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ON EXTON'S TRIPLE HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENTS-II

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Abstract
In the present paper we generalize a result of Mac Robert T.M. [12] for the case of matrix arguments. We also define the Exton's X_4 triple hypergeometric function of matrix arguments and utilizing our earlier definition of the Exton's X_3 triple hypergeometric function of matrix arguments [10] and this generalized result of Mac Robert T.M. [12], we generalize the results of Mathur Radha [11] for the Exton's triple hypergeometric functions X_3 and X_4 of matrix arguments by using the Mathai's matrix transform technique. At the end of the paper we also state corresponding results when the argument matrices are Hermitian positive definite (i.e. the corresponding results for complex matrix arguments) for which the steps of the proofs are parallel to those given by us here. We explicitly mention here that it has become most urgent for us to state these parallel results in order to foil any attempts of plagiarism, because in the past many results of the present author which were given and proved by this author for real symmetric positive definite matrices in his Ph.D. Thesis [9, Matrix Generalizations of Multiple Hypergeometric Functions By Using Mathai's Matrix Transform Techniques (Ph.D. Thesis, Kumaun University, Nainital, Uttarakhand, India, (2004)) #1943, *IMA Preprint Series*, University of Minnesota, Minneapolis, U.S.A. (2003)
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and his other Preprints as mentioned in the references of this Ph.D. thesis of his) have been plagiarized by some authors who have published the corresponding parallel results for complex matrix arguments in their names and the name(s) of their Research Supervisor(s) simply by making the trivial modifications and following verbatim and copying blindly the steps of proofs of this author verbatim and getting them published in some Refereed Research Journals of Mathematics of the Country and they seem , to this author, to have managed to obtain their Ph.D. Degrees from some University (ies) of the Country at least to some extent on this basis. When this author made written representations to the concerned University and the concerned Research Journal against this totally unacceptable and unethical academic malpractice about six years ago and requested the concerned University and Research Journal to order a very high level probe into this plagiarism scam and redress this author's grievances - nothing so far has been received by him in writing in this connection from any quarters, therefore, this approach is followed here.

Keywords: Exton's functions, triple hypergeometric functions, matrix arguments, matrix – transform, Mac Robert T.M.
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1. INTRODUCTION

In continuation of our previous paper [10] we carry ahead our study of the Exton's triple hypergeometric functions [2] of matrix arguments. In a series of papers on this topic in the future our aim is to define the twenty Exton's triple hypergeometric functions of matrix arguments and establish some integral representations for them. Here we propose to generalize an integral given by Mac Robert T.M. [12] for the matrix variables and then utilize this result to generalize the recent results given by Mathur Radha [11] for the Exton's triple hypergeometric series X_3 and X_4. The definition of the Exton's triple hypergeometric function X_3 of matrix arguments given by us earlier in [10] shall be recalled here and we propose to define the Exton's triple hypergeometric function X_4 of matrix arguments which shall be utilized in proving the generalizations of Mathur Radha's results [11] for the case of Exton's triple hypergeometric functions of matrix arguments, with the help of the Mathai's matrix transform technique. The matrices appearing in this paper are all real symmetric and positive definite with order $(p\times p)$ (except in the section 4 of the paper where all the matrices are understood to be Hermitian positive definite to state the corresponding parallel results for the case of functions with complex matrix arguments). $A>0$ will mean that the matrix A is positive definite. $A^{1/2}$ will represent the symmetric square root of the matrix A. While integrating over matrices $\int_X f(X)dX$ represents integral over X of the scalar function $f(X)$. $\mathrm{Re}(.)$ denotes the real part of (.) and $|A|$ denotes the determinant of the matrix A.

The paper is divided into four sections. After a brief introduction of the symbols utilized in the paper in the first section, the basic preliminary results and concepts necessary for the development of the results in this paper are stated in the present section. The definitions of the Exton's triple hypergeometric functions X_3 and X_4 of matrix arguments for the real case are stated in the second section of the paper. The third section contains the statements of the generalizations of the result of T.M. Mac Robert [12] and the generalizations of results of Mathur Radha [11] and their proofs. Finally, in section four of the paper we give the corresponding results when the argument matrices are Hermitian positive definite (i.e., for the case of complex matrix arguments).

Mathai [3] in 1978 defined the matrix transform (M- transform) of a function $f(X)$ of a $(p\times p)$ real symmetric positive definite matrix X as follows:

$$M_f(\rho)=\int_{X>0}|X|^{\rho-(p+1)/2}f(X)dX \tag{1.1}$$

for $X>0$ and $\mathrm{Re}(\rho)>(p-1)/2$ whenever $M_f(s)$ exists.

The following results and definitions will be used by us at various places in this paper.

Theorem 1.1: (Mathai [4], eq.(2.24), p.23)- Let X and Y be $(p\times p)$ symmetric matrices of functionally independent real variables and A be a $(p\times p)$ non singular matrix of constants. Then,

$$Y=AXA'\Rightarrow dY=|A|^{p+1}dX \tag{1.2}$$

and

$$Y=aX\Rightarrow dY=a^{p(p+1)/2}dX \tag{1.3}$$

where a is a scalar quantity.

Theorem 1.2: Type-2 Beta integral (Mathai [5], (eq. (2.2.4), p.36 and eq. (2.1.2), p.32)-

$$B_p(\alpha,\beta)=\int_{Y>0}|Y|^{\alpha-(p+1)/2}|I+Y|^{-(\alpha+\beta)}dY=\frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.4}$$

for $\mathrm{Re}(\alpha)>(p-1)/2, \mathrm{Re}(\beta)>(p-1)/2$, where,

$$\Gamma_p(\alpha)=\pi^{p(p-1)/4}\Gamma(\alpha)\Gamma\left(\alpha-\tfrac{1}{2}\right)\cdots\Gamma\left(\alpha-\tfrac{p-1}{2}\right) \tag{1.5}$$

for $\operatorname{Re}(\alpha) > (p-1)/2$. Here $\Gamma_p(\alpha)$ represents the well known matrix variate gamma function.

Theorem 1.3: Type-1 Beta integral (Mathai [5], (eq. 2.2.2, p.34 and eq. (2.1.2), p.32)-

$$B_p(\alpha,\beta) = \int_{0<X<I} |X|^{\alpha-(p+1)/2} |I-X|^{\beta-(p+1)/2} dX = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.6}$$

for $\operatorname{Re}(\alpha) > (p-1)/2, \operatorname{Re}(\beta) > (p-1)/2$.

Theorem 1.4: Type -2 Dirichlet integral (Mathai [5], (eq. 3.1.6, p.52)- Let $X_1, \cdots X_k$ be $(p \times p)$ real symmetric positive definite matrices. Then

$$\int_{X_1>0} \cdots \int_{X_k>0} |X_1|^{\alpha_1-(p+1)/2} \cdots |X_k|^{\alpha_k-(p+1)/2} |I + X_1 + \cdots + X_k|^{-(\alpha_1+\cdots+\alpha_{k+1})} dX_1 \cdots dX_k$$

$$= \frac{\Gamma_p(\alpha_1)\cdots\Gamma_p(\alpha_{k+1})}{\Gamma_p(\alpha_1+\cdots+\alpha_{k+1})} \tag{1.7}$$

for $\operatorname{Re}(\alpha_j) > \frac{(p-1)}{2}, j = 1, \cdots, k+1.$

2. DEFINITIONS OF THE EXTON'S FUNCTIONS

We proceed to define the Exton's triple hypergeometric functions X_3 and X_4 of matrix arguments through the matrix transform (M-transform) as below:

Definition 2.1: The Exton's function $X_4 = X_4\left[\begin{matrix} a_1, a_1; a_1, a_2; a_1, a_2 \\ c_1; \quad c_2; \quad c_3 \end{matrix} \middle| -X, -Y, -Z\right]$ of matrix arguments is defined as that class of functions which has the following matrix transform (M-transform):

$$M(X_4) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times$$

$$X_4\left[\begin{matrix} a_1, a_1; a_1, a_2; a_1, a_2 \\ c_1; \quad c_2; \quad c_3 \end{matrix} \middle| -X, -Y, -Z\right] dXdYdZ$$

$$= \frac{\Gamma_p(a_1-2\rho_1-\rho_2-\rho_3)\Gamma_p(a_2-\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)} \tag{2.1}$$

for $\operatorname{Re}(a_1 - 2\rho_1 - \rho_2 - \rho_3, a_2 - \rho_2 - \rho_3, c_i - \rho_i, \rho_i) > (p-1)/2, i = 1, 2, 3.$

Definition 2.2: Upadhyaya [10, eq.(2.2), p. 34] For Exton's function

$$X_3 = X_3\left[\begin{matrix} a_1, a_1; a_1, a_2; a_1, a_2 \\ c_1; \quad c_1; \quad c_2 \end{matrix} \middle| -X, -Y, -Z\right]$$

$$M(X_3) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times$$

$$X_3\left[\begin{matrix} a_1, a_1; a_1, a_2; a_1, a_2 \\ c_1; \quad c_1; \quad c_2 \end{matrix} \middle| -X, -Y, -Z\right] dXdYdZ$$

$$= \frac{\Gamma_p(a_1-2\rho_1-\rho_2-\rho_3)\Gamma_p(a_2-\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-\rho_1-\rho_2)\Gamma_p(c_2-\rho_3)} \tag{2.2}$$

for $\operatorname{Re}(a_1 - 2\rho_1 - \rho_2 - \rho_3, a_2 - \rho_2 - \rho_3, c_1 - \rho_1 - \rho_2, c_2 - \rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2$.

3. Mac ROBERT'S INTEGRAL AND INTEGRAL REPRESENTATIONS FOR THE EXTON'S FUNCTIONS X_3 AND X_4 OF MATRIX ARGUMENTS

In this section we prove the main results of the paper which will be stated as theorems. We present the proofs of the representative results and give statements of the other results which may be established on similar lines.

Theorem 3.1: The Mac Robert's Integral of Matrix Arguments - The following is the generalization of an integral due to Mac Robert [12] (see also Mathur Radha [11, eq. (1.1) p. 2357]).

$$\int_0^I |T|^{\alpha-(p+1)/2} |I-T|^{\beta-(p+1)/2} \left|I+T^{1/2}\Lambda T^{1/2}+(I-T)^{1/2} M (I-T)^{1/2}\right|^{-\alpha-\beta} dT = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} |I+\Lambda|^{-\alpha} |I+M|^{-\beta} \tag{3.1}$$

for $\mathrm{Re}(\alpha,\beta) > \frac{p-1}{2}$.

Proof: Taking the M-transform of the left side of eq.(3.1) with respect to the variables Λ and M the parameters ρ_1 and ρ_2 respectively we obtain,

$$\int_{\Lambda>0}\int_{M>0} |\Lambda|^{\rho_1-(p+1)/2} |M|^{\rho_2-(p+1)/2} \left|I+T^{1/2}\Lambda T^{1/2}+(I-T)^{1/2} M (I-T)^{1/2}\right|^{-\alpha-\beta} d\Lambda dM \tag{3.2}$$

Applying the transformations

$$U_1 = T^{1/2}\Lambda T^{1/2}, U_2 = (I-T)^{1/2} M (I-T)^{1/2}$$

with $dU_1 = |T|^{(p+1)/2} d\Lambda, dU_2 = |I-T|^{(p+1)/2} dM$ and $|U_1| = |T||\Lambda|, |U_2| = |I-T||M|$ in eq.(3.2) and then integrating out the variables U_1 and U_2 in the resulting expression by the help of a type -2 Dirichlet integral (theorem 1.4) leads to

$$|T|^{-\rho_1} |I-T|^{-\rho_2} \frac{\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha+\beta-\rho_1-\rho_2)}{\Gamma_p(\alpha+\beta)} \tag{3.3}$$

Substituting this expression on the left side of eq.(3.1) and then integrating out T with the help of a type-1 Beta integral (theorem (1.3)) yields

$$\frac{\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\alpha-\rho_1)\Gamma_p(\beta-\rho_2)}{\Gamma_p(\alpha+\beta)} \tag{3.4}$$

Now on taking the M-transform of the right side of eq. (3.1) with respect to the variables Λ and M, and the parameters ρ_1 and ρ_2 respectively to see that

$$\frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \int_{\Lambda>0}\int_{M>0} |\Lambda|^{\rho_1-(p+1)/2} |M|^{\rho_2-(p+1)/2} |I+\Lambda|^{-\alpha} |I+M|^{-\beta} d\Lambda dM \tag{3.5}$$

in which Λ and M can be integrated out by the help of the theorem (1.2) (type-2 Beta integral) to arrive at the result (3.4). This proves the theorem 3.1.

Theorem 3.2:

$$X_3\left[\begin{matrix} a_1,a_1;a_1,a_2;a_1,a_2 \\ c_1; \quad c_1; \quad c_2 \end{matrix}\middle| -X,-Y,-Z\right] = \frac{\Gamma_p(c_1)|I+\Lambda|^d |I+M|^{c_1-d}}{\Gamma_p(c_1-d)\Gamma_p(d)} \int_0^I |T|^{d-(p+1)/2} |I-T|^{c_1-d-(p+1)/2} \times$$

$$\left|I+T^{1/2}\Lambda T^{1/2}+(I-T)^{1/2} M (I-T)^{1/2}\right|^{-c_1} \times$$

$$X_3\begin{bmatrix} a_1, a_1; a_1, a_2; a_1, a_2 | & -U, -V, -Z \\ d; \quad d; \quad c_2 \quad | & \end{bmatrix} dT \tag{3.6}$$

where,

$$U = \left\{I + T^{1/2}\Lambda T^{1/2} + (I-T)^{1/2} M (I-T)^{1/2}\right\}^{-1/2} (I+\Lambda)^{1/2} T^{1/2} X T^{1/2} (I+\Lambda)^{1/2} \times \left\{I + T^{1/2}\Lambda T^{1/2} + (I-T)^{1/2} M (I-T)^{1/2}\right\}^{-1/2} \tag{3.7}$$

and

$$V = \left\{I + T^{1/2}\Lambda T^{1/2} + (I-T)^{1/2} M (I-T)^{1/2}\right\}^{-1/2} (I+\Lambda)^{1/2} T^{1/2} Y T^{1/2} (I+\Lambda)^{1/2} \times \left\{I + T^{1/2}\Lambda T^{1/2} + (I-T)^{1/2} M (I-T)^{1/2}\right\}^{-1/2} \tag{3.8}$$

for $\mathrm{Re}(c_1 - d, d) > \frac{p-1}{2}$.

Proof: Taking the M-transform of the left side of eq. (3.6) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 respectively we arrive at,

$$\int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times X_3\begin{bmatrix} a_1, a_1; a_1, a_2; a_1, a_2 | & -U, -V, -Z \\ d; \quad d; \quad c_2 \quad | & \end{bmatrix} dXdYdZ \tag{3.9}$$

where U and V are given respectively by eq. (3.7) and eq.(3.8).

Applying the transformations, $X_1 = U, Y_1 = V, Z_1 = Z$ with

$$dX_1 = \left|I + T^{1/2}\Lambda T^{1/2} + (I-T)^{1/2} M (I-T)^{1/2}\right|^{-\frac{(p+1)}{2}} |I+\Lambda|^{\frac{(p+1)}{2}} |T|^{\frac{(p+1)}{2}} dX$$

$$dY_1 = \left|I + T^{1/2}\Lambda T^{1/2} + (I-T)^{1/2} M (I-T)^{1/2}\right|^{-\frac{(p+1)}{2}} |I+\Lambda|^{\frac{(p+1)}{2}} |T|^{\frac{(p+1)}{2}} dY; \qquad dZ_1 = dZ$$

(keeping in mind the values of U and V given by eq. (3.7) and eq.(3.8) and utilizing the theorem (1.1)) and

$$|X_1| = \left|I + T^{1/2}\Lambda T^{1/2} + (I-T)^{1/2} M (I-T)^{1/2}\right|^{-1} |I+\Lambda||T||X|$$

$$|Y_1| = \left|I + T^{1/2}\Lambda T^{1/2} + (I-T)^{1/2} M (I-T)^{1/2}\right|^{-1} |I+\Lambda||T||Y|; \qquad |Z_1| = |Z|$$

to the eq. (3.9) and writing the M-transform of the X_3 function with the help of definition (2.2) yields

$$\left|I + T^{1/2}\Lambda T^{1/2} + (I-T)^{1/2} M (I-T)^{1/2}\right|^{\rho_1+\rho_2} |I+\Lambda|^{-\rho_1-\rho_2} |T|^{-\rho_1-\rho_2} \frac{\Gamma_p(a_1 - 2\rho_1 - \rho_2 - \rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)} \times \frac{\Gamma_p(a_2 - \rho_2 - \rho_3)\Gamma_p(d)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(d - \rho_1 - \rho_2)\Gamma_p(c_2 - \rho_3)} \tag{3.10}$$

Substituting this expression on the right side of eq.(3.6) and integrating out T in the consequent integral with the help of the Mac Robert's integral (theorem (3.1)) yields $M(X_3)$ as given by eq. (2.2), thereby proving the theorem. This theorem generalizes the result of eq.(2.1) p. 2358 of Mathur Radha [11] for the case of functions of matrix arguments.

In a similar manner, by utilizing the definition (2.1) we can prove the following matrix generalization of the result of eq.(2.2) p. 2358 of Mathur Radha [11].

Theorem 3.3:

$$X_4\left[\begin{matrix} a_1,a_1;a_1,a_2;a_1,a_2 \\ c_1; \quad c_2; \quad c_3 \end{matrix} \middle| -X,-Y,-Z\right]=\frac{\Gamma_p(b)\left|I+\Lambda\right|^{a_2}\left|I+M\right|^{b-a_2}}{\Gamma_p(b-a_2)\Gamma_p(a_2)}\int_0^I \left|T\right|^{a_2-(p+1)/2}\left|I-T\right|^{b-a_2-(p+1)/2}\times$$

$$\left|I+T^{1/2}\Lambda T^{1/2}+\left(I-T\right)^{1/2}M\left(I-T\right)^{1/2}\right|^{-b}\times$$

$$X_4\left[\begin{matrix} a_1,a_1;a_1,b;a_1,b \\ c_1; \quad c_2; \quad c_3 \end{matrix} \middle| -X,-V,-W\right]dT \tag{3.11}$$

where, V is given by eq.(3.8) and

$$W=\left\{I+T^{1/2}\Lambda T^{1/2}+\left(I-T\right)^{1/2}M\left(I-T\right)^{1/2}\right\}^{-1/2}\left(I+\Lambda\right)^{1/2}T^{1/2}ZT^{1/2}\left(I+\Lambda\right)^{1/2}\times$$
$$\left\{I+T^{1/2}\Lambda T^{1/2}+\left(I-T\right)^{1/2}M\left(I-T\right)^{1/2}\right\}^{-1/2} \tag{3.12}$$

for $\operatorname{Re}\left(b-a_2,a_2\right)>\frac{p-1}{2}$.

Theorem 3.4:

$$X_4\left[\begin{matrix} a_1,a_1;a_1,a_2;a_1,a_2 \\ c_1; \quad c_2; \quad c_3 \end{matrix} \middle| -X,-Y,-Z\right]=\frac{\Gamma_p(b)}{\Gamma_p(b-a_2)\Gamma_p(a_2)}\int_0^I \left|T\right|^{a_2-(p+1)/2}\left|I-T\right|^{b-a_2-(p+1)/2}\times$$

$$X_4\left[\begin{matrix} a_1,a_1;a_1,b;a_1,b \\ c_1; \quad c_2; \quad c_3 \end{matrix} \middle| -X,-T^{1/2}YT^{1/2},-T^{1/2}ZT^{1/2}\right]dT \tag{3.13}$$

for $\operatorname{Re}\left(b-a_2,a_2\right)>\frac{p-1}{2}$.

Proof: We take the M-transform of the right side of eq.(3.13) w.r.t. the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively to obtain

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\left|X\right|^{\rho_1-(p+1)/2}\left|Y\right|^{\rho_2-(p+1)/2}\left|Z\right|^{\rho_3-(p+1)/2}\times$$
$$X_4\left[\begin{matrix} a_1,a_1;a_1,b;a_1,b \\ c_1; \quad c_2; \quad c_3 \end{matrix} \middle| -X,-T^{1/2}YT^{1/2},-T^{1/2}ZT^{1/2}\right]dXdYdZ \tag{3.14}$$

Appealing to the transformations

$$X_1=X,Y_1=T^{1/2}YT^{1/2},Z_1=T^{1/2}ZT^{1/2}$$

which yields,

$$dX_1=dX,dY_1=\left|T\right|^{(p+1)/2}dY,dZ_1=\left|T\right|^{(p+1)/2}dZ$$

and $\left|X_1\right|=\left|X\right|,\left|Y_1\right|=\left|T\right|\left|Y\right|,\left|Z_1\right|=\left|T\right|\left|Z\right|$ in the eq.(3.14) and then writing the M-transform of the X_4 function involved, with the help of definition (2.1) leads us to

$$|T|^{-\rho_2-\rho_3}\frac{\Gamma_p(a_1-2\rho_1-\rho_2-\rho_3)\Gamma_p(b-\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(a_1)\Gamma_p(b)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)} \quad (3.15)$$

This expression on substituting on the right side of eq.(3.13) and integrating out T by using a type-1 Beta integral (theorem (1.3)) leads finally to $M\left(X_4\right)$ as given by eq. (2.1). This proves the theorem. It is pertinent to mention here that the theorem (3.4) provides a generalization of the result of eq.(4.2) p. 2359 of Mathur Radha[11].

Similarly the following theorem can be proved, which generalizes the result given in eq.(4.1) p. 2359 of Mathur Radha [11]:

Theorem 3.5:

$$X_3\left[\begin{matrix} a_1,a_1;a_1,a_2;a_1,a_2 \\ c_1; \quad c_1; \quad c_2 \end{matrix}\middle| -X,-Y,-Z\right]=\frac{\Gamma_p(c_1)}{\Gamma_p(d)\Gamma_p(c_1-d)}\int_0^I |T|^{d-(p+1)/2}|I-T|^{c_1-d-(p+1)/2}\times$$
$$X_3\left[\begin{matrix} a_1,a_1;a_1,a_2;a_1,a_2 \\ d; \quad d; \quad c_2 \end{matrix}\middle| -T^{1/2}XT^{1/2},-T^{1/2}YT^{1/2},-Z\right]dT \quad (3.16)$$

for $\operatorname{Re}\left(c_1-d,d\right)>\frac{p-1}{2}$.

4. CORRESPONDING RESULTS FOR FUNCTIONS WITH COMPLEX MATRIX ARGUMENTS

As mentioned in the abstract, now we mention the corresponding results when the argument matrices are Hermitian positive definite matrices. All the matrices appearing in this section of the paper are Hermitian positive definite matrices of order $\left(p\times p\right)$. We use the same notation for matrices as used in the previous sections of this paper, unlike Mathai [13], where matrices having complex entries are shown by placing a tilde $(\sim)$ sign over the notation of the matrix concerned. For results concerning the Jacobians of matrix transformations in the case of matrices when their elements are complex quantities, we refer the reader to Chapter 3 of Mathai [13]. The analogues of all the results mentioned in eqs. (1.1) to (1.7) of this paper for the corresponding cases of complex matrices can be found in Chapters 3 and 6 of Mathai [13]. This fact is of utmost importance for us to state here that once the results for functions of matrix arguments in the real case (with real symmetric positive definite matrices as arguments) are known then the corresponding results for the case of functions of matrix arguments in the complex case (with Hermitian positive definite matrices as arguments) can be most easily written merely by replacing the factor $\frac{p+1}{2}$ appearing in the power of the determinant of the matrix concerned in the integral of the function in the real case with the factor p in the power of the determinant of the matrix concerned in the corresponding integral of the function in the corresponding complex case and for the condition of convergence in the real case, which is stated as $\operatorname{Re}(.)>\frac{p-1}{2}$, the corresponding condition of convergence for the complex matrix argument case is $\operatorname{Re}(.)>p-1$. This seemingly very simple observation is very well established in the literature, for instance we refer the reader to Mathai [13] (see, for instance, Mathai [13], pp.364-365) and the various references mentioned therein. The author wants to emphasize and stress here with full force that with these two very trivial modifications almost all the results which were given and proved by this author in his doctoral dissertation [9] and various preprints mentioned in the references of this dissertation and about which a very clear proclamation was made by him in his paper [7], even then some research scholars along with their research supervisor(s) in some University(ies)/College(s) of our country have published the corresponding parallel results in some refereed research journal(s) of our country, some of which are even recognized by the Currently Approved U.G.C. List of Journals for the year 2017 – in their own names and in the names of their Research Supervisor(s) and in all these papers these authors have merely made these two aforesaid modifications in the results of this author and reproduced verbatim all his proofs which can be found either in his doctoral dissertation or in a number of his preprints which are all available at website of the I.M.A. Preprint Series, University of Minnesota, Minneapolis, U.S.A. (www.ima.umn.edu/preprints) during the months of November 2001 till November 2003 [7, 8, 9] and the CiteseerX of the University of Pennsylvania, U.S.A. besides a number of other websites also. This act, in the view of this author, amounts to a grave case of plagiarism. About six years ago, this author had sent his written representations against this act (by speed posts) to - the concerned University of the country and one of the concerned refereed research journals of the country demanding a very high level probe into the incident and for a proper redressal of his grievances, but till date he has not received anything in black and white in this connection from any quarters. This author had also demanded the photocopies of the Ph.D. theses of these Concerned Research Scholars form the Concerned University under the Right to Information Act of 2005 in order to assess completely the extent to which the results from his Ph.D. thesis [9] and other Preprints as mentioned in the references of this thesis [9] had been plagiarized but neither the Registrar nor the Vice Chancellor of the Concerned University have so far, ever, bothered to look into this matter of grave concern to the Academia of our Country and neither of these two Authorities of the Concerned University have so far bothered to inform this author of the results of the very high level probe, that this author had requested for, to look into this extremely serious matter nor was this author provided with the photocopies of the Ph.D. theses of the said two/three research scholars by the said Concerned University instead, this author was very strangely asked by the said University to 'see the said Ph.D. theses of the said candidates in the Research Section of the University during the office hours'! It is obvious that (due to various reasons which also includes, besides other things, the security of life) the author chose not to go to the headquarters of that University which is situated far off from this author's work place so he made yet another written representation dated 25th March 2012 against this order of the said University to the then Vice Chancellor of that University but the same representation has not been answered till date! It is also very obvious that the Concerned Authorities of the Concerned University would have definitely chosen to suppress this matter in the files - once very serious questions had been raised by this author on the originality and genuineness of the research work carried out and published by the said authors (who were all enrolled as Research Scholars from the said Concerned University and all of them had already been awarded their Ph.D. degrees in Mathematics by the said University by the time this author came to know of this plagiarism scam and sent his written representations against this scam to the said University) and their Research Supervisor in their names, when invincible evidences and proofs and irrefutable arguments had been sent by this author to the Registrar of the Concerned University in about two hundred twenty pages with his RTI (Right To Information (Act)) Application dated 15th December 2011. (This author definitely presumes that the said University has hushed up the matter of inquiry because - had the Concerned University ever conducted any inquiry into the matter- it was under a moral and legal obligation to inform this author in writing about the outcome of the inquiry and the action taken by its Authorities on the Report of the Inquiry Officer(s) - this belief of this author is based on the fact that for the past six years he has never received anything in writing about this matter from the Concerned University). Some two months back in September 2017, while searching the internet for certain research papers, this author was again shocked to find that two of the very same authors, who had earlier plagiarized and published the results of this authors' Ph.D. thesis [9] and his related Preprints had again published another similar plagiarized paper from the doctoral thesis [9] of this author in a yet another Refereed Research Journal of Mathematics of the Country in 2015 and to his utter dismay this author also discovered further that still another entirely new set of authors- who are different from the said two/three earlier authors- and also belonged to a different University of the Country had also published two plagiarized papers of this authors' doctoral thesis [9] and the related Preprints in still another Refereed Research Journal of Mathematics of our Country and that too in the year 2015! This author came to learn of these new episodes of plagiarism only in the month of September 2017 and in all these said papers of all these said authors- they have merely published the results of this author only for the case of 'functions of complex matrix arguments with simply the very same two trivial modifications as mentioned supra and in the proofs of these so called 'new results' all these authors have blindly copied verbatim the steps of proofs given by the present author in his doctoral dissertation [9] and its related Preprints ' while the present author had already given and proved these results for the case of functions of real matrix arguments way back during the years 2001-2003! All these utterly shocking incidents, which definitely bring a very bad name to the Country's Research Academia in the entire World of Academic Research have forced this author now to give the corresponding results for functions of matrix arguments in the complex case, here in this paper itself, with a view to foil any such attempt(s) of such a plagiarism in the future. It is extremely painful for this author to mention here that neither the examiners of the doctoral dissertations of these authors (which, in the view of this author, may most probably contain these plagiarized results as parts of some of the chapter(s) of these so called 'doctoral dissertations') nor the esteemed anonymous referee(s) of the concerned research journal(s) had been able to catch sight of this extremely simple manipulation as stated above in this paragraph. With these facts in the background, we now proceed to state the following results without proof for the case of complex matrix arguments. The author wants to point out here that, the steps of the proofs of all these results in the case of complex matrix arguments are exactly parallel to those in the case of real symmetric matrix arguments. For an authentic treatment of this statement the interested reader is referred by this author to Chapters 5 and 6 of Mathai [13].

Keeping in mind the statement made in the preceding paragraph, it is obvious that the definitions (2.1) and (2.2) in the case of complex matrix arguments would be rewritten respectively as follows in eqs. (4.1) and (4.2). For brevity we give only the mathematical equations without any formal statements which are easily understood in the relevant context. It being understood in this section that $|A|$ now represents the absolute value of the determinant of the matrix A of complex elements.

Definition 4.1: The Exton's function $X_4 = X_4\left[\begin{matrix} a_1, a_1; a_1, a_2; a_1, a_2 \,| \\ c_1; \quad c_2; \quad c_3 \,| \end{matrix} -X, -Y, -Z\right]$ of complex matrix arguments is defined as that class of functions which has the following matrix transform (M-transform):

$$M\left(X_4\right) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-p}|Y|^{\rho_2-p}|Z|^{\rho_3-p} \times$$
$$X_4\left[\begin{matrix} a_1, a_1; a_1, a_2; a_1, a_2 \,| \\ c_1; \quad c_2; \quad c_3 \,| \end{matrix} -X, -Y, -Z\right] dXdYdZ$$
$$= \frac{\Gamma_p(a_1-2\rho_1-\rho_2-\rho_3)\Gamma_p(a_2-\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2)\Gamma_p(c_3-\rho_3)} \quad (4.1)$$

for $\operatorname{Re}(a_1-2\rho_1-\rho_2-\rho_3, a_2-\rho_2-\rho_3, c_i-\rho_i, \rho_i) > (p-1), i = 1,2,3.$

Definition 4.2:

$$M\left(X_3\right) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-p}|Y|^{\rho_2-p}|Z|^{\rho_3-p} \times$$
$$X_3\left[\begin{matrix} a_1, a_1; a_1, a_2; a_1, a_2 \,| \\ c_1; \quad c_1; \quad c_2 \,| \end{matrix} -X, -Y, -Z\right] dXdYdZ$$
$$= \frac{\Gamma_p(a_1-2\rho_1-\rho_2-\rho_3)\Gamma_p(a_2-\rho_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-\rho_1-\rho_2)\Gamma_p(c_2-\rho_3)} \quad (4.2)$$

for $\operatorname{Re}(a_1-2\rho_1-\rho_2-\rho_3, a_2-\rho_2-\rho_3, c_1-\rho_1-\rho_2, c_2-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1).$

The following are respectively the mathematical statements of the theorems 3.1 to 3.5 for the corresponding results for the cases of functions of complex matrix arguments:

Theorem 4.3: The Mac Robert's Integral of Complex Matrix Arguments -

$$\int_0^I |T|^{\alpha-p}|I-T|^{\beta-p}\left|I+T^{1/2}\Lambda T^{1/2}+(I-T)^{1/2}M(I-T)^{1/2}\right|^{-\alpha-\beta} dT$$

$$=\frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)}\left|I+\Lambda\right|^{-\alpha}\left|I+M\right|^{-\beta} \tag{4.3}$$

for $\operatorname{Re}(\alpha,\beta)>(p-1)$.

Theorem 4.4:

$$\tilde{X}_3\left[\begin{matrix}a_1,a_1;a_1,a_2;a_1,a_2 \mid \\ c_1; \quad c_1; \quad c_2 \quad \mid\end{matrix}-X,-Y,-Z\right]=\frac{\Gamma_p(c_1)\left|I+\Lambda\right|^{d}\left|I+M\right|^{c_1-d}}{\Gamma_p(c_1-d)\Gamma_p(d)}\int_0^I\left|T\right|^{d-p}\left|I-T\right|^{c_1-d-p}\times$$

$$\left|I+T^{1/2}\Lambda T^{1/2}+\left(I-T\right)^{1/2}M\left(I-T\right)^{1/2}\right|^{-c_1}\times$$

$$\tilde{X}_3\left[\begin{matrix}a_1,a_1;a_1,a_2;a_1,a_2 \mid \\ d; \quad d; \quad c_2 \quad \mid\end{matrix}-U,-V,-Z\right]dT \tag{4.4}$$

for $\operatorname{Re}(c_1-d,d)>(p-1)$ where U and V are the same as given by eqs. (3.7) and (3.8) with the necessary modifications in this context being understood here.

Theorem 4.5:

$$\tilde{X}_4\left[\begin{matrix}a_1,a_1;a_1,a_2;a_1,a_2 \mid \\ c_1; \quad c_2; \quad c_3 \quad \mid\end{matrix}-X,-Y,-Z\right]=\frac{\Gamma_p(b)\left|I+\Lambda\right|^{a_2}\left|I+M\right|^{b-a_2}}{\Gamma_p(b-a_2)\Gamma_p(a_2)}\int_0^I\left|T\right|^{a_2-p}\left|I-T\right|^{b-a_2-p}\times$$

$$\left|I+T^{1/2}\Lambda T^{1/2}+\left(I-T\right)^{1/2}M\left(I-T\right)^{1/2}\right|^{-b}\times$$

$$\tilde{X}_4\left[\begin{matrix}a_1,a_1;a_1,b;a_1,b \mid \\ c_1; \quad c_2; \quad c_3 \quad \mid\end{matrix}-X,-V,-W\right]dT \tag{4.5}$$

where, V is given by eq.(3.8) and W by eq.(3.12) with the necessary modifications in this context being understood here, and for $\operatorname{Re}(b-a_2,a_2)>(p-1)$.

Theorem 4.6:

$$\tilde{X}_4\left[\begin{matrix}a_1,a_1;a_1,a_2;a_1,a_2 \mid \\ c_1; \quad c_2; \quad c_3 \quad \mid\end{matrix}-X,-Y,-Z\right]=\frac{\Gamma_p(b)}{\Gamma_p(b-a_2)\Gamma_p(a_2)}\int_0^I\left|T\right|^{a_2-p}\left|I-T\right|^{b-a_2-p}\times$$

$$\tilde{X}_4\left[\begin{matrix}a_1,a_1;a_1,b;a_1,b \mid \\ c_1; \quad c_2; \quad c_3 \quad \mid\end{matrix}-X,-T^{1/2}YT^{1/2},-T^{1/2}ZT^{1/2}\right]dT \tag{4.6}$$

for $\operatorname{Re}(b-a_2,a_2)>(p-1)$.

Theorem 4.7:

$$\tilde{X}_3\left[\begin{matrix}a_1,a_1;a_1,a_2;a_1,a_2 \mid \\ c_1; \quad c_1; \quad c_2 \quad \mid\end{matrix}-X,-Y,-Z\right]=\frac{\Gamma_p(c_1)}{\Gamma_p(d)\Gamma_p(c_1-d)}\int_0^I\left|T\right|^{d-p}\left|I-T\right|^{c_1-d-p}\times$$

$$X_3\left[\begin{matrix} a_1, a_1; a_1, a_2; a_1, a_2 \\ d; \quad d; \quad c_2 \end{matrix} \middle| -T^{1/2}XT^{1/2}, -T^{1/2}YT^{1/2}, -Z\right] dT \qquad (4.7)$$

for $\operatorname{Re}(c_1 - d, d) > (p-1)$.

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