-k)-...-\lambda_2C_{b+1}Q_{0,a-1}+\lambda_2E(I)\sum_{k=0}^{a-1}Q_{0,k}$$

$$+\lambda_2\sum_{k=0}^{a-1}Q_{0,k}k-\lambda_2\sum_{k=0}^{a-1}Q_{0,k}b\}+\sum_{k=0}^{b}(b-k)(P_{0,k}^{+}+q_{0,k}^{+})\}E[B_2]E(I)\lambda_2\}$$

$$-\{b(b-1)-(\lambda_2E(I))^2E[B_2^2]-E[B_2]\lambda_2E(I[I-1])\}\{d\{-\lambda_1E(I)$$

$$E(I_1)\sum_{k=0}^{a-1}Q_{0,k}+\lambda_2\sum_{n=1}^{b}\sum_{k=0}^{a-1}C_nQ_{0,k}(b-n-k)+\lambda_2C_b\sum_{k=1}^{a-1}Q_{0,k}k$$

$$-\lambda_2C_{b-1}\sum_{k=2}^{a-1}Q_{0,k}(1-k)-...-\lambda_2C_{b+1}Q_{0,a-1}+\lambda_2E(I)\sum_{k=0}^{a-1}Q_{0,k}$$

$$+\lambda_2\sum_{k=0}^{a-1}Q_{0,k}k-\lambda_2\sum_{k=0}^{a-1}Q_{0,k}b\}+\sum_{k=0}^{b}(b-k)(P_{0,k}^{+}+q_{0,k}^{+})\}.$$

10. THE AVERAGE WAITING TIME IN THE QUEUE

By using Little's formula, the average waiting tim of a customers in the high priority queue is

$$W_{q_1}=\frac{L_{q_1}}{\lambda_1} \tag{99}$$

and average waiting time of a customer in the non-priority queue is

$$W_{q_2}=\frac{L_{q_2}}{\lambda_2} \tag{100}$$

where L_{q_1} and L_{q_2} have been found in equations (97) and (98).

11. PARTICULAR CASES

Case I:

If there are no priority arrival and no batch arrivals for low priority customers that is, $\lambda_1=0$ and $C(z_2)=z$, then our model is reduced to $M/G^{(a,b)}/1$ queueing systems. Now,

$$L_{q_2}=\frac{\mu(1-\sum_{i=0}^{a-1}Q_i)}{2\lambda_2}\{(\lambda_2)^2E[B_2^2]q^{+}(1)+2E[B_2]\lambda_2q^{+'}(1)-2E[B_2]q^{+}(1)(\lambda_2)^2$$

$$E[B_2]\} + [\frac{\mu}{2\lambda_2}](1-\sum_{i=0}^{a-1} Q_i)\sum_{n=0}^{a-1}(a(a-1)-n(n-1))\{q_n^+\}$$

where

$$q^+(1) = \frac{\sum_{k=0}^{b}(b-k)q_k^+)}{b-E[B_2]\lambda_2},$$

$$q^{+'}(1) = \frac{\left\{ \begin{array}{l} b-E[B_2]\lambda_2\}\{\sum_{k=0}^{b}[b(b-1)-k(k-1)]q_k^+) + 2\sum_{k=0}^{b}(b-k)q_k^+)\} \\ \times \quad E[B_2]\lambda_2 - \sum_{k=0}^{b}(b-k)q_k^+)\{b(b-1)-\lambda_2)^2 E[B_2^2]\end{array} \right\}}{2\{b-E[B_2]\lambda_2\}^2}$$

this result is coincide with Holman. D.F. (1981) .

Case II:
If there are no priority arrival, no bulk service and no batch arrivals for low priority customers that is, $\lambda_1 = 0$, $a = b = 1$ and $C(z_2) = z$, then our model is reduced to $M/G/1$ queueing systems. Now,

$$L_{q_2} = \frac{\mu(1-Q)}{2\lambda_2}\{(\lambda_2)^2 E[B_2^2]q^+(1) + 2E[B_2]\lambda_2 q^{+'}(1) - 2E[B_2]q^+(1)(\lambda_2)^2 \; E[B_2]\}$$

where

$$q^+(1) = 1, q^{+'}(1) = \{\frac{\{2E[B_2]\lambda_2 q_0^+ + (\lambda_2)^2 E[B_2^2]q_0^+}{2\{1-E[B_2]\lambda_2\}}\}$$

this result is coincide with Medhi.J (1994) .

12. NUMERICAL RESULTS

The above queueing model is analysed numerically with the following assumptions. We consider the single arrival and single service for both priority and low-priority customers. Service time for both high-priority and low-priority customers follows the exponential and Erlang-2 distributions that is,

$$C(z_1) = z_1, C(z_2) = z_2, a = b = 1, E(I_1) = \frac{\lambda_2}{\mu-\lambda_1}, E(I_1[I_1-1]) = \{\frac{2\lambda_2\mu}{(\mu-\lambda_1)^3}\}.$$

We assume arbitrary values to the parameters such that the stability condition is satisfied. MATLAB software has been used to illustrate the results numerically. Note that the exponential distribution is $f(x) = ve^{-vx}, x > 0$, Erlang-2 stage distribution is $f(x) = v^2xe^{-vx}, x > 0$.

In table 1 and 2 shows that increasing the arrival rate of high priority customers decrease the idle time and also increase the busy period and queue lengths for the values of λ_2 =0.8, λ_1 =1.2,1.4,1.6,1.8,2. $\mu = 6$

In table 3 and 4 shows that increasing the arrival rate of low-priority customers decrease the idle time and queue length of high priority customers, and also increase the busy period, queue length of low-priority customers for the values of λ_1 =1, λ_2 =1.2,1.4,1.6,1.8,2. $\mu = 6$

All the trends shown by this tables are as expected.

Results are presented for the values of λ_1 and λ_2 in the following tables with their corresponding graphical representations of the system performance measures of both exponential and Erlangian-2 distributions.

Table 1: Effect of λ_1 on various queue characteristics

Exponential Distribution				
λ_1	Q	ρ	L_{q_1}	L_{q_2}
1.2	0.6667	0.3333	0.7528	0.2840
1.4	0.6333	0.3667	0.7673	0.3229
1.6	0.6000	0.4000	0.7997	0.3593
1.8	0.5667	0.4333	0.8546	0.3921
2.0	0.5333	0.4667	0.9374	0.4199

Table 2: Effect of λ_1 on various queue characteristics

Erlang-2 stage Distribution				
λ_1	Q	ρ	L_{q_1}	L_{q_2}
1.2	0.6667	0.3333	0.7406	0.2606
1.4	0.6333	0.3667	0.7467	0.2988
1.6	0.6000	0.4000	0.7670	0.3344
1.8	0.5667	0.4333	0.8048	0.3662
2.0	0.5333	0.4667	0.8642	0.3930

Table 3: Effect of λ_2 on various queue characteristics

	Exponential Distribution			
λ_2	Q	ρ	L_{q_1}	L_{q_2}
1.2	0.6333	0.3667	0.6892	0.7460
1.4	0.6000	0.4000	0.6568	1.1473
1.6	0.5667	0.4333	0.6238	1.6643
1.8	0.5333	0.4667	0.5902	2.3039
2.0	0.5000	0.5000	0.5559	3.0681

Table 4: Effect of λ_2 on various queue characteristics

Erlang-2 stage Distribution				
λ_2	Q	ρ	L_{q_1}	L_{q_2}
1.2	0.5500	0.4500	0.5802	0.5306
1.4	0.5000	0.5000	0.5290	0.8055
1.6	0.4500	0.5500	0.4772	1.1369
1.8	0.4000	0.6000	0.4249	1.5073
2.0	0.3500	0.6500	0.3721	1.8873

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