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TERMINATION CRITERION AND ERROR ANALYSIS OF A MIXED RULE USING AN ANTI-GAUSSIAN RULE IN WHOLE INTERVAL AND ADAPTIVE ALGORITHM

Bibhu Prasad Singh*, Dr. Rajani Ballav Dash**

Author Affiliation:
* Institute of Mathematics and Application, Andharua, Bhubaneswar, Odisha, India
E-mail:bpsmath78@gmail.com
** Ex Reader in Mathematics, Ravenshaw University, Cuttack, Odisha, India.
E-mail: rajani_bdash@rediffmail.com

Corresponding Author:
Bibhu Prasad Singh, Institute of Mathematics and Application, Andharua, Bhubaneswar, Odisha 751003, India.

E-mail: bpsmath78@gmail.com

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Abstract

A mixed quadrature rule of higher precision for approximate evaluation of real definite integrals has been constructed using an anti-Gaussian rule. The analytical convergence of the rule has been studied. The error bounds have been determined asymptotically. In adaptive quadrature routines not before mixed quadrature rules basing on anti-Gaussian quadrature rule have been used for fixing termination criterion. Adaptive quadrature routines being recursive by nature, a termination criterion is formed taking in to account a mixed quadrature rule. The algorithm presented in this paper and successfully tested on different integrals by C program. The relative efficiency of the mixed quadrature rule is reflected in the table at the end.

Keywords: Anti-Gaussian rule, Gaussian rule, Boole's rule, mixed rule, adaptive algorithm, error analysis, termination criterion

2000 Mathematics Subject Classification: 65D30, 65D32

1. INTRODUCTION

The Concept of mixed quadrature was first coined by R.N Das and G.Pradhan [15]. The method of mixing quadrature rules is based on forming a mixed quadrature rule of higher precision by taking linear/convex combination of two quadrature rules of lower precision. Though in literature we find precision enhancement through Richardson Extrapolation and Kronrod extension [11,17,18] taking respectively trapezoidal rule and Gaussian quadrature as base rules, these methods are quite cumbersome. On the other hand, the precision enhancement through mixed quadrature method is very simple and easy to handle. Authors [14-16] have also developed mixed quadrature rules for approximate evaluation of the integrals of analytic functions following F. Lether [10].

So far in this paper in which an anti-Gaussian quadrature rule has been used to construct a mixed quadrature rule.

Dirk P. Laurie [1-3, 5] is first to coin the idea of anti-Gaussian quadrature formula. An anti-Gaussian quadrature formula is an (n+1) point formula of degree (2n-1) which integrates all polynomials of degree up to (2n+1) with an error equal in magnitude but opposite in sign to that of n-point Gaussian formula.

If $H^{(n+1)}(p)=\sum_{i=1}^{(n+1)}\lambda_i\, f(\xi_i)$ be (n+1) point anti-Gaussian formula and $G^{(n)}(p)$ be n point Gaussian formula then by hypothesis,
$I(p)-H^{(n+1)}(p)$ = - $(I(p)-G^{(n)}(p))$, $p\in P_{2n+1}$ where p is a polynomial of degree $\leq 2n+1$..

In this paper we design a four point anti-Gaussian rule following Laurie. We mix this anti-Gaussian four point rule with Boole's five point rule.

The relative efficiency of the mixed rule has been shown by numerically evaluating some test integrals.

2. CONSTRUCTION OF ANTI-GAUSSIAN FOUR POINT RULE FROM GAUSS-LEGENDRE THREE POINT RULE

We choose the Gauss-Legendre three point rule,

$$G_w^3(f)=\frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right)+\frac{8}{9}f(0)+\frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)\qquad(1)$$

and develop a four point anti-Gaussian rule $RH_w^4(f)$ from three point Gaussian rule $G_w^3(f)$.
Using the principle $I(p)-RH_w^{n+1}(f)=-\left(I(p)-G_w^n(f)\right)$ as adopted in Laurie [1], we get

$$RH_w^4(f)=2\int_{-1}^{1}f(x)dx-\left(G_w^3(f)\right)\qquad(2)$$

$$\Rightarrow \alpha_1 f(\xi_1)+\alpha_2 f(\xi_2)+\alpha_3 f(\xi_3)+\alpha_4 f(\xi_4)=2\int_{-1}^{1}f(x)dx-\left(G_w^3(f)\right),\qquad(3)$$

Taking $RH_w^4(f)=\alpha_1 f(\xi_1)+\alpha_2 f(\xi_2)+\alpha_3 f(\xi_3)+\alpha_4 f(\xi_4)$..........................(4)

In order to obtain the unknown weights and nodes, we assume that
(i)The rule is exact for all polynomial of degree ≤ 4.
(ii)The rule integrates all polynomials of degree up to six with an error equal in magnitude and opposite in sign to that of Gaussian rule. Thus we obtain a system of eight equations having eight unknowns using

$$\alpha_i,\xi_i\ ,\ i=1,2,3,4$$

For $f(x)=x^i, i=0,1,2,3,4,5,6,7$.

Solving the systems of equation, we get

$$\alpha_1 = \frac{35(3+\sqrt{681})}{3\times\sqrt{681(39+\sqrt{681})}} = \alpha_4, \alpha_2 = \frac{35(\sqrt{681}-3)}{3\times\sqrt{681}(39+\sqrt{681})} = \alpha_3$$

$$\xi_1 = \sqrt{\frac{39+\sqrt{681}}{70}} = -\xi_4 \;,\xi_2 = \sqrt{\frac{39-\sqrt{681}}{70}} = -\xi_3$$

$$RH_w^4(f) = \frac{35(3+\sqrt{681})}{3\times\sqrt{681(39+\sqrt{681})}} f(\sqrt{\frac{39+\sqrt{681}}{70}}) + \frac{35(\sqrt{681}-3)}{3\times\sqrt{681}(39+\sqrt{681})} f(\sqrt{\frac{39-\sqrt{681}}{70}})$$
$$+\frac{35(\sqrt{681}-3)}{3\sqrt{681}(39-\sqrt{681})} f(-\sqrt{\frac{39-\sqrt{681}}{70}}) + \frac{35(3+\sqrt{681})}{3\sqrt{681}(39+\sqrt{681})} f(-\sqrt{\frac{39+\sqrt{681}}{70}})$$

But the anti-Gaussian four point rules computed as

$$RH_w^4(f) = \alpha_1\{f(-\xi_1) + f(\xi_1)\} + \alpha_2\{f(-\xi_2) + f(\xi_2)\} \text{..................................(5)}$$

Hence, by Taylor series expansion, we have

$$RH_w^4(f) = 2(\alpha_1+\alpha_2)f(0) + 2(\alpha_1\xi_1^2 + \alpha_2\xi_2^2)\frac{f^{ii}(0)}{2!} + 2(\alpha_1\xi_1^4 + \alpha_2\xi_2^4)\frac{f^{iv}(0)}{4!}$$
$$+2(\alpha_1\xi_1^6 + \alpha_2\xi_2^6)\frac{f^{vi}(0)}{6!} + 2(\alpha_1\xi_1^8 + \alpha_2\xi_2^8)\frac{f^{viii}(0)}{8!} + \ldots$$

By putting the values of $\alpha_1, \alpha_2 and \xi_1, \xi_2$ in the above equation, we have

$$RH_w^4(f) = 2f(0) + 2\frac{f^{ii}(0)}{3!} + 2\frac{f^{iv}(0)}{5!} + 2\frac{29\times f^{vi}(0)}{6!\times 175} + 2\frac{297105136\times f^{viii}(0)}{8!} + \ldots$$

We have,

$$I(f) = \int_{-1}^{1} f(x)dx = 2f(0) + 2\frac{f^{ii}(0)}{3!} + 2\frac{f^{iv}(0)}{5!} + 2\frac{f^{vi}(0)}{7!} + 2\frac{f^{viii}(0)}{9!} + \ldots$$

The error of the anti-Gaussian four point rule is computed as

$$EH_w^4(f) = \int_{-1}^{1} f(x)dx - RH_w^4(f) = -\frac{2\times 28\times f^{vi}(0)}{7!\times 175} - \frac{2\times 2673946223\times f^{viii}(0)}{9!} + \ldots$$
$$EH_w^4(f) = I(f) - RH_w^4(f) = -\frac{f^{vi}(0)}{75\times 210} - \frac{5347892446\times f^{viii}(0)}{9!} + \text{...............(6)}$$

3. CONSTRUCTION OF MIXED QUADRATURE RULE BY USING ANTI-GAUSSIAN FOUR POINT RULE WITH BOOLE'S RULE

We have the Boole's rule,

$$BL_5(f) = \frac{1}{45}[7\{f(-1) + f(1)\} + 32\{f(-\frac{1}{2}) + f(\frac{1}{2})\} + 12f(0)]\text{............(7)}$$

Hence, by taylors series expansion, we have

$$BL_5(f) = 2f(0) + 2\frac{f^{ii}(0)}{3!} + 2\frac{f^{iv}(0)}{5!} + \frac{1\times f^{vi}(0)}{3\times 6!} + \frac{19\times f^{viii}(0)}{60\times 8!} + \text{..........} \quad \text{...(8)}$$

The error associated with Boole's rule is computed as

$$EBL_5(f) = I(f) - BL_5(f) = -\frac{1}{7!\times 3} f^{vi}(0) - \frac{51}{9!\times 60} f^{viii}(0) - \text{.........................(9)}$$

The error associated with the anti-Gaussian four point rule is

$$EH_w^4(f) = I(f) - RH_w^4(f) = -\frac{f^{vi}(0)}{75\times 210} - \frac{534789244\times f^{viii}(0)}{9!} + \text{...............(10)}$$

Eliminating $f^{vi}(0)$ from the equation (9) and (10), we have

$$(\frac{1}{72} - \frac{1}{75})I(f) = \frac{1}{72}RH_w^4(f) - \frac{1}{75}BL_5(f) + (\frac{51}{9!\times 75\times 60} - \frac{534789244}{9!\times 72}) f^{viii}(0) + \ldots$$

$$I(f) = [25RH_w^4(f) - 24BL_5(f)] + (25\times 72)(\frac{51}{9!\times 75\times 60} - \frac{534789244}{9!\times 72}) f^{viii}(0) + \ldots$$

$$I(f) = [25RH_w^4(f) - 24BL_5(f)] + [25EH_w^4(f) - 24EBL_5(f)]$$

$$I(f) = RH_w^4 BL_5(f) + EH_w^4 BL_5(f)$$

$$RH_w^4 BL_5(f) = [25RH_w^4(f) - 24BL_5(f).]\text{..(11)}$$

$$EH_w^4 BL_5(f) = [25EH_w^4(f) - 24EBL_5(f).]\text{..(12)}$$

$$RH_w^4(f) = 25[\frac{(3+\sqrt{681})\times 35}{3\times\sqrt{681}(39+\sqrt{681})} f(\sqrt{\frac{39+\sqrt{681}}{70}}) + \frac{35(\sqrt{681}-3)}{3\times\sqrt{681}(39+\sqrt{681})}$$

$$f(\sqrt{\frac{39-\sqrt{681}}{70}}) + \frac{35(\sqrt{681}-3)}{3\sqrt{681}(39-\sqrt{681})} f(-\sqrt{\frac{39-\sqrt{681}}{70}}) + \frac{35(3+\sqrt{681})}{3\sqrt{681}(39+\sqrt{681})}$$

$$f(-\sqrt{\frac{39+\sqrt{681}}{70}})] - \frac{24}{25}[7\{f(-1) + f(1)\} + 32\{f(-\frac{1}{2}) + f(\frac{1}{2})\} + 12f(0)]$$

This is the desired mixed quadrature rule of precision seven. For the approximate evaluation of $I(f)$. The truncation error generated in this approximation is given by.

$$EH_w^4 BL_5(f) = [25EH_w^4(f) - 24EBL_5(f).]\text{...(13)}$$

or

$$EH_w^4 BL_5(f) = (25\times 72)(\frac{51}{9!\times 75\times 60} - \frac{534789244}{9!\times 72}) f^{viii}(0) + \text{.........................(14)}$$

$$\left| EH_w^4 BL_5(f) \right| \le (25\times 72)(\frac{51}{9!\times 75\times 60} - \frac{534789244}{9!\times 72}) \left| f^{viii}(\eta) \right| + \text{................(15)}$$

$$-1 \langle \eta \langle 1$$

The rule $RH_w^4 BL_5(f)$ is called a mixed type rule of precision seven as it is constructed from two different types of the rules of the same precision (i.e. precision 5).

4. ERROR ANALYSIS

An asymptotic error estimate and an error bound of the rule (13) are given by.

Theorem 4.1:

Let $f(x)$ be sufficiently differentiable function in the closed interval $[-1,1]$. Then the error $EH_w^4BL_5(f)$ associated with the rule $RH_w^4BL_5(f)$ is given by

$$\left| EH_w^4BL_5(f) \right| \le \frac{50}{8!}\left(\frac{17}{375} - \frac{534789244}{18}\right)\left| f^{viii}(\eta) \right|, -1\langle\eta\langle 1$$

Proof: Theorem follows from equation (11) and (13)

We have $RH_w^4BL_5(f) = [25RH_w^4(f) - 24BL_5(f)]$

And the truncation error generated in this approximation is given by

Hence we have,

$$EH_w^4BL_5(f) = [25EH_w^4(f) - 24EBL_5(f).]$$

$$\left| EH_w^4BL_5(f) \right| \le \frac{50}{8!}\left(\frac{17}{375} - \frac{534789244}{18}\right)\left| f^{viii}(\eta) \right|, -1\langle\eta\langle 1$$

Theorem 4.2:

The bound of the truncation error

$EH_w^4BL_5(f) = I(f) - RH_w^4BL_5(f)$ is given by

$$\left|EH_w^4BL_5(f)\right| \le \frac{M}{630}\left|\eta_2 - \eta_1\right|, \eta_1, \eta_2 \in [-1,1]$$

Where $M = \max_{-1 \le x \le 1} \left| f^{vii}(x) \right|$

Proof: We have $EH_w^4(f) = -\frac{1}{75 \times 210} f^{vi}(\eta_1)$ …………………………………..(16)

and

$$EBL_5(f) = -\frac{1}{72 \times 210} f^{vi}(\eta_1) \quad \text{...(17)}$$

$$EH_w^4BL_5(f) = [25EH_w^4(f) - 24EBL_5(f)] \quad \text{...(18)}$$

Putting the values of equation (16) and (17) in equation (18) we have,

$$\left|EH_w^4BL_5(f)\right| \le \frac{1}{630}\left| f^{vi}(\eta_2) - f^{vi}(\eta_1) \right|, \eta_1, \eta_2 \in [-1,1]$$

$$= \frac{1}{630}\int_{\eta_1}^{\eta_2} f^{vii}(x)dx \text{, where } \eta_1, \eta_2 \in [-1,1]$$

$$\le \frac{M}{630}\left|\eta_2 - \eta_1\right|$$

Where $M = \max_{-1 \le x \le 1} f^{vii}(x)$

Which gives a theoretical error bound as η_1, η_2 are unknown points in $[-1,1]$. From this theorem it is clear that the error in approximation will be less if points are η_1, η_2 closer to each other.

Corollary 1: The error bound for the truncation error $EH_w^4 BL_5(f)$ is given by

$$\left| EH_w^4 BL_5(f) \right| \leq \frac{2M}{630}$$

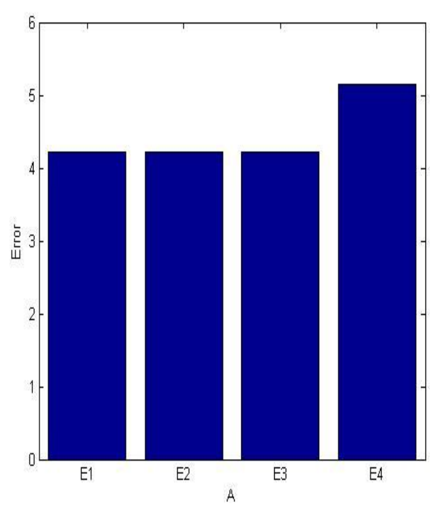
Proof: The proof follows from theorem (4.2) and $|\eta_1 - \eta_2| \leq 2.$

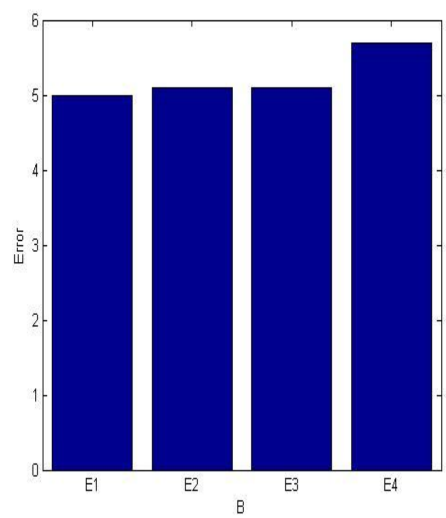
5. NUMERICAL VERIFICATION BY TABLE AND GRAPHS

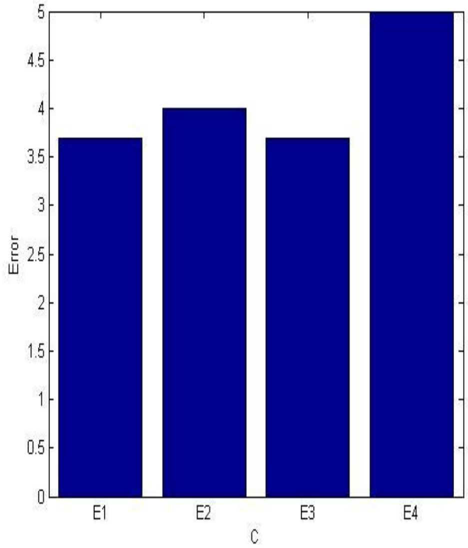
Table - 1:

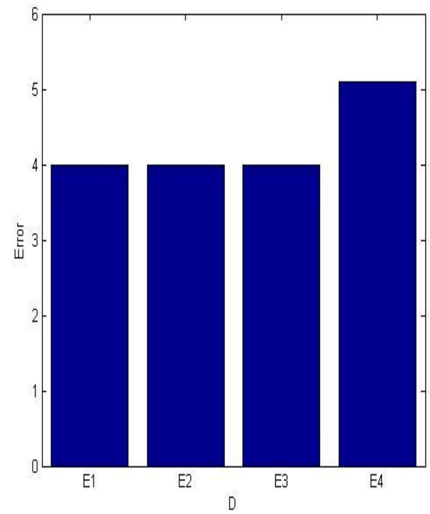
Sl No	Integrals	Exact Value	$G_w^3(f)$/ |E1|	$RH_w^4(f)$ / |E2|	$BL_5(f)$/ |E3|	$RH_w^4 BL_5(f)$/ | E4 |
1	$I_1 = \int_{-1}^{1} e^x dx$	2.350402387	2.350333692/ 0.000068	2.3504678/ 0.000065	2.3504709/ 0.000068	2.35039464/ 0.0000077
2	$I_2 = \int_0^1 e^{-x^2} dx$	0.746825	0.74681458/ 0.0000104	0.74683367/ 0.0000086	0.746833709/ 0.0000087	0.746832809/ 0.0000028
3	$I_3 = \int_0^1 e^{x^2} dx$	1.4627	1.4624097/ 0.00029	1.46289391/ 0.00019	1.46290943/ 0.000209	1.46252134/ 0.0000178
4	$I_4 = \int_1^3 \left(\frac{\sin^2 x}{x} \right) dx$	0.7948251	0.79465267/ 0.000172	0.79499761/ 0.000172	0.795001365/ 0.000176	0.79480762/ 0.0000082
5	$I_5 = \int_0^1 \sqrt{x} dx$	0.666666	0.669174/ 0.002508	0.66429729/ 0.002369	0.6577566/ 0.008909	0.82127385/ 0.000627

Where $E_1 = | I(f) - G_w^3(f) |$, $E_2 = | I(f) - RH_w^4(f) |$,
$E_3 = | I(f) - BL_5(f) |$, $E_4 = | I(f) - RH_w^4 BL_5(f) |$ are errors of various rules.
The graphical representation of these errors is given below in figures: A, B, C, D.









Using the results of the table and the notations for the errors of different methods given above the table , four bar graphs for the errors of the mixed quadrature rule and its constituent rules have been constructed in figures A, B, C and D correspond to $I_1 = \int_{-1}^{1} e^x dx$, $I_2 = \int_0^1 e^{-x^2} dx$, $I_3 = \int_0^1 e^{x^2} dx$ and $I_4 = \int_1^3 (\frac{\sin^2 x}{x}) dx$
respectively.

In the four graphs ,the error names of the mixed quadrature rule and its constituent rules have been embedded along X-axis and the respective values of the errors depicting heights of the bars are given along Y-axis. The graphical representation of these errors is given above in figures: A, B, C, D. From the above four graphs the unit in Y-axis is:

$$1 = -\log 10^{-1}, 2 = -\log 10^{-2}, 3 = -\log 10^{-3}, 4 = -\log 10^{-4}, 5 = -\log 10^{-5}, 6 = -\log 10^{-6}.$$

Thus from the graphs, we conclude that larger the height of the bar the smaller is the error. Here we derived most significant result that our mixed rule is more accurate than its constituent rules.

6. ADAPTIVE QUADRATURE ALGORITHM

A simple Adaptive Strategy

Given a real integrable function f an interval $[a,b]$ and a prescribed tolerance ε, it is desired to compute an approximation P to the integral $I = \int_a^b f(x)dx$, So that $|P - I| \le \varepsilon$. This can be done following adaptive integration schemes developed in papers [4-7,9,12,13]. In adaptive integration, the points at which the integrand is evaluated are chosen in a way that depends on the nature of the integrand. The basic principle of adaptive quadrature routines is discussed in the following manner.

If c is any point between a and b then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

The idea is that if we can approximate each of the two integrals on the right to within a specified tolerance, then the sum gives us the desired result. If not we can recursively apply the adaptive property to each of the intervals $[a,c]$ and $[c,b]$. Adaptive subdivision of course has geometrical appeal. It seems intuitive that points should be concentrated in regions where the integrand is badly behaved. The whole interval rules can take no direct account of this.

In this paper we design an algorithm for numerical computation of integrals in the adaptive quadrature routines involving mixed rules. The literature of the mixed quadrature rule [9,14-16] involves construction of a symmetric quadrature rule of higher precision as a linear/convex combination of two other rules of equal lower precision.

Algorithm for Adaptive quadrature routines:

The input to this schemes is a, b, ε, n, f, the output $I \approx \int_a^b f(x)dx$ with the error hopefully less than ε , n is the number of intervals initially chosen. A Simple adaptive strategy is out lined in the following step algorithm.

Step - 1: An approximation I_1 to $I \approx \int_a^b f(x)dx$ is computed.

Step - 2: The interval is divided into pieces $[a,c]$ and $[c,b]$.

Where $c = \dfrac{a+b}{2}$ and then $I_2 = \int_a^c f(x)dx$ and

$I_3 = \int_c^b f(x)dx$ are computed.

Step - 3: $I_2 + I_3$ is computed with to I_1 estimate the error in $I_2 + I_3$.

Step - 4: If | estimated error $| \le \varepsilon/2$ (termination-criterion), then $I_2 + I_3$ is accepted as an approximation to $I \approx \int_a^b f(x)dx$. Otherwise the same procedure is applied to $[a,c]$ and $[c,b]$ **allowing each pieces a tolerance of** $\varepsilon/2$.

Table 2:

Sl No	Integrals	Exact Value	$G_w^3(f)$ No. of. step |Error|	$RH_w^4(f)$ No. of. step |Error|	$BL_5(f)$ No .of. step |Error|	$RH_w^4 BL_5(f)$ No .of. step |Error|	Prescribed tolerance
1	$I_1 = \int_{-1}^{1} e^x dx$	2.35040238	2.350402369 03 0.00000001	2.350402405 03 0.000000025	2.350402406 03 0.000000026	2.3504023871 03 0.00000000015	0.00001
2	$I_2 = \int_0^1 e^{-x^2} dx$	0.746825	0.74682413 03 0.0000008	0.74682413321 03 0.0000008	0.746824133 03 0.0000008	0.746824132 03 0.00000086	0.00001

3	$I_3 = \int_0^1 e^{x^2} dx$	1.4627	1.46265164 03 0.000048	1.46265184 03 0.000048	1.4626517 05 0.000048	1.462651742 03 0.000048	0.00001
4	$I_4 = \int_1^3 \left(\frac{\sin^2 x}{x} \right) dx$	0.7948251	0.79482515 03 0.00000005	0.794825203 03 0.0000001	0.7948252 03 0.0000001	0.794825181 03 0.00000008	0.00001
5	$I_5 = \int_0^1 \sqrt{x} dx$	0.666666	0.666666 13 0.0000024	0.666665 13 0.00000098	0.6666644 15 0.0000015	0.6666683 13 0.0000023	0.00001

Adaptive quadrature routines essentially consist of applying the mixed rule $RH_w^4BL_5(f)$ and its constituents rules $G_w^3(f)$, $RH_w^4(f)$ and $BL_5(f)$ are to each of the sub intervals covering until the termination criterion is satisfied. If the termination criterion is not satisfied on one or more the sub intervals, then those subintervals must be further sub divided and the entire process repeated. The result obtained by a shorter program in standard CPP which should be more transportable and efficient.

7. OBSERVATION

In whole interval routine from the Table-1 as well as from the bar graph it is observed that the absolute error corresponding to the mixed rule $RH_w^4BL_5(f)$ is lesser than those corresponding to its constituent rules $G_w^3(f), RH_w^4(f), BL_5(f)$ are compared and mixed rule is better than its constituent's rules, when the test integrals are evaluated. However when these rules are used in adaptive mode, table-2 depict that the mixed quadrature rule using anti-Gaussian rule give very good result and less number of steps than its constituent rules when tested on a number of integrals.

8. CONCLUSION

After observation one can smartly draw conclusion over the efficiency of the rule formed in this paper as follows:

(1) The mixed $RH_w^4BL_5(f)$ rule is more efficient than its constituent rules $G_w^3(f), RH_w^4(f), BL_5(f)$ and previously developed mixed rules.

(2) In this paper we have concentrated mainly on computation of definite integrals in the adaptive quadrature routines involving mixed quadrature rule. We observed that mixed quadrature rule so formed can be very well used for evaluating real definite integrals than its constituent rules in the adaptive quadrature routines.

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