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Application of Rishi transform to the solution of nonlinear Volterra integral equation of the second kind *

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Abstract This paper aims to investigate the solution of non-linear Volterra integral equation of the second kind by using the Rishi transform. The solutions of two numerical problems in compact form are determined by applying the Rishi transform, which suggests that the Rishi transform can be used as a tool for solving these and the other types of related real world problems across various disciplines.

Key words Rishi transform, inverse Rishi transform, convolution, Upadhyaya transform, Volterra integral equation; Dirac Delta function..

2020 Mathematics Subject Classification 35A22, 44A05, 44A35, 45D05, 45G10.

Nomenclature of symbols

$\mathfrak{R}$, the Rishi transform operator;
$\mathfrak{R}^{-1}$,the inverse Rishi transform operator;
$\mathbb{N}$, the set of natural numbers ;
$\in$, belongs to;
!, the usual factorial notation;
Γ, the classical Gamma function;
$\mathbb{R}$, the set of real numbers;

1 Introduction

Volterra integral equations are used to describe various real world problems across various disciplines such as physics, biology, mechanics and the medical sciences [1–3]. Volterra integral equations are solved by numerous numerical [4–9] and analytical methods [10–20]. The selection of these methods depends on the nature of the Volterra integral equations being studied. If the Volterra integral equations are linear (homogeneous or non-homogeneous) then researchers solve these problems by using analytical

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Table 1: Some operational properties of the Rishi transform [20].

S.N.	Name of the property	Mathematical form
1	Linearity	$\Re\left\{\sum_{i=1}^{n} a_i\eta_i(t)\right\} = \sum_{i=1}^{n} a_i\Re\{\eta_i(t)\}$, where a_i's are arbitrary constants.
2	Change of Scale	If $\Re\{\eta(t)\} = \mathcal{T}(\varepsilon,\sigma)$ then $\Re\{\eta(at)\} = \frac{1}{a^2}\mathcal{T}\left(\frac{\varepsilon}{a},\sigma\right)$.
3	Translation	If $\Re\{\eta(t)\} = \mathcal{T}(\varepsilon,\sigma)$ then $\left\{\Re\left\{e^{at}\eta(t)\right\} = \left(\frac{\varepsilon-a\sigma}{\varepsilon}\right)\mathcal{T}(\varepsilon-a\sigma,\sigma)\right\}$.
4	Convolution	If $\Re\{\eta_1(t)\} = \mathcal{T}_1(\varepsilon,\sigma)$ and $\Re\{\eta_2(t)\} = \mathcal{T}_2(\varepsilon,\sigma)$ then $\Re\{\eta_1(t)*\eta_2(t)\}$ $= \left(\frac{\varepsilon}{\sigma}\right)\mathcal{T}_1(\varepsilon,\sigma)\mathcal{T}_2(\varepsilon,\sigma)$.

methods. But if the Volterra integral equations are non-linear (homogeneous or non-homogeneous) then the researchers solve them by using the numerical methods. The problem of finding the solutions of nonlinear Volterra integral equations becomes more complicated due to the presence of nonlinearity of the unknown functions involved. Recently researchers [21–27] used various integral transforms and solved the problems of this type in science and engineering. The authors of the papers [28–35] established the duality relation of recently developed integral transforms and all these results are well documented. This paper presents a new approach for handling the nonlinear Volterra integral equations of the second kind by using a recently developed integral transform called the 'Rishi transform'.

2 The Rishi transform and its inverse

Definition 2.1. Rishi transform: The Rishi transform [18] of a sectionally continuous function of exponential order $\eta(t), t \geq 0$ is given by

$$\Re\{\eta(t)\} = \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty \eta(t)e^{-\left(\frac{\varepsilon}{\sigma}\right)t}dt = \mathcal{T}(\varepsilon,\ \sigma),\quad \varepsilon > 0,\ \sigma > 0 \tag{2.1}$$

Definition 2.2. Inverse Rishi transform: The inverse Rishi transform of $\mathcal{T}(\varepsilon,\sigma)$, denoted by $\Re^{-1}\{\mathcal{T}(\varepsilon,\sigma)\}$, is a function $\eta(t)$ having the property that $\Re\{\eta(t)\} = \mathcal{T}(\varepsilon,\ \sigma)$.

We must mention here that the Rishi transform of (2.1) *is a particular case of the Upadhyaya transform [21, 22], which is by far the most versatile unification and generalization of all the integral transforms of the Laplace type today existing in the mathematics research literature.* For the convenience of the readers we summarize in Table 1, Table 2 and Table 3 some useful operational properties of the Rishi transfrom, the Rishi transform of some commonly occurring functions and their inverse Rishi transforms.

3 Mean value theorem for integrals [36]

If a function $\eta(t)$ is continuous on $[\ell,\ m]$, then there exists a number β in $[\ell,\ m]$ such that

$$\int_\ell^m \eta(t)dt = \eta(\beta)(m-\ell). \tag{3.1}$$

Table 2: The Rishi transform of some elementary functions [20]

S.N.	$\eta(t), t > 0$	$\mathfrak{R}\{\eta(t)\} = \mathcal{T}(\varepsilon, \sigma)$
1	1	$\left(\frac{\sigma}{\varepsilon}\right)^2$
2	e^{at}	$\frac{\sigma^2}{\varepsilon(\varepsilon - a\sigma)}$
3	$t^a, a \in \mathbb{N}$	$a!\left(\frac{\sigma}{\varepsilon}\right)^{a+2}$
4	$t^a,\ a > -1, a \in \mathbb{R}$	$\left(\frac{\sigma}{\varepsilon}\right)^{a+2}\Gamma(a+1)$
5	$\sin(at)$	$\frac{a\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2a^2)}$
6	$\cos(at)$	$\frac{\sigma^2}{\left(\varepsilon^2+\sigma^2a^2\right)}$
7	$\sinh(at)$	$\frac{\sigma^3}{\varepsilon(\varepsilon^2-\sigma^2a^2)}$
8	$\cosh(at)$	$\frac{\sigma^2}{(\varepsilon^2-\sigma^2a^2)}$

Table 3: The inverse Rishi transform of some elementary functions [25]

S.N.	$\mathcal{T}(\varepsilon, \sigma)$	$\eta(t) = \mathfrak{R}^{-1}\{\mathcal{T}(\varepsilon, \sigma)\}$
1	$\left(\frac{\sigma}{\varepsilon}\right)^2$	1
2	$\frac{\sigma^2}{\varepsilon(\varepsilon - a\sigma)}$	e^{at}
3	$\left(\frac{\sigma}{\varepsilon}\right)^{a+2}, a \in \mathbb{N}$	$\frac{t^a}{a!}$
4	$\left(\frac{\sigma}{\varepsilon}\right)^{a+2},\ a > -1, a \in \mathbb{R}$	$\frac{t^a}{\Gamma(a+1)}$
5	$\frac{\sigma^3}{\varepsilon(\varepsilon^2+\sigma^2a^2)}$	$\frac{\sin(at)}{a}$
6	$\frac{\sigma^2}{\left(\varepsilon^2+\sigma^2a^2\right)}$	$\cos(at)$
7	$\frac{\sigma^3}{\varepsilon(\varepsilon^2-\sigma^2a^2)}$	$\frac{\sinh(at)}{a}$
8	$\frac{\sigma^2}{(\varepsilon^2-\sigma^2a^2)}$	$\cosh(at)$

4 Dirac Delta function [37]

The Dirac delta function is considered as the limiting form of the function

$$\delta_\nu (t-\ell) = \begin{cases} \frac{1}{\nu}, \ell \le t \le \ell+\nu, \\ 0, \text{ otherwise}, \end{cases}$$

as $\nu \to 0$.

5 Rishi transform of the Dirac Delta function

If $\eta(t)$ is a continuous function at $t=\ell$, then
$\int_0^\infty \eta(t)\delta_\nu(t-\ell)\,dt = \int_\ell^{\ell+\nu} \eta(t)\frac{1}{\nu}dt = \eta(\beta)(\ell+\nu-\ell)\frac{1}{\nu} = \eta(\beta)$, $\ell < \beta < \ell+\nu$ by (3.1).
As $\nu \to 0$, we have $\int_0^\infty \eta(t)\delta(t-\ell)\,dt = \eta(\ell)$.
In particular, when $\eta(t) = e^{-\left(\frac{\varepsilon}{\sigma}\right)t}$, we have

$$\int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t}\delta(t-\ell)\,dt = e^{-\left(\frac{\varepsilon}{\sigma}\right)\ell}$$

$$\Rightarrow \left(\frac{\sigma}{\varepsilon}\right)\int_0^\infty e^{-\left(\frac{\varepsilon}{\sigma}\right)t}\delta(t-\ell)\,dt = \left(\frac{\sigma}{\varepsilon}\right)e^{-\left(\frac{\varepsilon}{\sigma}\right)\ell}$$

$$\Rightarrow \mathfrak{R}\{\delta(t-\ell)\} = \left(\frac{\sigma}{\varepsilon}\right)e^{-\left(\frac{\varepsilon}{\sigma}\right)\ell}$$

$$\Rightarrow \mathfrak{R}\{\delta(t)\} = \left(\frac{\sigma}{\varepsilon}\right).$$

6 Rishi transform for the solution of the nonlinear Volterra integral equation of the second kind

In this work, we have assumed the following form of non-linear Volterra integral equation of the second kind

$$\mathcal{G}(t) = \eta(t) + \int_0^t \mathcal{G}(t-x)\mathcal{G}(x)\,dx, \tag{6.1}$$

where $\mathcal{G}(t)$ and $\eta(t)$ are unknown and known functions respectively.
Operating by the Rishi transform on both the sides of (6.1), we get

$$\mathfrak{R}\{\mathcal{G}(t)\} = \mathfrak{R}\{\eta(t)\} + \mathfrak{R}\left\{\int_0^t \mathcal{G}(t-x)\mathcal{G}(x)\,dx\right\}$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = \mathfrak{R}\{\eta(t)\} + \mathfrak{R}\{\mathcal{G}(t)*\mathcal{G}(t)\} \tag{6.2}$$

Use of the convolution theorem in (6.2) gives

$$\mathfrak{R}\{\mathcal{G}(t)\} = \mathfrak{R}\{\eta(t)\} + \left(\frac{\varepsilon}{\sigma}\right)\mathfrak{R}\{\mathcal{G}(t)\}\mathfrak{R}\{\mathcal{G}(t)\}$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = \mathfrak{R}\{\eta(t)\} + \left(\frac{\varepsilon}{\sigma}\right)[\mathfrak{R}\{\mathcal{G}(t)\}]^2$$

$$\Rightarrow \left(\frac{\varepsilon}{\sigma}\right)[\mathfrak{R}\{\mathcal{G}(t)\}]^2 - \mathfrak{R}\{\mathcal{G}(t)\} + \mathfrak{R}\{\eta(t)\} = 0$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = \frac{1 \pm \sqrt{1-4\left(\frac{\varepsilon}{\sigma}\right)\mathfrak{R}\{\eta(t)\}}}{2\left(\frac{\varepsilon}{\sigma}\right)}$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)\left[1 \pm \left(\sqrt{1-4\left(\frac{\varepsilon}{\sigma}\right)\mathfrak{R}\{\eta(t)\}}\right)\right] \tag{6.3}$$

On taking the inverse Rishi transform of (6.3), the solution of (6.1) is given by

$$\mathcal{G}(t) = \mathfrak{R}^{-1}\left\{\frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)\left[1 \pm \left(\sqrt{1-4\left(\frac{\varepsilon}{\sigma}\right)\mathfrak{R}\{\eta(t)\}}\right)\right]\right\}. \tag{6.4}$$

7 Numerical applications

We discuss two numerical problems is this section for a better understanding of the complete procedure of the method of determination of the solution of nonlinear Volterra integral equation of second kind in detail by the application of the Rishi transform.

Problem 7.1. Consider the following nonlinear Volterra integral equation of the second kind

$$\mathcal{G}(t) = 1 - \frac{t}{2} + \frac{1}{2}\int_0^t \mathcal{G}(t-x)\mathcal{G}(x)\,dx, \tag{7.1}$$

which we propose to solve by applying the method of the Rishi transform.

Solution: On taking the Rishi transform of (7.1), we get

$$\mathfrak{R}\{\mathcal{G}(t)\} = \mathfrak{R}\{1\} - \frac{1}{2}\mathfrak{R}\{t\} + \frac{1}{2}\mathfrak{R}\left\{\int_0^t \mathcal{G}(t-x)\mathcal{G}(x)\,dx\right\}$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = \left(\frac{\sigma}{\varepsilon}\right)^2 - \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)^3 + \frac{1}{2}\mathfrak{R}\{\mathcal{G}(t) * \mathcal{G}(t)\} \tag{7.2}$$

Using the convolution theorem in (7.2), we have

$$\mathfrak{R}\{\mathcal{G}(t)\} = \left(\frac{\sigma}{\varepsilon}\right)^2 - \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)^3 + \frac{1}{2}\left(\frac{\varepsilon}{\sigma}\right)\mathfrak{R}\{\mathcal{G}(t)\}\,\mathfrak{R}\{\mathcal{G}(t)\}$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = \left(\frac{\sigma}{\varepsilon}\right)^2 - \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)^3 + \frac{1}{2}\left(\frac{\varepsilon}{\sigma}\right)[\mathfrak{R}\{\mathcal{G}(t)\}]^2$$

$$\Rightarrow \frac{1}{2}\left(\frac{\varepsilon}{\sigma}\right)[\mathfrak{R}\{\mathcal{G}(t)\}]^2 - \mathfrak{R}\{\mathcal{G}(t)\} + \left(\frac{\sigma}{\varepsilon}\right)^2 - \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)^3 = 0$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = \frac{1 \pm \sqrt{1 - 4\left(\frac{1}{2}\right)\left(\frac{\varepsilon}{\sigma}\right)\left[\left(\frac{\sigma}{\varepsilon}\right)^2 - \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)^3\right]}}{2\left(\frac{1}{2}\right)\left(\frac{\varepsilon}{\sigma}\right)}$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = \frac{1 \pm \sqrt{1 - 2\left[\left(\frac{\sigma}{\varepsilon}\right) - \frac{1}{2}\left(\frac{\sigma}{\varepsilon}\right)^2\right]}}{\left(\frac{\varepsilon}{\sigma}\right)}$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = \frac{1 \pm \sqrt{1 - 2\left(\frac{\sigma}{\varepsilon}\right) + \left(\frac{\sigma}{\varepsilon}\right)^2}}{\left(\frac{\varepsilon}{\sigma}\right)}$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = \frac{1 \pm \left[1 - \left(\frac{\sigma}{\varepsilon}\right)\right]}{\left(\frac{\varepsilon}{\sigma}\right)}$$

$$\Rightarrow \mathfrak{R}\{\mathcal{G}(t)\} = 2\left(\frac{\sigma}{\varepsilon}\right) - \left(\frac{\sigma}{\varepsilon}\right)^2 \text{ and } \left(\frac{\sigma}{\varepsilon}\right)^2. \tag{7.3}$$

Now taking the inverse Rishi transform of (7.3), the solutions of (7.1) are given by

$$\mathcal{G}(t) = \mathfrak{R}^{-1}\left\{2\left(\frac{\sigma}{\varepsilon}\right) - \left(\frac{\sigma}{\varepsilon}\right)^2\right\} \text{ and } \mathfrak{R}^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^2\right\},$$

$$\Rightarrow \mathcal{G}(t) = 2\mathfrak{R}^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)\right\} - \mathfrak{R}^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^2\right\} \text{ and } \mathfrak{R}^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)^2\right\},$$

$$\Rightarrow \mathcal{G}(t) = [2\delta(t) - 1] \text{ and } 1.$$

Problem 7.2. Consider the following nonlinear Volterra integral equation of the second kind

$$\mathcal{G}(t) = (4t+2)\,e^t - \int_0^t \mathcal{G}(t-x)\mathcal{G}(x)\,dx, \tag{7.4}$$

which we propose to solve by the application of the Rishi transform.

Solution: Taking the Rishi transform of (7.4), we get

$$\Re\{\mathcal{G}(t)\} = 4\Re\{te^t\} + 2\Re\{e^t\} - \Re\left\{\int_0^t \mathcal{G}(t-x)\mathcal{G}(x)\,dx\right\},$$

$$\Rightarrow \Re\{\mathcal{G}(t)\} = 4\left[\frac{\sigma^3}{\varepsilon(\varepsilon-\sigma)^2}\right] + 2\left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right] - \Re\{\mathcal{G}(t) * \mathcal{G}(t)\}. \tag{7.5}$$

Using the convolution theorem in (7.5), we have

$$\Re\{\mathcal{G}(t)\} = 4\left[\frac{\sigma^3}{\varepsilon(\varepsilon-\sigma)^2}\right] + 2\left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right] - \left(\frac{\varepsilon}{\sigma}\right)\Re\{\mathcal{G}(t)\}\Re\{\mathcal{G}(t)\}$$

$$\Rightarrow \Re\{\mathcal{G}(t)\} = 4\left[\frac{\sigma^3}{\varepsilon(\varepsilon-\sigma)^2}\right] + 2\left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right] - \left(\frac{\varepsilon}{\sigma}\right)[\Re\{\mathcal{G}(t)\}]^2$$

$$\Rightarrow \left(\frac{\varepsilon}{\sigma}\right)[\Re\{\mathcal{G}(t)\}]^2 + \Re\{\mathcal{G}(t)\} - 2\left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right] - 4\left[\frac{\sigma^3}{\varepsilon(\varepsilon-\sigma)^2}\right] = 0$$

$$\Rightarrow \Re\{\mathcal{G}(t)\} = \frac{-1 \pm \sqrt{1 - 4\left(\frac{\varepsilon}{\sigma}\right)\left[-2\left\{\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right\} - 4\left\{\frac{\sigma^3}{\varepsilon(\varepsilon-\sigma)^2}\right\}\right]}}{2\left(\frac{\varepsilon}{\sigma}\right)}$$

$$\Rightarrow \Re\{\mathcal{G}(t)\} = \frac{-1 \pm \sqrt{1 - \left[-8\left\{\frac{\sigma}{(\varepsilon-\sigma)}\right\} - 16\left\{\frac{\sigma^2}{(\varepsilon-\sigma)^2}\right\}\right]}}{2\left(\frac{\varepsilon}{\sigma}\right)}$$

$$\Rightarrow \Re\{\mathcal{G}(t)\} = \frac{-1 \pm \sqrt{1 + 8\left\{\frac{\sigma}{(\varepsilon-\sigma)}\right\} + 16\left\{\frac{\sigma^2}{(\varepsilon-\sigma)^2}\right\}}}{2\left(\frac{\varepsilon}{\sigma}\right)}$$

$$\Rightarrow \Re\{\mathcal{G}(t)\} = \frac{-1 \pm \left[1 + 4\left\{\frac{\sigma}{(\varepsilon-\sigma)}\right\}\right]}{2\left(\frac{\varepsilon}{\sigma}\right)}$$

$$\Rightarrow \Re\{\mathcal{G}(t)\} = 2\left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right] \text{ and } -\left(\frac{\sigma}{\varepsilon}\right) - 2\left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right]. \tag{7.6}$$

After taking the inverse Rishi transform of (7.6), the solutions of (7.4) are given by

$$\mathcal{G}(t) = 2\Re^{-1}\left\{\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right\} \text{ and } \Re^{-1}\left\{-\left(\frac{\sigma}{\varepsilon}\right) - 2\left[\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right]\right\},$$

$$\Rightarrow \mathcal{G}(t) = 2\Re^{-1}\left\{\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right\} \text{ and } -\Re^{-1}\left\{\left(\frac{\sigma}{\varepsilon}\right)\right\} - 2\Re^{-1}\left\{\frac{\sigma^2}{\varepsilon(\varepsilon-\sigma)}\right\},$$

$$\Rightarrow \mathcal{G}(t) = 2e^t \text{ and } \left[-\delta(t) - 2e^t\right].$$

8 Conclusions

The nonlinear Volterra integral problem of the second kind is effectively solved in compact form by the authors in this work by employing the Rishi transform. The results show that the Rishi transform is a very effective integral transform for obtaining the compact form solution of the nonlinear Volterra integral equation of the second kind without arduous and substantial computational work. In future the Rishi transform may be used to solve complicated scientific and engineering problems that may be coverted into either a single or several nonlinear Volterra integral equations of the second kind.

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