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Parallel correspondences of bi-conservative surfaces in E^3 with fractional order derivative *

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Abstract In this paper, we study the bi-conservative property of parallel revolution surface in Euclidean 3-space E^3 using a new definition of fractional derivative known as the conformable fractional derivative, which is a natural extension of the usual derivative and its definition is the simplest and the most natural definition of a fractional derivative which satisfies formulas of derivative of the product and the quotient of two functions and which has a simpler chain rule depending just on the basic limit definition of the derivative (see, R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh. A new definition of fractional derivative, Journal of Computational and Aapplied Mathematics, 264, 65–70, 2014).

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1 Introduction

Fractional calculus has become lately a branch of pure mathematics with many application in physics and engineering [2–4]. The detailed properties of fractional derivatives may be found in [5,6]. Fractional derivatives have many definitions. Unlike the traditional ones, they are not local. Certain derivatives have certain advantages over other derivatives. However, they all have the non-locality attribute in common.

The definition of a local kind derivative known as the conformable fractional derivative was first presented by Khalil et al. [1] in 2014. The fact that this derivative meets a significant portion of the well-known features of the integer order derivatives is stated as a key factor in its introduction. Abdeljawad [7] conducted a thorough investigation of the recently developed conformable fractional calculus in 2015. Martynyuk provided a physical explanation of the conformable derivative in [8]. More than a hundred research articles are already published in recent years utilizing this derivative, like [9–13].

Comparing the conformable fractional derivative to the classical fractional derivatives reveals two benefits. Firstly, the definition of the conformable fractional derivative is natural and it satisfies most of the properties of the classical integral derivative, including the linearity, the Rolle's theorem, the

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mean value theorem, the product rule, the quotient rule, the power rule, the chain rule, and the vanishing derivatives for constant functions. Secondly, there are many advantages of using the conformable derivative in the modeling of various physical problems. This is because the differential equations involving the conformable fractional derivative are simpler to solve numerically than those involving the Riemann-Liouville or Caputo fractional derivatives. The conformable fractional derivative has actually already been used extensively by researchers in a wide range of domains, and numerous corresponding approaches are developed [14].

The conformable fractional derivative seems to be a natural extension of the usual derivative, and it coincides with the known fractional derivatives on polynomials (up to a constant multiple). Further, if $\alpha = 1$, the definition coincides with the classical definition of the first derivative. The conformable definition is the simplest and the most natural and efficient definition of fractional derivative of order $\alpha \in (0, 1]$. We should remark that the definition can be generalized to include any α. However, the case $\alpha \in (0, 1]$ is the most important one, and once it is established, the other cases are simple [1].

The conformable fractional derivative of f of order α introduced by Khalil et al. [1] is defined in (2.4). They then defined the fractional derivative of higher order (i.e., of order $\alpha > 1$). They also defined the fractional integral of order $0 < \alpha \le 1$ only. They also proved the product rule, and the fractional mean value theorem and solved some (conformable) fractional differential equations where the fractional exponential function $e^{\frac{t^\alpha}{\alpha}}$ played an important rule. Katugampola introduced in [15] the new derivative which is defined in (2.6). As a consequence of the above definitions, the authors in [1, 15] showed that the α derivatives obey the product rule and quotient rule and have results similar to Rolle's theorem and the mean value theorem in classical calculus.

These days, a lot of papers are being written on the study of bi-conservative surfaces. In the past ten years, numerous pleasing outcomes on these kinds have been achieved. The theory of bi-harmonic submanifolds gave rise to the study of bi-conservative submanifolds in recent years, which has established itself as a very promising and fascinating area of study. In the realm of differential geometry, the study of bi-conservative submanifolds is a fascinating and relatively new topic that is closely linked to the theory of bi-harmonic submanifolds [16, 17].

The study of bi-conservative submanifolds is derived from the theory of bi-harmonic submanifolds which has been of large interest in the last decade. The bi-conservative submanifolds were studied for the first time in 1995 by T. Hasanis and T. Valchos [18].

In [18], Hasanis and Vlachos classified all biconservative hypersurfaces in E^4. They called such hypersurfaces as H-hypersurfaces. After that, biconservative hypersurfaces and submanifolds in space forms have been studied by several geometers ([19–21], etc).

In this article we look into the potential of getting the conditions required for a revolution parallel surface to acquire the property of being bi-conservative in Euclidean 3-space E^3. The general conditions of this surface are established. By providing special examples and employing a new approach, we are able to resolve the equations in the applications section. We find the theoretical or numerical solutions to those equations. Then we convert these results into geometric shapes using Matlab v.18 and also create a comparison between the results of applications for the original surface and the parallel to it [22].

2 The basic concepts

In this section we define the revolution surfaces, the parallel surfaces and the relation between the Gaussian and mean curvatures to the surface and the parallel to it in Euclidean 3-space. We also define the conformable fractional derivative that we mentioned earlier in the introductory section.

Definition 2.1. [23, 24] We say M a revolution surface which is generated by a plane curve $w(u)$ when it is rotated around a straight line in the same plane.The parametrization of the plane curve is given by

$$w(u) = \Big(\phi(u), \psi(u)\Big). \tag{2.1}$$

Then the parametrization of the revolution surface is given by

$$M : X(u, \upsilon) = \Big(\phi(u)\ cos\ \upsilon, \phi(u)\ sin\ \upsilon, \psi(u)\Big),\ 0 < \upsilon < 2\pi,\ \phi(u) > 0. \tag{2.2}$$

The coefficients of the first and second fundamental forms of the surface M are given by

$$\left.\begin{array}{l} g_{11} = \psi'^2 + \phi'^2, \quad g_{22} = \phi^2, \quad g_{12} = 0, \\ h_{11} = \phi'\psi'' - \psi'\phi'', \quad h_{22} = \psi'\phi, \quad h_{12} = 0. \end{array}\right\} \tag{2.3}$$

It is convenient to assume that the rotating curve is parameterized by arc length, that is, that

$$\phi'^2 + \psi'^2 = 1. \tag{2.4}$$

The Gaussian curvature G is given by

$$G = -\frac{\psi'(\psi'\phi'' - \phi'\psi'')}{\phi} \tag{2.5}$$

and ϕ is always positive, it follows that the parabolic points are given by either $\psi' = 0$ (the tangent line to the generator curve is perpendicular to the axis of rotation) or, $\phi'\psi'' - \psi'\phi'' = 0$ (the curvature of the generator curve is zero). A point which satisfies both conditions is a planar point, since these conditions imply that $h_{11} = h_{12} = h_{22}$.
It is convenient to put the Gaussian curvature in still another form. By differentiating (2.4) we obtain $\phi'\phi'' = -\psi'\psi''$. Thus,

$$G = -\frac{\psi'(\psi'\phi'' - \phi'\psi'')}{\phi} = -\frac{\psi'^2\phi'' + \phi'^2\phi''}{\phi} = -\frac{\phi''}{\phi}(\psi'^2 + \phi'^2) = -\frac{\phi''}{\phi}. \tag{2.6}$$

The principal curvatures of a surface of revolution are given by

$$k_1 = \phi'\psi'' - \psi'\phi'', \; k_2 = \frac{\psi'}{\phi}, \tag{2.7}$$

hence, the mean curvature of such surface is

$$H = \frac{\psi' + \phi(\phi'\psi'' - \psi'\phi'')}{2\phi}. \tag{2.8}$$

Definition 2.2. [25] Let M be an oriented surface and let N be a unit normal vector field of M. A surface $\overline{M}$ is said to be parallel to M if there is a normal geodesic congruence between M and $\overline{M}$ such that the distance between M and $\overline{M}$ such that the distance between corresponding points is constant, i.e. for each $X \in M$ we have

$$\overline{M} : \overline{X}(u, v) = X(u, v) + cN(u, v), \tag{2.9}$$

where $c \neq 0$ is a real constant. We can say that M and $\overline{M}$ are parallel surfaces at distance c. The relation between the Gaussian and mean curvature $G, H, \overline{G}$ and $\overline{H}$ of M and $\overline{M}$ respectively are given by

$$\overline{G} = \frac{G}{\eta}, \tag{2.10}$$

$$\overline{H} = \frac{H - c\,G}{\eta}, \tag{2.11}$$

where $\eta = 1 - 2cH + c^2\,G \neq 0$.

Also, if k_1, k_2 and $\overline{k}_1$, $\overline{k}_2$ denote the principal curvatures of M and $\overline{M}$, respectively, then we have

$$\overline{k}_1 = \frac{k_1}{1 - ck_1}, \; \overline{k}_2 = \frac{k_2}{1 - ck_2}. \tag{2.12}$$

Definition 2.3. [26]. A surface $\overline{M}$ in Euclidean 3-space is bi-conservative if the mean curvature function $\overline{H}$ satisfies

$$\overline{A}\;(\mathrm{grad}\;\overline{\mathrm{H}}) = -\overline{\mathrm{H}}\;\mathrm{grad}\;\overline{\mathrm{H}}. \tag{2.13}$$

This condition can be split into two differential equations as follows

$$\overline{a}_{11}\;\overline{H}_u + \overline{a}_{12}\;\overline{H}_v + \overline{H}\;\overline{H}_u = 0, \tag{2.14}$$

$$\overline{a}_{21}\ \overline{H}_u + \overline{a}_{22}\ \overline{H}_v + \overline{H}\ \overline{H}_v = 0, \tag{2.15}$$

where $\overline{A} = (\ \overline{a}_{ij}\ ),\ i, j = 1, 2$ is given by

$$\left.\begin{array}{ll} \overline{a}_{11} = (\overline{h}_{11}\ \overline{g}_{22} - \overline{h}_{12}\ \overline{g}_{12})/\ \overline{g}, & \overline{a}_{12} = (\overline{h}_{12}\ \overline{g}_{11} - \overline{h}_{11}\ \overline{g}_{12})/\ \overline{g} \\ \overline{a}_{21} = (\overline{h}_{12}\ \overline{g}_{22} - \overline{h}_{22}\ \overline{g}_{12})/\ \overline{g}, & \overline{a}_{22} = (\overline{h}_{22}\ \overline{g}_{11} - \overline{h}_{12}\ \overline{g}_{12})/\ \overline{g}. \end{array}\right\} \tag{2.16}$$

Definition 2.4. [1] Given a function $f : [0, \infty) \to R$. Then the conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \text{ for all } t > 0, \alpha \in (0, 1).$$

We will, sometimes, write $f^\alpha(t)$ for $T_\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

One can easily show that T_α satisfies all the properties in the following theorem.

Theorem 2.5. *Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t > 0$. Then*
(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.
(2) $T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.
(3) $T_\alpha(c) = 0$, for all constant functions $f(t) = c$.
(4) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
(5) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
(6) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Conformable fractional derivative of certain functions
(1) $T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.
(2) $T_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}, c \in \mathbb{R}$.
(3) $T_\alpha(\sin bx) = bx^{1-\alpha}\cos bx, b \in \mathbb{R}$.
(4) $T_\alpha(\cos bx) = -bx^{1-\alpha}\sin bx, b \in \mathbb{R}$.
(5) $T_\alpha(\tan bx) = bx^{1-\alpha}\sec^2 bx, b \in \mathbb{R}$.
(6) $T_\alpha(\cot bx) = -bx^{1-\alpha}\csc^2 bx, b \in \mathbb{R}$.
(7) $T_\alpha(\sec bx) = bx^{1-\alpha}\sec bx \tan bx, b \in \mathbb{R}$.
(8) $T_\alpha(\csc bx) = -bx^{1-\alpha}\csc bx \cot bx, b \in \mathbb{R}$.

Definition 2.6. [15] Let $f : [0, \infty) \to \mathbb{R}$. Then the fractional derivative of f of order α is defined by

$$D^\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(te^{\epsilon t^{-\alpha}}) - f(t)}{\epsilon}, \tag{2.17}$$

for $t > 0$, $\alpha \in (0, 1)$. If f is α- differentiable in some $(0, a)$, $a > 0$ *and* $\lim_{\epsilon \to 0} D^\alpha(f)(t)$ exists, then define

$$D^\alpha(f)(0) = \lim_{t \to 0^+} D^\alpha(f)(t).$$

3 General properties of revolution surface M and $\overline{M}$ using conformable fractional calculus

In this section, we shall describe and derive the fundamental quantities of revolution surface M and $\overline{M}$ using conformable fractional calculus. The general conditions for these surfaces to become bi-conservative are derived.

Corollary 3.1. *The condition (2.4) will become*

$$\psi'^2 + \phi'^2 = u^{2\alpha-2}. \tag{3.1}$$

Proof. Since $\psi = \psi(u)$ and $\phi = \phi(u)$, then, $T_\alpha(\psi) = u^{1-\alpha}\psi'$, $T_\alpha(\phi) = u^{1-\alpha}\phi'$

$$\Rightarrow \left(u^{1-\alpha}\psi'\right)^2 + \left(u^{1-\alpha}\phi'\right)^2 = 1 \text{ then } \psi'^2 + \phi'^2 = u^{2\alpha-2}.$$

□

By differentiating this condition by using conformable fractional calculus, we get

$$\psi'\psi'' + \phi'\phi'' = (\alpha - 1)u^{2\alpha-3}. \tag{3.2}$$

Using conformable fractional calculus, (3.1) and (3.2) we will compute the properties of the surface (2.2) as follows:

The coefficients of the fractional first fundamental form are given by

$$g_{11}^\alpha = 1, \;\; g_{22}^\alpha = \phi^2 \; \upsilon^{2-2\alpha} \;\; \text{and} \;\; g_{12}^\alpha = 0. \tag{3.3}$$

Therefore, the fractional metric of the first fundamental form g^α of the surface (2.2) is given by

$$g^\alpha = \phi^2 \; \upsilon^{2-2\alpha}. \tag{3.4}$$

The fractional unit normal vector field N^α is given by

$$N^\alpha = u^{1-\alpha}(-\psi' \cos \upsilon, -\psi' \sin \upsilon, \phi'). \tag{3.5}$$

Consequently, the coefficients of the fractional second fundamental form are written as follows:

$$h_{11}^\alpha = u^{3-3\alpha} \; (\phi'\psi'' - \psi'\phi''), \;\; h_{22}^\alpha = \psi'\phi \; u^{1-\alpha} \; \upsilon^{2-2\alpha} \;\; \text{and} \;\; h_{12}^\alpha = 0. \tag{3.6}$$

Hence the fractional metric of the second fundamental form h^α is given by

$$h^\alpha = \psi' \; \phi \; u^{4-4\alpha} \; \upsilon^{2-2\alpha} \; (\phi'\psi'' - \psi'\phi''). \tag{3.7}$$

Corollary 3.2. *The fractional Gaussian and mean curvature G^α and H^α of surface (2.2) are given by*

$$G^\alpha = \frac{u^{1-2\alpha}}{\phi}\Big((\alpha - 1)\phi' - u\phi''\Big), \tag{3.8}$$

$$H^\alpha = \frac{\psi' \; u^{1-\alpha} + \phi \; u^{3-3\alpha} \; (\phi'\psi'' - \psi'\phi'')}{2 \; \phi}, \tag{3.9}$$

and a_{ij}^α are given by

$$a_{11}^\alpha = u^{3-3\alpha} \; (\phi'\psi'' - \psi'\phi''), \;\; a_{22}^\alpha = \frac{\psi'}{\phi}u^{1-\alpha}, \;\; a_{12}^\alpha = a_{21}^\alpha = 0. \tag{3.10}$$

Lemma 3.3. *If we put $\alpha = 1$ in (3.8) and (3.9), we obtain the Gaussian and mean curvature in integer case (2.6) and (2.8).*

From (2.2) if this is taken as axis of z and u denotes perpendicular distance from it, the parametrization of the surface M is given by

$$M_0 : X(u, \upsilon) = (u \cos \upsilon, u \sin \upsilon, \psi(u)). \tag{3.11}$$

Corollary 3.4. *The fractional Gaussian and mean curvature G and H of surface (3.11) are given by*

$$G^\alpha = (\alpha - 1)u^{-2\alpha}, \tag{3.12}$$

$$H^\alpha = \tfrac{1}{2}(u^{-\alpha}\psi' + u^{3-3\alpha}\psi''), \tag{3.13}$$

and a_{ij}^α are given by

$$a_{11}^\alpha = u^{3-3\alpha} \; \psi'', \;\; a_{22}^\alpha = \psi' u^{-\alpha}, \;\; a_{12}^\alpha = a_{21}^\alpha = 0. \tag{3.14}$$

Now, we will study the parallel surface to M_0 and we denote this surface by $\overline{M}_0$. Thus, using (2.9), (3.5) and (3.11), the parallel surface $\overline{M}_0$ has the following form

$$\overline{M}_0 : \overline{X}(u,v) = \Big((u - c\psi' u^{1-\alpha})\cos v,\ (u - c\psi' u^{1-\alpha})\sin v,\ \psi + cu^{1-\alpha}\Big). \tag{3.15}$$

The fractional Gaussian and mean curvature $\overline{G}$ and $\overline{H}$ of surface (3.15) are given by

$$\overline{G}^{\alpha} = \frac{(\alpha-1)u^{-2\alpha}}{1 - c\,(u^{-\alpha}\psi' + u^{3-3\alpha}\psi'') + c^2(\alpha-1)u^{-2\alpha}}, \tag{3.16}$$

$$\overline{H}^{\alpha} = \frac{\frac{1}{2}(u^{-\alpha}\psi' + u^{3-3\alpha}\psi'') - c(\alpha-1)u^{-2\alpha}}{1 - c\,(u^{-\alpha}\psi' + u^{3-3\alpha}\psi'') + c^2(\alpha-1)u^{-2\alpha}}, \tag{3.17}$$

and $\overline{a}^{\alpha}_{ij}$ are given by

$$\overline{a}^{\alpha}_{11} = \frac{u^{3-3\alpha}\psi'' - c\lambda_1}{1 - 2cu^{3-3\alpha}\psi'' + c^2\lambda_1},\ \overline{a}^{\alpha}_{22} = \frac{u^{-\alpha}\psi'}{1 - cu^{-\alpha}\psi'},\ \overline{a}^{\alpha}_{12} = \overline{a}^{\alpha}_{21} = 0, \tag{3.18}$$

where λ_1 is given by

$$\lambda_1 = u^{2-4\alpha}\Big((u\psi'' + (1-\alpha)\psi')^2 + (1-\alpha)^2\Big). \tag{3.19}$$

Corollary 3.5. *The parallel revolution surface $\overline{M}_0$ is bi-conservative if the following two equations are valid*

$$\begin{aligned}\overline{a}^{\alpha}_{11}\overline{H}^{\alpha}_{u} + \overline{a}^{\alpha}_{12}\overline{H}^{\alpha}_{v} + \overline{H}^{\alpha}\ \overline{H}^{\alpha}_{u} = &\Big[2\lambda_2(u^{3-3\alpha}\psi'' - c\lambda_1) + \lambda_5(1 - 2cu^{3-3\alpha}\psi'' + c^2\lambda_1)\Big] \\ &\Big[\lambda_2\Big(\lambda_3 + \lambda_4 + 4c\alpha(\alpha-1)u^{-3\alpha}\Big) + c\lambda_5\Big(\lambda_3 + \lambda_4 + 2c\alpha(\alpha-1)u^{-3\alpha}\Big)\Big] = 0\end{aligned} \tag{3.20}$$

$$\overline{a}^{\alpha}_{21}\overline{H}^{\alpha}_{u} + \overline{a}^{\alpha}_{22}\overline{H}^{\alpha}_{v} + \overline{H}^{\alpha}\ \overline{H}^{\alpha}_{v} = 0, \tag{3.21}$$

where $\overline{H}^{\alpha}_{u}, \overline{H}^{\alpha}_{v},\ \lambda_2,\ \lambda_3,\ \lambda_4$ and λ_5 *are given by*

$$\left.\begin{aligned}&\overline{H}^{\alpha}_{u} = \frac{\lambda_2\Big(\lambda_3 + \lambda_4 + 4c\alpha(\alpha-1)u^{-3\alpha}\Big) + c\lambda_5\Big(\lambda_3 + \lambda_4 + 2c\alpha(\alpha-1)u^{-3\alpha}\Big)}{2\lambda_2^2}, \overline{H}^{\alpha}_{v} = 0, \\ &\lambda_2 = 1 - c\Big(u^{-\alpha}\psi' + u^{3-3\alpha}\psi'' - c(\alpha-1)u^{-2\alpha}\Big),\ \lambda_3 = u^{1-2\alpha}\psi'' - \alpha u^{-2\alpha}\psi', \\ &\lambda_4 = u^{4-4\alpha}\psi''' + (3-3\alpha)u^{3-4\alpha}\psi'',\ \lambda_5 = u^{-\alpha}\psi' + u^{3-3\alpha}\psi'' - 2c(\alpha-1)u^{-2\alpha}.\end{aligned}\right\} \tag{3.22}$$

Remark 3.6. We have computed the bi-conservative property of the original surface M_0 [22].

4 Applications on the parallel revolution surface $\overline{M}_0$

In this section we investigate the possibility of obtaining the conditions for parallel revolution surfaces to become fractional bi-conservative in Euclidean 3-space E^3. Since the general conditions for these surfaces take the form of nonlinear differential equations, we can solve them using special cases and obtain the theoretical or numerical solutions.

Here, we give the following cases:

4.1. Case 1. If we put $\psi(u) = e^u$, we denote this surface by $\overline{M}_1$. So using (3.20) and (3.21), we have the following corollary:

(i) $\overline{M}_1$ is bi-conservative if the following equation is valid

$$\begin{aligned}&\left[2\left(1-c\left[u^{-\alpha}e^{u}+u^{3-3\alpha}e^{u}-c\left(\alpha-1\right)u^{-2\alpha}\right]\right)\left(u^{3-3\alpha}e^{u}-cu^{2-4\alpha}[e^{2u}(u+1-\alpha)^{2}\right.\\&+(1-\alpha)^{2}])+(u^{-\alpha}e^{u}+u^{3-3\alpha}e^{u}-2c(\alpha-1)u^{-2\alpha})(1-2cu^{3-3\alpha}e^{u}+\\&c^{2}u^{2-4\alpha}[e^{2u}(u+1-\alpha)^{2}+(1-\alpha)^{2}])][(1-c[u^{-\alpha}e^{u}+u^{3-3\alpha}e^{u}-\\&c(\alpha-1)u^{-2\alpha}])(e^{u}[u^{1-2\alpha}-\alpha u^{-2\alpha}+u^{4-4\alpha}+3(1-\alpha)u^{3-4\alpha}]+4c\alpha(\alpha-1)u^{-3\alpha})\\&+c(u^{-\alpha}e^{u}+u^{3-3\alpha}e^{u}-2c(\alpha-1)u^{-2\alpha})(e^{u}[u^{1-2\alpha}-\alpha u^{-2\alpha}+u^{4-4\alpha}\\&+3(1-\alpha)u^{3-4\alpha}]+2c\alpha(\alpha-1)u^{-3\alpha})]=0.\end{aligned} \tag{4.1}$$

We shall take some different values of α in (4.1) as below:

(a) At $\alpha=1,\ c=3$, we have

$$\begin{aligned}&\Big(\big[3e^{u}-1\big]\big[6e^{u}+6u^{-1}e^{u}-2\big]+\big[1+u^{-1}\big]\big[9e^{2u}-6e^{u}+1\big]\Big)\\&\Big([3e^{u}+3u^{-1}e^{u}][u^{-1}-u^{-2}+1]-[u^{-1}-u^{-2}+1][3e^{u}+3u^{-1}e^{u}-1]\Big)=0.\end{aligned} \tag{4.2}$$

(b) At $\alpha=0.75,\ c=3$, we have

$$\begin{aligned}&\Big(\big[u^{-3/4}e^{u}+u^{3/4}e^{u}+\tfrac{3}{2}u^{-3/2}\big]\big[9u^{-1}e^{2u}(u+\tfrac{1}{4})^{2}+\tfrac{9}{16}u^{-1}-6u^{3/4}+1\big]\\&-\big[u^{3/4}e^{u}-3u^{-1}e^{2u}(u+\tfrac{1}{4})^{2}-\tfrac{3}{16}u^{-1}\big]\big[6u^{-3/4}e^{u}+6u^{3/4}e^{u}+\tfrac{9}{2}u^{-3/2}-2\big]\Big)\\&\Big(\big[e^{u}\big(u+u^{-1/2}-\tfrac{3}{4}u^{-3/2}+\tfrac{3}{4}\big)-\tfrac{9}{4}u^{-9/4}\big]\big[3u^{-3/4}e^{u}+3u^{3/4}e^{u}+\tfrac{9}{4}u^{-3/2}-1\big]\\&-\big[e^{u}\big(u+u^{-1/2}-\tfrac{3}{4}u^{-3/2}+\tfrac{3}{4}\big)-\tfrac{9}{8}u^{-9/4}\big]\big[3u^{-3/4}e^{u}+3u^{3/4}e^{u}+\tfrac{9}{2}u^{-3/2}\big]\Big)=0.\end{aligned} \tag{4.3}$$

(c) At $\alpha=0.5,\ c=3$, we have

$$\begin{aligned}&\Big(\big[e^{u}\big(\tfrac{3}{2}u-\tfrac{1}{2}u^{-1}+u^{2}+1\big)-\tfrac{3}{2}u^{-3/2}\big]\big[3u^{-1/2}e^{u}+3u^{3/2}e^{u}+9u^{-1}\big]\\&-\big[e^{u}\big(\tfrac{3}{2}u-\tfrac{1}{2}u^{-1}+u^{2}+1\big)-3u^{-3/2}\big]\big[3u^{-1/2}e^{u}+3u^{3/2}e^{u}+\tfrac{9}{2}u^{-1}-1\big]\Big)\\&\Big(\big[3e^{2u}(u+\tfrac{1}{2})^{2}-u^{3/2}e^{u}+\tfrac{3}{4}\big]\big[6u^{-1/2}e^{u}+6u^{3/2}e^{u}+9u^{-1}-2\big]\\&+\big[9e^{2u}(u+\tfrac{1}{2})^{2}-6u^{3/2}e^{u}+\tfrac{13}{4}\big]\big[u^{-1/2}e^{u}+u^{3/2}e^{u}+3u^{-1}\big]\Big)=0.\end{aligned} \tag{4.4}$$

The numerical solution of (4.2), (4.3) and (4.4) is the same and equals $u=4231/43$.

(d) At $\alpha = 0.1,\ c = 3$, we have

$$
\begin{aligned}
&([u^{-1/10}e^{u} + u^{27/10}e^{u} + \frac{27}{5}u^{-1/5}][9u^{8/5}e^{2u}(u+\frac{9}{10})^{2} + \frac{729}{100}u^{8/5} - 6u^{27/10}e^{u} + 1]\\
&-[u^{27/10}e^{u} - 3u^{8/5}e^{2u}(u+\frac{9}{10})^{2} - \frac{243}{100}u^{8/5}][6u^{-1/10}e^{u} + 6u^{27/10}e^{u} + \frac{81}{5}u^{-1/5} - 2])\\
&([e^{u}(u^{4/5} - \frac{1}{10}u^{-1/5} + \frac{27}{10}u^{13/5} + u^{18/5}) - \frac{27}{25}u^{-3/10}][3u^{-1/10}e^{u} + 3u^{27/10}e^{u}\\
&+\frac{81}{10}u^{-1/5} - 1] - [e^{u}(u^{4/5} - \frac{1}{10}u^{-1/5} + \frac{27}{10}u^{13/5} + u^{18/5}) - \frac{27}{50}u^{-3/10}]\\
&[3u^{-1/10}e^{u} + 3u^{27/10}e^{u} + \frac{81}{5}u^{-1/5}]) = 0.
\end{aligned}
\tag{4.5}
$$

The numerical solution of (4.5) is $u = -\frac{1905}{179} - \frac{672}{493}i$.

Corollary 4.1. *The surface* $\overline{M}_1$ *is bi-conservative when* $\alpha = 1,\ 0.75,\ 0.5$ *while it is not bi-conservative when* $\alpha = 0.1$.

The eqs. (4.2) - (4.5) and their corresponding of the original surface M_1 are illustrated in (Figs. 1, 2, 3, 4), respectively.

4.2. Case 2. If we put $\psi(u) = log\ u$, we denote this surface by $\overline{M}_2$. Then using (3.20) and (3.21), we have the following corollary:
(i) $\overline{M}_2$ is bi-conservative if the following equation is valid

$$
\begin{aligned}
&\left[2(1 - c[u^{-1-\alpha} - u^{1-3\alpha} - c(\alpha-1)u^{-2\alpha}])\,(-u^{1-3\alpha} - cu^{2-4\alpha}[\alpha^{2}u^{-2} + (1-\alpha)^{2}])\right.\\
&+(u^{-1-\alpha} - u^{1-3\alpha} - 2c(\alpha-1)u^{-2\alpha})(1 + 2cu^{1-3\alpha} + c^{2}u^{2-4\alpha}[\alpha^{2}u^{-2} + (1-\alpha)^{2}])]\\
&[(1 - c[u^{-1-\alpha} - u^{1-3\alpha} - c(\alpha-1)u^{-2\alpha}])(-(1+\alpha)u^{-1-2\alpha} + (3\alpha-1)u^{1-4\alpha}\\
&+4c\alpha(\alpha-1)u^{-3\alpha}) + c(u^{-1-\alpha} - u^{1-3\alpha} - 2c(\alpha-1)u^{-2\alpha})(-(1+\alpha)u^{-1-2\alpha}\\
&+(3\alpha-1)u^{1-4\alpha} + 2c\alpha(\alpha-1)u^{-3\alpha})] = 0.
\end{aligned}
\tag{4.6}
$$

We shall take some different values of α in (4.6) as the following:

(a) At $\alpha = 1,\ c = 3$, the condition (4.6) is valid.
(b) At $\alpha = 0.75,\ c = 3$, we have

$$
\begin{aligned}
&([\frac{7}{4}u^{-5/2} - \frac{5}{4}u^{-2} + \frac{9}{4}u^{-9/4}][\frac{9}{4}u^{-3/2} - 3u^{-5/4} + 3u^{-7/4} - 1] - [\frac{9}{2}u^{-3/2}\\
&-3u^{-5/4} + 3u^{-7/4}][\frac{7}{4}u^{-5/2} - \frac{5}{4}u^{-2} + \frac{9}{8}u^{-9/4}])([\frac{27}{16}u^{-3} + \frac{3}{16}u^{-1} + u^{-5/4}]\\
&[\frac{9}{2}u^{-3/2} - 6u^{-5/4} + 6u^{-7/4} - 2] + [\frac{81}{16}u^{-3} + \frac{9}{16}u^{-1} + 6u^{-5/4} + 1][\frac{3}{2}u^{-3/2}\\
&-u^{-5/4} + u^{-7/4}]) = 0.
\end{aligned}
\tag{4.7}
$$

The numerical solution of (4.7) is $u = 8905/272$.

(c) At $\alpha = 0.5,\ c = 3$, we have

$$\Big(\big[\tfrac{3}{2}u^{-2} - \tfrac{1}{2}u^{-1} + \tfrac{3}{2}u^{-3/2}\big]\big[9u^{-1} - 3u^{-1/2} + 3u^{-3/2}\big] - \big[\tfrac{3}{2}u^{-2} - \tfrac{1}{2}u^{-1} + 3u^{-3/2}\big]$$
$$\big[\tfrac{9}{2}u^{-1} - 3u^{-1/2} + 3u^{-3/2} - 1\big]\Big)\Big(\big[\tfrac{3}{4}u^{-2} + u^{-1/2} + \tfrac{3}{4}\big]\big[9u^{-1} - 6u^{-1/2} + 6u^{-3/2} - 2\big]$$
$$+\big[3u^{-1} - u^{-1/2} + u^{-3/2}\big]\big[\tfrac{9}{4}u^{-2} + 6u^{-1/2} + \tfrac{13}{4}\big]\Big) = 0. \tag{4.8}$$

The numerical solution of (4.8) is $u = 8805/176$.
(d) At $\alpha = 0.1,\ c = 3$, we have

$$\begin{aligned}&([\frac{7}{10}u^{3/5} + \frac{11}{10}u^{-6/5} + \frac{27}{25}u^{-3/10}][\frac{81}{10}u^{-1/5} - 3u^{7/10} + 3u^{-11/10} - 1]\\ &-[\frac{81}{5}u^{-1/5} - 3u^{7/10} + 3u^{-11/10}][\frac{7}{10}u^{3/5} + \frac{11}{10}u^{-6/5} + \frac{27}{50}u^{-3/10}])\\ &([9u^{8/5}(\frac{1}{100}u^{-2} + \frac{81}{100}) + 6u^{7/10} + 1][\frac{27}{5}u^{-1/5} - u^{7/10} + u^{-11/10}]\\ &+[3u^{8/5}(\frac{1}{100}u^{-2} + \frac{81}{100}) + u^{7/10}][\frac{81}{5}u^{-1/5} - 6u^{7/10} + 6u^{-11/10} - 2]) = 0.\end{aligned} \tag{4.9}$$

The numerical solution of (4.9) is $u = 173/41$.

Corollary 4.2. *The surface $\overline{M}_2$ is bi-conservative in both integer and fractional alpha cases.*

The eqs. (4.7), (4.8), (4.9) and their corresponding are illustrated in (Figs. 5, 6, 7), respectively.

4.3. Case 3. If we put $\psi(u) = cos\ u$, we denote this surface by $\overline{M}_3$. So using (3.20) and (3.21), we have the following corollary:
(i) $\overline{M}_3$ is bi-conservative if the following equation is valid

$$\begin{aligned}&\big[2(1 + c[u^{-\alpha}\sin u + u^{3-3\alpha}\cos u + c(\alpha-1)u^{-2\alpha}])\ (-u^{3-3\alpha}\cos u\\ &-cu^{2-4\alpha}[(u\cos u + (1-\alpha)\sin u)^2 + (1-\alpha)^2]) - (u^{-\alpha}\sin u +\\ &u^{3-3\alpha}\cos u + 2c(\alpha-1)u^{-2\alpha})(1 + 2cu^{3-3\alpha}\cos u + c^2u^{2-4\alpha}[(u\cos u\\ &+(1-\alpha)\sin u)^2 + (1-\alpha)^2])\big]\ [(1 + c[u^{-\alpha}\sin u + u^{3-3\alpha}\cos u +\\ &c(\alpha-1)u^{-2\alpha}])(-u^{1-2\alpha}\cos u + \alpha u^{-2\alpha}\sin u + u^{4-4\alpha}\sin u\\ &-3(1-\alpha)u^{3-4\alpha}\cos u + 4c\alpha(\alpha-1)u^{-3\alpha}) - c(u^{-\alpha}\sin u\\ &+u^{3-3\alpha}\cos u + 2c(\alpha-1)u^{-2\alpha})(-u^{1-2\alpha}\cos u + \alpha u^{-2\alpha}\sin u\\ &+u^{4-4\alpha}\sin u - 3(1-\alpha)u^{3-4\alpha}\cos u + 2c\alpha(\alpha-1)u^{-3\alpha})] = 0.\end{aligned} \tag{4.10}$$

We shall take some different values of α in (4.10) as the following

(a) At $\alpha = 1,\ c = 3$, we have

$$\Big(\big[\cos u + u^{-1}\sin u\big]\big[9\cos^2 u + 6\cos u + 1\big] + \big[3\cos^2 u + \cos u\big]\big[6\cos u + 6u^{-1}\sin u + 2\big]\Big)$$
$$\Big(\big[3\cos u + 3u^{-1}\sin u\big]\big[\sin u - u^{-1}\cos u + u^{-2}\sin u\big]$$
$$-\big[\sin u - u^{-1}\cos u + u^{-2}\sin u\big]\big[3\cos u + 3u^{-1}\sin u + 1\big]\Big) = 0. \tag{4.11}$$

The numerical solution of (4.11) is $u = -30293/133$.

(b) At $\alpha = 0.75,\ c = 3$, we have

$$([6u^{3/4}\cos u + 9u^{-1}(\frac{1}{4}\sin u + u\cos u)^2 + \frac{9}{16}u^{-1} + 1][u^{3/4}\cos u + u^{-3/4}\sin u$$
$$-\frac{3}{2}u^{-3/2}] + [u^{3/4}\cos u + 3u^{-1}(\frac{1}{4}\sin u + u\cos u)^2 + \frac{3}{16}u^{-1}][6u^{3/4}\cos u +$$
$$6u^{-3/4}\sin u - \frac{9}{2}u^{-3/2} + 2])([3u^{3/4}\cos u + 3u^{-3/4}\sin u - \frac{9}{4}u^{-3/2} + 1]$$
$$[\frac{3}{4}\cos u + u^{-1/2}\cos u - \frac{3}{4}u^{-3/2}\sin u - u\sin u + \frac{9}{4}u^{-9/4}] - [3u^{3/4}\cos u +$$
$$3u^{-3/4}\sin u - \frac{9}{2}u^{-3/2}][\frac{3}{4}\cos u + u^{-1/2}\cos u - \frac{3}{4}u^{-3/2}\sin u - u\sin u + \frac{9}{8}u^{-9/4}]) = 0. \tag{4.12}$$

The numerical solution of (4.12) is $u = -\frac{18221}{80} + \frac{45}{14113}i$.

(c) At $\alpha = 0.5,\ c = 3$, we have

$$([3u^{3/2}\cos u + 3u^{-1/2}\sin u - 9u^{-1}][\cos u - \frac{1}{2}u^{-1}\sin u - u^2\sin u + \frac{3}{2}u\cos u$$
$$+\frac{3}{2}u^{-3/2}] - [3u^{3/2}\cos u + 3u^{-1/2}\sin u - \frac{9}{2}u^{-1} + 1][\cos u - \frac{1}{2}u^{-1}\sin u$$
$$-u^2\sin u + \frac{3}{2}u\cos u + 3u^{-3/2}])([u^{3/2}\cos u + 3(\frac{1}{2}\sin u + u\cos u)^2 + \frac{3}{4}]$$
$$[6u^{3/2}\cos u + 6u^{-1/2}\sin u - 9u^{-1} + 2] + [6u^{3/2}\cos u + 9(\frac{1}{2}\sin u + u\cos u)^2$$
$$+\frac{13}{4}][u^{3/2}\cos u + u^{-1/2}\sin u - 3u^{-1}]) = 0. \tag{4.13}$$

The numerical solution of (4.13) is $u = 14938/151$.

(d) At $\alpha = 0.1,\ c = 3$, we have

$$([3u^{27/10}\cos u + 3u^{-1/10}\sin u - \frac{81}{10}u^{-1/5} + 1][u^{4/5}\cos u + \frac{27}{10}u^{13/5}\cos u$$
$$-\frac{1}{10}u^{-1/5}\sin u - u^{18/5}\sin u + \frac{27}{25}u^{-3/10}] - [3u^{27/10}\cos u + 3u^{-1/10}\sin u$$
$$-\frac{81}{5}u^{-1/5}][u^{4/5}\cos u + \frac{27}{10}u^{13/5}\cos u - \frac{1}{10}u^{-1/5}\sin u - u^{18/5}\sin u +$$
$$\frac{27}{50}u^{-3/10}])([6u^{27/10}\cos u + 9u^{9/8}(\frac{9}{10}\sin u + u\cos u)^2 + \frac{729}{100}u^{8/5} + 1]$$
$$[u^{27/10}\cos u + u^{-1/10}\sin u - \frac{27}{5}u^{-1/5}] + [u^{27/10}\cos u + 3u^{8/5}(\frac{9}{10}\sin u + u\cos u)^2$$
$$+\frac{243}{100}u^{8/5}][6u^{27/10}\cos u + 6u^{-1/10}\sin u - \frac{81}{5}u^{-1/5} + 2]) = 0. \tag{4.14}$$

The numerical solution of (4.14) is $u = 22365/226$.

Corollary 4.3. *The surface $\overline{M}_3$ is bi-conservative when $\alpha = 1, 0.5, 0.1$ while it is not bi-conservative when $\alpha = 0.75$.*

The eqs. (4.11) – (4.14) and their corresponding of M_3 are illustrated in (Figs. 8, 9, 10, 11), respectively.

4.4. Case 4. If we put $\psi(u) = \sin u$, we denote this surface by $\overline{M}_4$. So using (3.20) and (3.21), we have the following corollary:

(i) $\overline{M}_4$ is bi-conservative if the following equation is valid

$$\begin{aligned}&[2(1 - c[u^{-\alpha}\cos u - u^{3-3\alpha}\sin u - c(\alpha-1)u^{-2\alpha}])(-u^{3-3\alpha}\sin u\\ &-cu^{2-4\alpha}[(-u\sin u + (1-\alpha)\cos u)^2 + (1-\alpha)^2]) + (u^{-\alpha}\cos u -\\ &u^{3-3\alpha}\sin u - 2c(\alpha-1)u^{-2\alpha})(1 + 2cu^{3-3\alpha}\sin u + c^2u^{2-4\alpha}[(-u\sin u +\\ &(1-\alpha)\cos u)^2 + (1-\alpha)^2])]\ [(1 - c[u^{-\alpha}\cos u - u^{3-3\alpha}\sin u -\\ &c(\alpha-1)u^{-2\alpha}])(-u^{1-2\alpha}\sin u - \alpha u^{-2\alpha}\cos u - u^{4-4\alpha}\cos u -\\ &3(1-\alpha)u^{3-4\alpha}\sin u + 4c\alpha(\alpha-1)u^{-3\alpha}) + c(u^{-\alpha}\cos u - u^{3-3\alpha}\sin u -\\ &2c(\alpha-1)u^{-2\alpha})(-u^{1-2\alpha}\sin u - \alpha u^{-2\alpha}\cos u - u^{4-4\alpha}\cos u -\\ &3(1-\alpha)u^{3-4\alpha}\sin u + 2c\alpha(\alpha-1)u^{-3\alpha})] = 0.\end{aligned} \tag{4.15}$$

We shall take some different values of α in (4.15) as the following:

(a) At $\alpha = 1,\ c = 3$, we have

$$\begin{aligned}&([\sin u + u^{-1}\cos u][9\sin^2 u + 6\sin u + 1] + [3\sin^2 u + \sin u][6\sin u -\\ &6u^{-1}\cos u + 2])([3\sin u - 3u^{-1}\cos u][\cos u + u^{-1}\sin u + u^{-2}\cos u] -\\ &[\cos u + u^{-1}\sin u + u^{-2}\cos u][3\sin u - 3u^{-1}\cos u + 1]) = 0.\end{aligned} \tag{4.16}$$

The numerical solution of (4.16) is $u = -18123/80$.

(b) At $\alpha = 0.75,\ c = 3$, we have

$$
\begin{aligned}
&([6u^{3/4}\sin u + 9u^{-1}(\frac{1}{4}\cos u - u\sin u)^2 + \frac{9}{16}u^{-1} + 1][u^{-3/4}\cos u - \\
&u^{3/4}\sin u + \frac{3}{2}u^{-3/2}] + [u^{3/4}\sin u + 3u^{-1}(\frac{1}{4}\cos u - u\sin u)^2 + \frac{3}{16}u^{-1}] \\
&[6u^{-3/4}\cos u - 6u^{3/4}\sin u + \frac{9}{2}u^{-3/2} - 2])([3u^{-3/4}\cos u - 3u^{3/4}\sin u \\
&+\frac{9}{4}u^{-3/2} - 1][\frac{3}{4}\sin u + u^{-1/2}\sin u + \frac{3}{4}u^{-3/2}\cos u + u\cos u + \frac{9}{4}u^{-9/4}] \\
&-[3u^{-3/4}\cos u - 3u^{3/4}\sin u + \frac{9}{2}u^{-3/2}][\frac{3}{4}\sin u + u^{-1/2}\sin u + \frac{3}{4}u^{-3/2}\cos u \\
&+u\cos u + \frac{9}{8}u^{-9/4}]) = 0.
\end{aligned}
\tag{4.17}
$$

The numerical solution of (4.17) is $u = -42314/187 + 205/2534i$.

(c) At $\alpha = 0.5,\ c = 3$, we have

$$
\begin{aligned}
&([-3u^{3/2}\sin u + 3u^{-1/2}\cos u + 9u^{-1}][\sin u + \frac{1}{2}u^{-1}\cos u + u^2\cos u + \\
&\frac{3}{2}u\sin u + \frac{3}{2}u^{-3/2}] - [-3u^{3/2}\sin u + 3u^{-1/2}\cos u + \frac{9}{2}u^{-1} - 1][\sin u + \\
&\frac{1}{2}u^{-1}\cos u + u^2\cos u + \frac{3}{2}u\sin u + 3u^{-3/2}])([u^{3/2}\sin u + 3(\frac{1}{2}\cos u - u\sin u)^2 \\
&+\frac{3}{4}][-6u^{3/2}\sin u + 6u^{-1/2}\cos u + 9u^{-1} - 2] + [6u^{3/2}\sin u + \\
&9(\frac{1}{2}\cos u - u\sin u)^2 + \frac{13}{4}][-u^{3/2}\sin u + u^{-1/2}\cos u + 3u^{-1}]) = 0.
\end{aligned}
\tag{4.18}
$$

The numerical solution of (4.18) is $u = 19373/209$.

(d) At $\alpha = 0.1,\ c = 3$, we have

$$
\begin{aligned}
&([-3u^{27/10}\sin u + 3u^{-1/10}\cos u + \frac{81}{10}u^{-1/5} - 1][u^{4/5}\sin u + \frac{27}{10}u^{13/5}\sin u \\
&+\frac{1}{10}u^{-1/5}\cos u + u^{18/5}\cos u + \frac{27}{25}u^{-3/10}] - [-3u^{27/10}\sin u + 3u^{-1/10}\cos u \\
&+\frac{81}{5}u^{-1/5}][u^{4/5}\sin u + \frac{27}{10}u^{13/5}\sin u + \frac{1}{10}u^{-1/5}\sin u + u^{18/5}\cos u \\
&+\frac{27}{50}u^{-3/10}])([6u^{27/10}\sin u + 9u^{9/8}(\frac{9}{10}\cos u - u\sin u)^2 + \frac{729}{100}u^{8/5} + 1] \\
&[-u^{27/10}\sin u + u^{-1/10}\cos u + \frac{27}{5}u^{-1/5}] + [u^{27/10}\sin u + 3u^{8/5}(\frac{9}{10}\cos u \\
&-u\sin u)^2 + \frac{243}{100}u^{8/5}][-6u^{27/10}\sin u + 6u^{-1/10}\cos u + \frac{81}{5}u^{-1/5} - 2]) = 0.
\end{aligned}
\tag{4.19}
$$

The numerical solution of (4.19) is $u = 11005/113$.

Corollary 4.4. *The surface* $\overline{M}_4$ *is bi-conservative when* $\alpha = 1,\ 0.5,\ 0.1$ *while it is not bi-conservative when* $\alpha = 0.75$.

The eqs. (4.16), (4.17), (4.18), (4.19) and their corresponding are illustrated in (Figs. 12, 13, 14, 15) respectively.
The previous results can be summarized in the following table:

Property	Case 1	Case 1	Case 2	Case 3	Case 4
	α of M_1	α of $\overline{M}_1$	α of $M_2,\ \overline{M}_2$	α of $M_3,\ \overline{M}_3$	α of $M_4,\ \overline{M}_4$
Bi-conservative	1, 0.5, 0.1	1, 0.75, 0.5	1, 0.75, 0.5, 0.1	1, 0.5, 0.1	1, 0.5, 0.1
Not bi-conservative	0.75	0.1	No results	0.75	0.75

5 Conclusion

In this study, we have studied the bi-conservative of parallel revolution surface using a conformable fractional derivative, which is a natural extension of the usual derivative and its definition is the simplest and most natural definition of fractional derivative.

In section 2 we defined M and $\overline{M}$ as parallel surfaces and some properties which are related to our work, such as the bi-conservative of surfaces. We also defined the conformable fractional derivative of any function f and introduced the rules of differentiation related to it.

In section 3 we studied the basic properties of parallel revolution surface $\overline{M}_0$ and further we studied the bi-conservative property of this surface using conformable fractional derivative.

In section 4 we presented four examples of the surface $\overline{M}_0$ and plotted the results using Matlab program v.18, also, we made a comparison between the results of applications for the original surface and the parallel to it [22], which makes it easier for the reader to understand the applications results easily.

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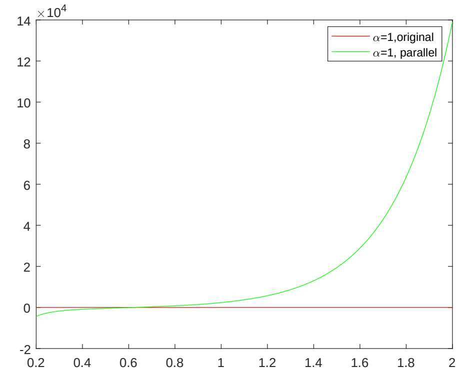

Fig. 1: Bi-conservative of M_1 with $\overline{M_1}$, $\alpha = 1$

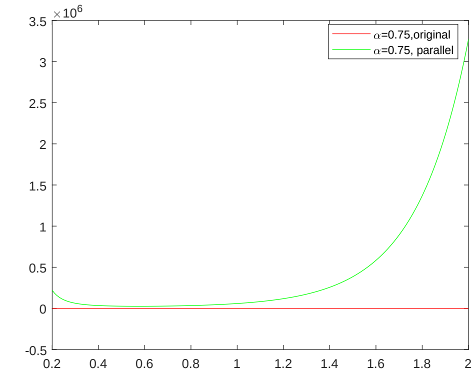

Fig. 2: Bi-conservative of M_1 with $\overline{M_1}$, $\alpha = 0.75$

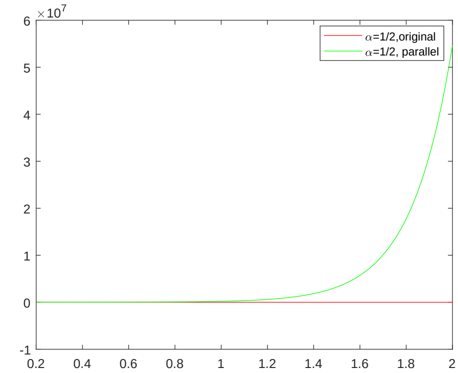

Fig. 3: Bi-conservative of M_1 with $\overline{M_1}$, $\alpha = 0.5$

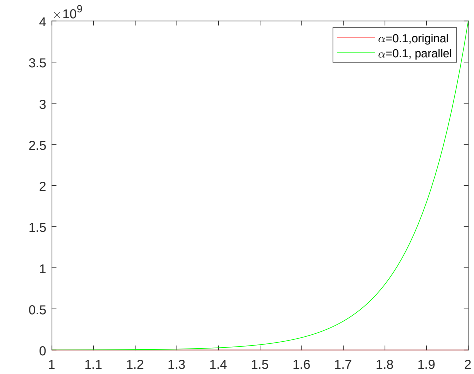

Fig. 4: Bi-conservative of M_1 with $\overline{M_1}$, $\alpha = 0.1$

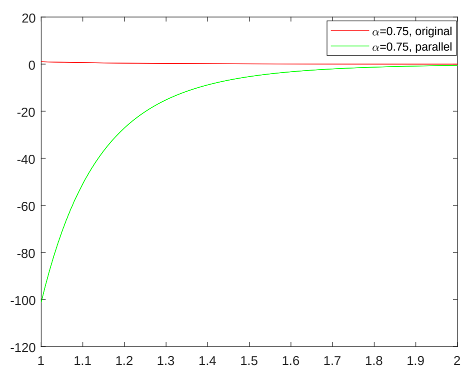

Fig. 5: Bi-conservative of M_2 with $\overline{M_2}$, $\alpha = 0.75$

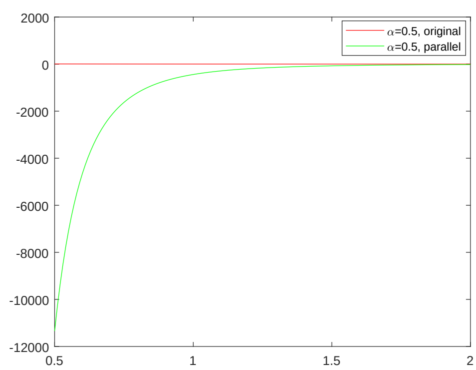

Fig. 6: Bi-conservative of M_2 with $\overline{M_2}$, $\alpha = 0.5$

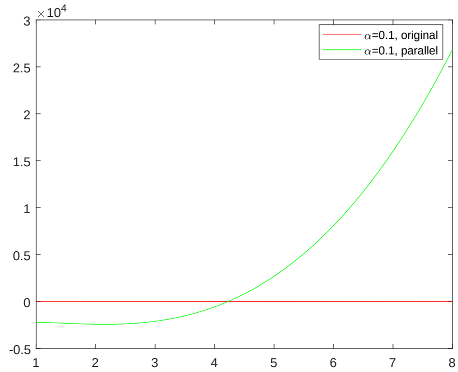

Fig. 7: Bi-conservative of M_2 with $\overline{M_2}$, $\alpha = 0.1$

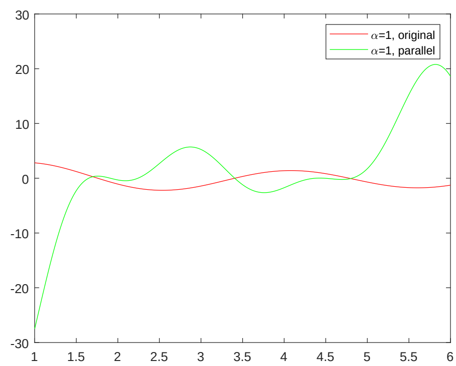

Fig. 8: Bi-conservative of M_3 with $\overline{M_3}$, $\alpha = 1$

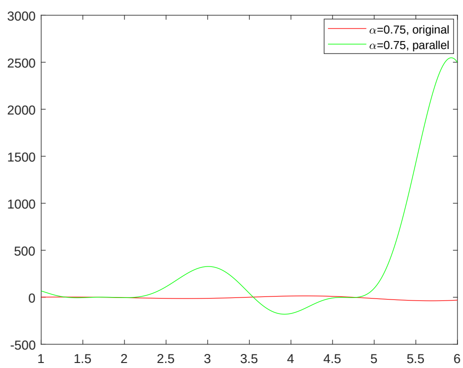

Fig. 9: Bi-conservative of M_3 with $\overline{M_3}$, $\alpha = 0.75$

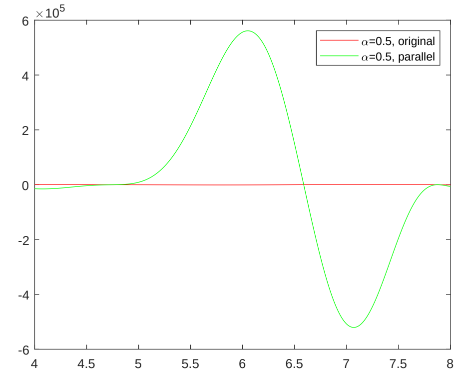

Fig. 10: Bi-conservative of M_3 with $\overline{M_3}$, $\alpha = 0.5$

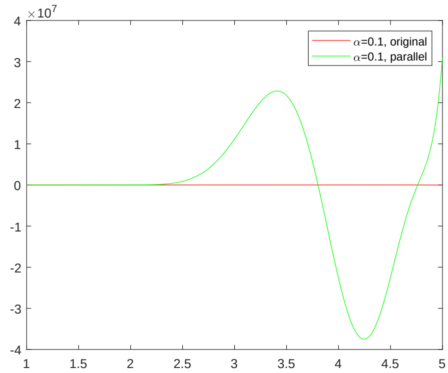

Fig. 11: Bi-conservative of M_3 with $\overline{M_3}$, $\alpha = 0.1$

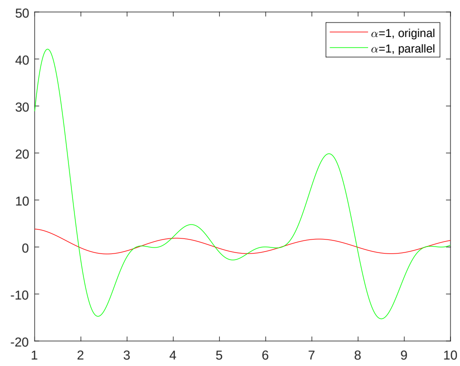

Fig. 12: Bi-conservative of M_4 with $\overline{M_4}$, $\alpha = 1$

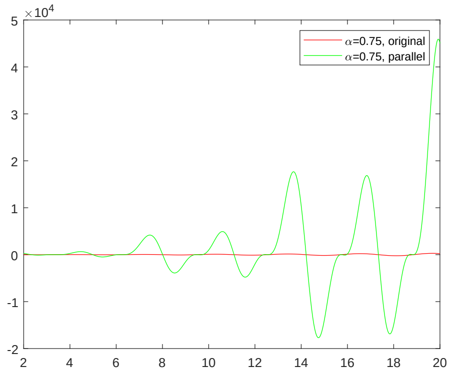

Fig. 13: Bi-conservative of M_4 with $\overline{M_4}$, $\alpha = 0.75$

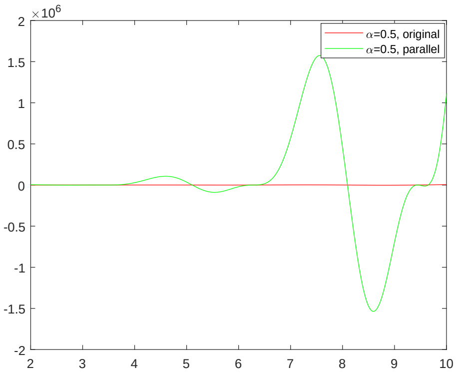

Fig. 14: Bi-conservative of M_4 with $\overline{M_4}$, $\alpha = 0.5$

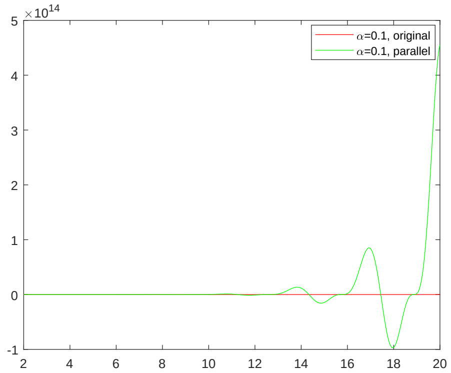

Fig. 15: Bi-conservative of M_4 with $\overline{M_4}$, $\alpha = 0.1$

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