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FIXED POINT THEOREMS IN G-METRIC SPACES USING Φ-MAPS

Pardeep Kumar*, Nawneet Hooda, Pankaj Kumar*****

Author Affiliation:
*Government College for Girls, Sec 14, Gurgaon, India
**Department of Mathematics, DCR University of Science & Technology, Murthal, India
***Department of Mathematics,GJ University of Science & Technology, Hisar, India

Corresponding Author:
Nawneet Hooda, Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonipat, Haryana 131039, India

E-mail: nawneethooda@gmail.com

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Abstract

In this paper, we prove some fixed point theorems in G-metric spaces for contraction mappings. Our results generalize some well known results in literature.

Keywords and Phrases: Weakly compatible mappings, Φ-maps, fixed point.

Mathematics Subject Classification. 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

The fixed point theorems in metric spaces are playing a major role to construct methods in mathematics to solve problems in applied mathematics and sciences. The study of common fixed points of mappings satisfying various contractive conditions has been the centre of rigorous research activity.

Mustafa and Sims [4] introduced the concept of G-metric space as a generalization of metric space.

Definition 1.1: Let X be a nonempty set and $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality).

The function G is called a generalized metric or a G – metric on X , and the pair (X, G) is called a G - metric space.

Definition 1.2: Let (X, G) be a G – metric space, $\{x_n\}$ a sequence of points in X. We say that $\{x_n\}$ is G – convergent to x in X if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$; i.e., for any $\in > 0$ there exists an N such that $G(x, x_n, x_m) < \in$ for all m, n ≥ N and a G-Cauchy sequence if , for any $\in > 0$, there exists an N such that $G(x_n, x_m, x_l) < \in$ for all n, m , $l \geq N$.

Proposition 1.3: Let (X, G) be a G – metric space and $\{x_n\}$be a sequence in X. Then the following are equivalent:

(i) $\{x_n\}$ is G -convergent to x,
(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Proposition 1.4: In a G – metric space (X, G) the following are equivalent:

(i) The sequence $\{x_n\}$ is G – Cauchy,
(ii) for each $\in > 0$ there exists an N such that $G(x_n, x_m, x_m) < \in$ for all $n, m \geq N$.

A G-metric space X is said to complete if every G-Cauchy sequence in X is a G-convergent sequence in X.

Proposition 1.5: Let (X, G) be a G – metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.

Definition 1.6: A G – metric space (X, G) is called a symmetric G – metric space if

$$G(x, y, y) = G(y, x, x) \text{ for all } x, y \text{ in } X.$$

Definition 1.7: Let f and g be single-valued self -mappings on a set X. If w = fx = gx for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Abbas and Rhoades [1] proved the existence of the unique common fixed points of a pair of weakly compatible mappings in G-metric spaces.

Proposition 1.8: ([1, Proposition 1.5]) Let f and g be weakly compatible self-mappings on a set X . If f and g have a unique point of coincidence w = fx = gx, then w is the unique common fixed point of f and g.

Matkowski [3] introduced the Φ-map as :

Definition 1.9: Let Φ be theset of all function φ such that $\varphi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function satisfying $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\varphi \in \Phi$, then φ is called a Φ-map. Furthermore, if φ is a Φ-map, then

(i) $\varphi(t) < t$ for all $t \in (0, +\infty)$,
(ii) $\varphi(0) = 0$.

2. MAIN RESULTS

Now we prove our main results in G-metric space using Φ-map.

Theorem 2.1: Let f and g be self -maps of a G-metric space (X, G) satisfying

$$f(X) \subseteq g(X) \tag{2.1}$$

where G(X) is complete subspace of X , and

$$G(fx, fy, fz) \leq \varphi\left(\max\left\{\begin{matrix} G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy), G(gy, fx, fx) \\ G(fx, gy, gz), G(gy, fz, fz), G(gz, fz, fz), G(gz, fx, fx), G(gz, fy, fy) \end{matrix}\right\}\right), \tag{2.2}$$

for all $x, y, z \in X$;

Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof.: Let f and g satisfy (2.1) and x_0 be any arbitrary point in X. Then one can choose a point x_1 in X such that $fx_0 = gx_1$ and in general x_{n+1} such that $fx_n = gx_{n+1}$, for all $n \in \mathbb{N}$.
If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a point of coincidence.
Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, by (2.2) we obtain that

$$G(gx_n, gx_{n+1}, gx_{n+1}) = G(fx_{n-1}, fx_n, fx_n)$$
$$\leq \varphi\left(\max\left\{\begin{array}{c} G(gx_{n-1}, gx_n, gx_n), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_n, gx_n, gx_n) \\ G(gx_n, gx_n, gx_n), G(gx_n, gx_{n+1}, gx_{n+1}), \\ G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_n, gx_n, gx_n), G(gx_n, gx_{n+1}, gx_{n+1}) \end{array}\right\}\right)$$

$$= \varphi(\max\{G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n-1}, gx_n, gx_n), 0\})$$
$$= \varphi(\max\{G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n-1}, gx_n, gx_n)\}).$$

If $\max\{G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n-1}, gx_n, gx_n)\} = G(gx_n, gx_{n+1}, gx_{n+1})$ then $G(gx_n, gx_{n+1}, gx_{n+1}) \leq \varphi(G(gx_n, gx_{n+1}, gx_{n+1})) < G(gx_n, gx_{n+1}, gx_{n+1})$ which leads to a contradiction.
This implies that $G(gx_n, gx_{n+1}, gx_{n+1}) \leq \varphi(G(gx_{n-1}, gx_n, gx_n))$.
That is, for each $n \in \mathbb{N}$, we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) = G(fx_{n-1}, fx_n, fx_n)$$
$$\leq \varphi(G(gx_{n-1}, gx_n, gx_n))$$
$$\leq \varphi^2(G(gx_{n-2}, gx_{n-1}, gx_{n-1}))$$
$$\vdots$$
$$\leq \varphi^n(G(gx_0, gx_1, gx_1)).$$

Since $\lim_{n\to\infty} \varphi^n\big(G(gx_0, gx_1, gx_1)\big) = 0$ and $\varphi(\varepsilon) < \varepsilon$, there exists $N \in \mathbb{N}$ such that

$$\varphi^n\big(G(gx_0, gx_1, gx_1)\big) < \varepsilon - \varphi(\varepsilon) \text{ for all } n \geq N.$$

This implies that $G(gx_n, gx_{n+1}, gx_{n+1}) < \varepsilon - \varphi(\varepsilon)$ for all $n \geq N$. (2.3)
Now we will show that $\{gx_n\}$ is G-Cauchy. Let $\varepsilon > 0$ be any number.
Let $m, n \in \mathbb{N}$ with $m > n$.

We will prove that $G(gx_n, gx_m, gx_m) < \varepsilon$ for all $m \geq n \geq N$ by induction on m. (2.4)
Since $\varepsilon - \varphi(\varepsilon) < \varepsilon$ and by (2.3), we obtain that (2.4) holds for $m = n+1$.
Suppose that (2.4) holds for $m = k$. Therefore, for $m = k+1$, we have by rectangle inequality

$$G(gx_n, gx_{k+1}, gx_{k+1}) \leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{k+1}, gx_{k+1})$$
$$< \varepsilon - \varphi(\varepsilon) + G(fx_n, fx_k, fx_k)$$
$$\leq \varepsilon - \varphi(\varepsilon) +$$
$$\varphi\left(\max\left\{\begin{array}{c} G(gx_n, gx_k, gx_k), G(gx_n, fx_n, fx_n), G(gx_k, fx_k, fx_k), G(gx_k, fx_n, fx_n) \\ G(fx_n, gx_k, gx_k), G(gx_k, fx_k, fx_k), G(gx_k, fx_k, fx_k), G(gx_k, fx_n, fx_n), G(gx_k, fx_k, fx_k) \end{array}\right\}\right)$$
$$= \varepsilon - \varphi(\varepsilon) + \varphi\left(\max\left\{\begin{array}{c} \varepsilon, \varepsilon, \varepsilon, \varepsilon, \\ \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon \end{array}\right\}\right)$$
$$= \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon.$$

Thus (2.4) holds for all $m \geq n \geq N$. It follows that $\{gx_n\}$ is G-Cauchy and by the completeness of g(X), we obtain that $\{gx_n\}$ is G-convergent to some $q \in g(X)$, so there exists $p \in X$ such that $gp = q$.

We will show that $gp = fp$. By (2.2), we have

$$G(gp, gp, fp) \leq G(gp, gp, gx_n) + G(gx_n, gx_n, fp)$$
$$= G(gp, gp, gx_n) + G(fx_{n-1}, fx_{n-1}, fp)$$
$$\leq G(gp, gp, gx_n) +$$
$$\varphi\left(\max\left\{\begin{array}{c} G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, fx_{n-1}, fx_{n-1}), G(gx_{n-1}, fx_{n-1}, fx_{n-1}), \\ G(gx_{n-1}, fx_{n-1}, fx_{n-1}), G(fx_{n-1}, gx_{n-1}, fp), G(gx_{n-1}, fp, fp), \\ G(gp, fp, fp), G(gp, fx_{n-1}, fx_{n-1}), G(gp, fx_{n-1}, fx_{n-1}) \end{array}\right\}\right)$$
$$= G(gp, gp, gx_n) + \varphi\left(\max\left\{\begin{array}{c} G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n-1}, fp), G(gx_{n-1}, fp, fp), \\ G(gp, fp, fp), G(gp, gx_n, gx_n), G(gp, gx_n, gx_n) \end{array}\right\}\right)$$
$$= G(gp, gp, gx_n) + \varphi\left(\max\left\{\begin{array}{c} G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_n, gx_{n-1}, fp), G(gx_{n-1}, fp, fp), \\ G(gp, fp, fp), G(gp, gx_n, gx_n) \end{array}\right\}\right)$$

Case 1. If $\max\left\{\begin{matrix} G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_n, gx_{n-1}, fp), G(gx_{n-1}, fp, fp), \\ G(gp, fp, fp), G(gp, gx_n, gx_n) \end{matrix}\right\} = G(gx_{n-1}, gx_{n-1}, gp)$, we obtain that

$$G(gp, gp, fp) \le G(gp, gp, gx_n) + \varphi(G(gx_{n-1}, gx_{n-1}, gp)) < G(gp, gp, gx_n) + G(gx_{n-1}, gx_{n-1}, gp).$$

By taking $n \to \infty$, we have G(gp, gp, fp) = 0 and so gp = fp.

Case 2. If $\max\left\{\begin{matrix} G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_n, gx_{n-1}, fp), G(gx_{n-1}, fp, fp), \\ G(gp, fp, fp), G(gp, gx_n, gx_n) \end{matrix}\right\} = G(gx_{n-1}, gx_n, gx_n)$, we obtain that

$$G(gp, gp, fp) \le G(gp, gp, gx_n) + \varphi(G(gx_{n-1}, gx_n, gx_n)) < G(gp, gp, gx_n) + G(gx_{n-1}, gx_n, gx_n).$$

By taking $n \to \infty$, we have G(gp, gp, fp) = 0 and so gp = fp.

Case 3. If $\max\left\{\begin{matrix} G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_n, gx_{n-1}, fp), G(gx_{n-1}, fp, fp), \\ G(gp, fp, fp), G(gp, gx_n, gx_n) \end{matrix}\right\} = G(gx_n, gx_{n-1}, fp)$, we obtain that

$$G(gp, gp, fp) \le G(gp, gp, gx_n) + \varphi(G(gx_n, gx_{n-1}, fp)) < G(gp, gp, gx_n) + G(gx_n, gx_{n-1}, fp).$$

By taking $n \to \infty$, we have G(gp, gp, fp) = 0 and so gp = fp.

Case 4. If $\max\left\{\begin{matrix} G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_n, gx_{n-1}, fp), G(gx_{n-1}, fp, fp), \\ G(gp, fp, fp), G(gp, gx_n, gx_n) \end{matrix}\right\} = G(gx_{n-1}, fp, fp)$, we obtain that

$$G(gp, gp, fp) \le G(gp, gp, gx_n) + \varphi(G(gx_{n-1}, fp, fp)) < G(gp, gp, gx_n) + G(gx_{n-1}, fp, fp).$$

By taking $n \to \infty$, we have G(gp, gp, fp) < 0 + G(gp, fp, fp). But by (G3) $G(gp, fp, fp) < G(gp, gp, fp)$ and so gp = fp.

Case 5. If $\max\left\{\begin{matrix} G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_n, gx_{n-1}, fp), G(gx_{n-1}, fp, fp), \\ G(gp, fp, fp), G(gp, gx_n, gx_n) \end{matrix}\right\} = G(gp, fp, fp)$, we obtain that

$$G(gp, gp, fp) \le G(gp, gp, gx_n) + \varphi(G(gp, fp, fp)) < G(gp, gp, gx_n) + G(gp, fp, fp).$$

By taking $n \to \infty$, we have G(gp, gp, fp) < 0 + G(gp, fp, fp), a contradiction.

Case 6. If $\max\left\{\begin{matrix} G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), \\ G(gx_n, gx_{n-1}, fp), G(gx_{n-1}, fp, fp), \\ G(gp, fp, fp), G(gp, gx_n, gx_n) \end{matrix}\right\} = G(gp, gx_n, gx_n)$, we obtain that

$$G(gp, gp, fp) \le G(gp, gp, gx_n) + \varphi(G(gp, gx_n, gx_n)) < G(gp, gp, gx_n) + G(gp, gx_n, gx_n)).$$

By taking $n \to \infty$, we have G(gp, fp, fp) = 0 and so gp = fp.
Now we show that f and g have a unique point of coincidence. Suppose that fq = gq for some q ∈ X. Assume that gp ≠ gq.
By applying (2.2), it follows that
G(gp, gp, gq) = G(fp, fp, fq)

$$\le \varphi\left(\max\left\{\begin{matrix} G(gp, gp, gq), G(gp, fp, fp), G(gp, fp, fp), \\ G(gp, fp, fp), G(fp, gp, gq), G(gp, fq, fq), G(gq, fq, fq), G(gq, fp, fp), G(gq, fp, fp) \end{matrix}\right\}\right)$$

$$\le \varphi\left(\max\left\{\begin{matrix} G(gp, gp, gq), 0, 0, 0, \\ G(fp, gp, gq), G(gp, fq, fq), G(gq, fq, fq) \end{matrix}\right\}\right)$$

$$= \varphi\left(\max\left\{\begin{matrix} G(gp, gp, gq), G(fp, gp, gq), \\ G(gp, fq, fq), G(gq, fq, fq) \end{matrix}\right\}\right)$$

$$\le \varphi\left(\max\left\{\begin{matrix} G(gp, gp, gq) \\ G(gp, gq, gq) \end{matrix}\right\}\right)$$

$$\le \varphi(G(gp, gp, gq))$$

$< G(gp, gp, gq)$ which leads to a contradiction.

Therefore gp = gq.

This implies that f and g have a unique point of coincidence. By Proposition 1.8 , we can conclude that f and g have a unique common fixed point.

Our result is of more general nature than that of Anchalee Kaewcharoen's following theorem:

Corollary 2.2: [2:Theorem 2.6]. Let f and g be self -maps of a G-metric space (X, G) satisfying

$f(X) \subseteq g(X)$ where G(X) is complete subspace of X , (2.5)

$$G(fx, fy, fz) \leq \varphi(\max\{G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy), G(fx, gy, gz)\}), \quad (2.6)$$

for all x, y, z ∈ X;

Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

If we take g as the identity function on X in corollary (2.2), we have the following corollary:

Corollary 2.3: [5] Let (X, G) be a complete G-metric space. Suppose that the function f: X → X satisfies

$$G(fx,fy,fz) \leq \varphi(max\{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(fx, y, z)\}) \quad (2.7)$$

for all x, y, z ∈ X. Then f has a unique fixed point.

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