Original Article

Bulletin of Pure and Applied Sciences.
Vol.36E (Math & Stat.),No.2, 2017.P.227-233
Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2017.00025.X

APPLICATION OF THE DIFFERENTIAL TRANSFORMATION METHOD IN THE STUDY OF BLOOD FLOW IN HUMAN ARTERIES

Radhika TSL*, Raja Rani T, Brindha S***

Author Affiliation:
*Department of Mathematics, Birla Institute of Technology & Science, Pilani, Hyderabad, Telangana 500078, India
**Department of General Studies, Military Technological College (MTC), Portsmouth University, UK
Corresponding Author:
Dr. (Mrs) Radhika TSL, Assistant Professor,Department of Mathematics,Birla Institute of Technology & Science, Pilani, Hyderabad, Telangana 500078, India

E-mail: radhikatsl@hyderabad.bits-pilani.ac.in

Received on 28.09.2017, **Accepted on** 28.10.2017

Abstract

In this paper, we applied the differential transformation method to solve the mathematical model developed for the study of blood flow in human arteries. In our model, we have assumed power law fluid model for human blood and the blood flow as a pulsatile flow. This model resulted in a non-linear, unsteady partial differential equation (PDE) in terms of the fluid (or blood) velocity. This PDE with appropriate initial-boundary conditions was solved using the differential transformation method. The differential transformation method is an iterative method that generates a recurrence relation involving lower order approximations of the velocity function, zeroth approximation being the initial approximation. We then, obtained an expression for the velocity of blood flow and used this expression to predict the flow velocity in the radial artery of the human. Data required for this prediction i.e data related to radial artery, values of the parameters in the power law model and also data for simulating pulsatile flow are taken from published results. Through the velocity plots, we could observe a natural mechanism of the human circulatory system: Decrease in the volume of the blood flow increases the heart rate which is to compensate the deficiency in the supply of required amount of blood to the various parts of the body. We also presented and discussed other results.

Keywords:Non-linear PDE, Differential Transformation method, non-Newtonian fluid, Power law fluid.

1. INTRODUCTION

The study of the mechanism of blood circulation in human is vital in a better understanding of the clinical and pathological observations on the human circulatory system. Thus the problem of blood flow in arteries has become a problem of great interest to Physicists, Biologists, Engineers as well as Mathematicians down the time for several decades. In literature, we find the following approaches to solve this problem - the Engineering models, Clinical methods and Mathematical models. References [1-9] present the early work on understanding the physics of the blood flow in humans using each of these three methods. While Bernoulli worked on the mathematical models, Poiseuilli, Young and others worked on the theoretical and experimental methods. Hales was the first to design an engineering model called the lumped parameter model for this study. It was not until the early nineteenth century that much rigorous work has been initiated. Considerable work has been reported on these methods [10- 18] down the years, but still there is a growing need for more research in this area for the reason that many diseases that affect the functioning of circulatory system are life style diseases that develop over time depending on the lifestyle choices a person makes. Thus, one needs to modify or reconstruct the models from time to time to arrive at better results.

Our focus in this paper is on developing a mathematical model that not only gives an expression for the flow velocity but also on applying this model to understand the flow dynamics in a specific artery. In our model, we have described blood as a non-Newtonian fluid (Power law fluid model) [19, 20]. We also modeled the pulsatile nature of the blood flow as described in [21]. The resultant mathematical model with appropriate boundary and initial conditions has been solved using the differential transformation method. The results are then applied to predict the velocity of blood flow in the radial artery (an artery in the human arm). An interesting feature of our present study is that, we could observe a natural mechanism that is being adopted by the human circulatory system that: " The human circulatory system can regulate itself so as to attain normalcy unless some pathological conditions develop"[16].

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Let us consider an unsteady (time-dependent) flow of a fluid (blood in our case) in a circular tube (artery). We consider the cylindrical polar coordinate system (r, θ, z) (where *r* and *z* are the radial and axial coordinates, respectively, and θ is the azimuthal angle) to describe the geometry of the problem, and assume the fluid flow to be in the *z* direction as shown in Fig 1. Further, the flow is taken to be axi-symmetric so that the flow variables i.e the velocity and pressure are independent of θ .

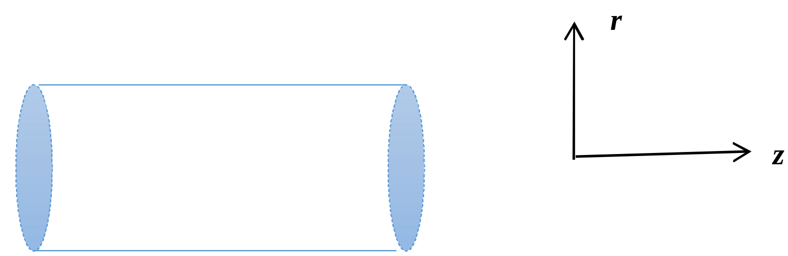

Fig. 1: Schematic diagram of blood flow in tube

Let the only non-vanishing component of the fluid velocity be in the *z*- direction and taken as $w(r,z,t)$ and the thermodynamicpressure is $p(r,z,t)$, where*t* is the time. The continuity and momentum equations take the form:

$$\frac{\partial w}{\partial z} = 0 \quad (1)$$

$$-\frac{\partial p}{\partial r} = 0 \quad (2)$$

$$\rho \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 w}{\partial r^2} + T_{rx} \quad (3)$$

where μ is the viscosity of the fluid (blood in our case) and T_{rx} is the *z*- component of stress tensor (T_{ij})in *r*-direction.

Since the flow of blood in arteries is due to a pulsatile pressure gradient, we adopt the following model to describe blood flow.

$$-\frac{\partial p}{\partial z} = a_0 + a_1 \cos \omega t, t > 0, \tag{4}$$

where $p(z,t)$ is the pressure (in view of equation (2), pressure is independent of *r*), a_0 is the constant amplitude of the pressure gradient, a_1 is the amplitude of the pulsatile component giving rise to systolic and diastolic pressures, $\omega = 2\pi f$, f being the number of heart beats per minute [5].
Now, for modeling blood, we take the power law model whose constitutive equation is given by

$$T_{rz} = \kappa\left(-\frac{\partial w}{\partial r}\right)^n \tag{5}$$

where
T_{rz} is the stress component, κ is the consistency index which is a measure of the consistency of the substance, n is the power law index [21].

Using expression (5) in equation (3), we have

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - \frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\left(-\frac{\partial w}{\partial r}\right)^n\right) \tag{6}$$

and the conditions are:

$w(r,t) = 0$ on $r = R$ (no-slip boundary condition)

at t=0, $w(r,0) = w_0(r)$ which is the time independent solution of equation (6). (7)

This initial condition is computed by assuming no-slip boundary condition at $r = R$ with $-\frac{\partial p}{\partial z}$ as the pressure gradient given by expression (4) when *t*=0. After straight forward calculations, we get

$$w_0(r) = \left(-\frac{R}{2\kappa}\left(\frac{\partial p}{\partial z}\right)_{t=0}\right)^{1/n} \frac{R}{1/n+1}\left(1-\left(\frac{r}{R}\right)^{1/n+1}\right) \tag{8}$$

2.1 Solution to the Problem using the Differential Transformation method[22-25]

Using the differential transformation method for equation (6), we get the recursive relation as

$$(k+1)w_{k+1}(r) = \frac{1}{\rho}\left(a_0\delta(k-0) + a_1\left(\frac{(iw)^k + (-i\omega)^k}{2\times k!}\right) - \frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\left(-\frac{\partial w_k(r)}{\partial r}\right)^n\right)\right) \tag{9}$$

where the initial approximation is given by expression (8).

For $k = 0$, $$w_1(r) = \frac{1}{\rho}\left(a_0 + a_1 - \frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\left(-\frac{\partial w_0(r)}{\partial r}\right)^n\right)\right) = 0 \tag{10}$$

For $k = 1$, $$w_2(r) = \frac{1}{2\rho}\left(-\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\left(-\frac{\partial w_1(r)}{\partial r}\right)^n\right)\right) = 0 \tag{11}$$

For $k=2$, $w_3(r)=\frac{1}{3\rho}\left(-\frac{a_1\omega^2}{2}-\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\left(-\frac{\partial w_2(r)}{\partial r}\right)^n\right)\right)=-\frac{a_1\omega^2}{6\rho}$ (12)

For $k=3$, $w_4(r)=\frac{1}{3\rho}\left(-\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\left(-\frac{\partial w_3(r)}{\partial r}\right)^n\right)\right)=0$ (13)

For $k=4$, $w_5(r)=\frac{1}{5\rho}\left(\frac{a_1\omega^4}{2\times 4!}-\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\left(-\frac{\partial w_3(r)}{\partial r}\right)^n\right)\right)=\frac{a_1\omega^4}{10\times 4!}$ (14)

In this way,we find *m* approximations of the solution to equation (6) and thus an *m*-approximate solution to the velocity function is $\tilde{w}(r,t)=\sum_{k=0}^{m} w_k(r)t^k$. The exact solution is then given by $w(r,t)=\lim_{m\to\infty}\tilde{w}(r,t)$. We developed a code in MATHEMATICA software to compute the first 10 approximations and the plots for the approximate solution for the velocity function have been presented below.

3. APPLICATION OF THE PRESENT MODEL FOR HUMAN CIRCULATORY SYSTEM

To apply our model to the human artery, data related to the radial artery is taken from published work [26, 27]. Other data such as Blood pressure (BP) and heart rate are noted for a healthy Indian individual and is presented in Table-1. Also the coefficients in the power law model are taken for a healthy individual as cited in [28].

Table 1: Experimental Data from literature

BP (sys/ dias) *mmHg*	120/80
Heart Rate/min	72
Radius of radial artery R	1.6 mm
Density of blood ρ	1050 kg/m^3
Consistency inded κ	0.98Pa.S^n
Power law index *n*	0.708
Length of the radial artery *l*	20 cm-25 cm

Table 2: Data calculated to simulate the model

a_0 (*Pa/m*)	12441.3
a_1 (*Pa/m*)	5332
Flexibility coefficient α (*Pa*$^{-1}$)	2x 10(-7)

The constant and pulsatile components of the pressure gradient given in Table 2 (a_0 and a_1, respectively) have been estimated (using data from Table 1) using the systolic (*sys*) and diastolic (*dias*) components of the blood pressure (BP) as:

$$a_0=\left(\frac{1}{3}sys+\frac{2}{3}dias\right)/l$$
$$a_1=(sys-dias)/l \quad (15)$$

Where *l* is taken to be 25*cm*.

For finding the flexibility coefficient range specified in Table 2 we carried out simulations in MATHEMATICA. This value has been found in such a way so as to get values of the radius of the artery*R*, that donot deviate much from the average radius of the radial artery (of a healthy individual) . We may note that, if theradius of the radial artery is known for an individual, the flexibility coefficient αcan be determined accurately for that particular individual.

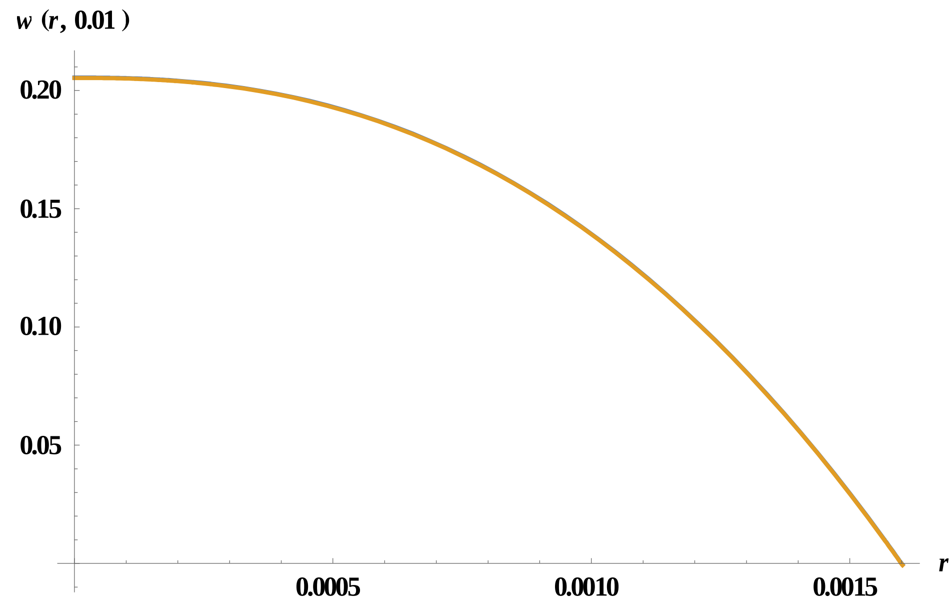

Fig. 2: Plot of velocity of blood flow versus the radius of the artery at t=0.01 sec

Figure 2shows the velocity profile of blood flow (heart rate is 72 beats/min) in a radial artery. The velocity of the flow is maximum at the center and decreases to zero at the boundary of the artery. Further, the flow profile is flatter in the middle and decays faster towards the wall which is a the behavior of blood like fluid. It is interesting to note that the present model could efficiently predict this behavior.

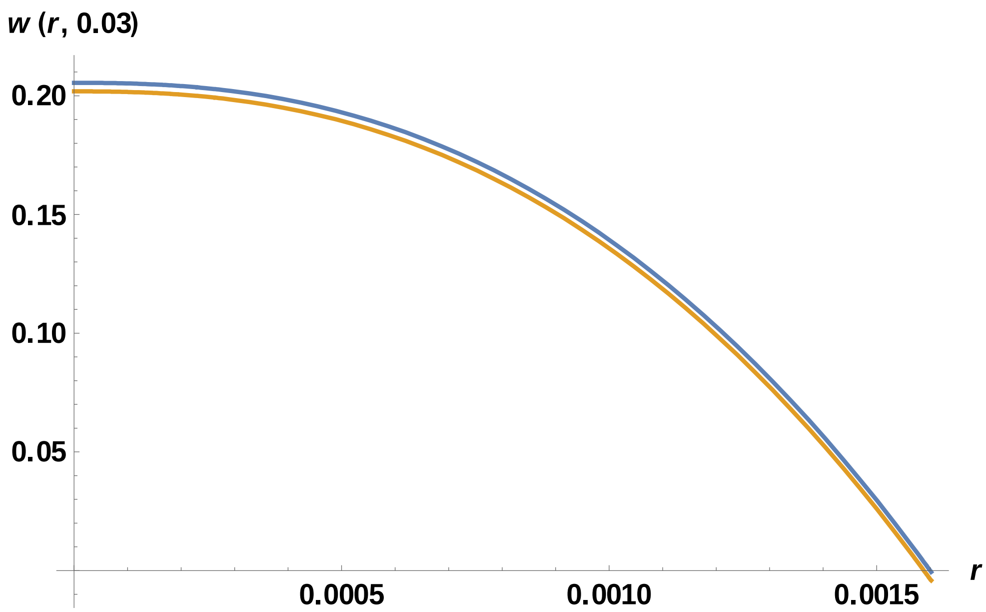

Fig. 3: Plot of velocity of blood flow versus radius of the artery at t=0.03 sec

Yellow line indicates velocity of blood flow when heartbeat is 72 beats/min

Blue line indicates velocity of blood flow when heartbeat is 60 beats/min

It is clear from the Fig. 3 that when the heartbeat is more, the velocity of the blood flow is less. Since the volumetric flux or the volume of blood flow is in terms of the velocity, we can thus conclude that when the volume of the blood flow is less (i.e in case of in-sufficient stroke volume), the frequency of heart beat increases so as to compensate the deficient blood supply to the different parts of the body.

4. CONCLUSION

In this paper, we presented a mathematical model to calculate the velocity of blood flow in human artery. The model uses a non-Newtonian model called Power law model for describing human blood and the blood flow as pulsatile. Under certain realistic assumptions that are valid for arteries such as radial artery, this model resulted in a non-linear time dependent PDE (in terms of blood velocity), which, together with appropriate initial- boundary conditions has been solved using the differential transformation method. The model is then applied to understand the flow dynamics of the human radial artery. An important observation made in our study is that when the velocity of the blood flow is low, heart rate increases in order to compensate the deficiency of blood supply to the different parts of the body.

REFERENCES

1. Mc. Donald D A., 1955. "The relation of pulsatile pressure to flow in arteries ", *J. Physiol*, 127, pp 533-552.
2. Womersley J R., 1955. "Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known", *J. Physiol*, 127, pp553-563.
3. Womersley J R., 1957. "An elastic tube theory of pulse transmission and oscillatory flow in mammalian arteries", *W.A.D.C Technical report*.
4. Rudinger G., 1966." Review of current mathematical methods for the analysis of blood flow", *Proc. Biomed Fluid Mech. Symposium*, ASME, NewYork,
5. Burton A.C., 1966. "Physiology and Bio Physics of the circulation", introductory text, *Year Book Medical Publisher*, Chicago.
6. Goldwyn RM, Watt T., 1967. "Arterial pressure pulse contour analysis via a mathematical model for the clinical quantification of human vascular properties" *IEEE* Trans Biomed Eng., 14: pp 11–17.
7. Westerhof N, Bosman F, De Vries CJ et al., 1969. "Analog studies of the human systemic arterial tree", *J Biomech*, 2: pp121–143.
8. Ling S C., Atabek H B., 1972. "A non- linear analysis of pulsatile flow in arteries", *J Fluid Mech*, 55, pp 493- 511.
9. Caro C G., Pedley T J., SchroterRc, Seed W A., 1978. "The mechanics of circulation", *Oxford Press.*
10. He X, Ku D N, Moore J E., 1993. "Simple calculations of the velocity profiles for pulsatile flow in a blood vessel using MATHEMATICA", *Ann. Bio med Eng,*, 21, pp 45-49.
11. Lance Jonathan Myers, Wayne Logan Capper. m., 2001. "Analytical Solution for Pulsatile Axial Flow velocity Waveforms in Curved Elastic Tubes", *IEEE TRANSACTIONS ON BIOMEDICAL ENGINEERING*, 48(8).
12. Chaturani P., Upadhya V S., 1978. " Pulsatile flow of a couple stress fluid through circular tubes with applications to blood flow", *Biorheology*, 15, pp3-4.
13. Chaturani P., Upadhya V S., 1979. "On Micropolar model for blood flow through narrow tubes", *Biorheology*, pp419-428.
14. Gamble G., Zorn J., Sanders G., MacMahon S., Sharpe N., 1994. "Estimation of arterial stiffness, compliance, and distensibility from M-mode ultrasound measurements of the common carotid artery", *Stroke, Journal of the American Heart Association*.
15. Guglielmi G., 2006. "Electrical Models in the Analysis of Hemodynamic Characteristics of Arteriovenous Malformations", *Interventional Neuroradiology*, 12, pp 9-15.
16. O'Rourke M F., Avalio A P., 1980. "Pulsatile flow and pressure in human systemic arteries. Studies in man and in multi branched model of the human systemic arterial tree", *Circulation Research*, 46, pp 363-372.
17. Bryon Bird R., Stewart. Warren E., Lightfoot, Edwin N., 2005. "Transport Phenomena", Second edition, *Wiley*.

18. Radhika T S L., Raja Rani T., 2015. "Pulsatile flow of micropolar fluid in an elastic tube – a model developed for estimating arterial stiffness", *Asian Journal of Mathematics and Computer Research*, 7(2), pp 140-147.
19. Fung Y. C., 1993. "Biomechanics- mechanical properties of living tissues", Second edition, *Springer*.
20. Surendra Kumar., 2015. "Study of blood flow using power law and Harschel-Bulkley non-Newtonian fluid model through elastic artery", March 26-28, 2015, Gauhati University, Guwahati, Assam, India.
21. Ostwald. 1929. Waele-Ostwald equation: KolloidZeitschrift, 47 (2) pp 176-187.
22. HessameddinYaghoobi, Mohsen Torabi., 2011. "The application of differential transformation method to nonlinear equations arising in heat transfer", *International Communications in Heat and Mass Transfer*, 38, pp 815–820.
23. BushraA.Taha., 2011. "The Use of Reduced Differential Transform Method for Solving Partial Differential Equations with Variable Coefficients", *Journal of Basrah Researches (Sciences*), 37(4) C.
24. Mohammed O., Al-Amr., 2014. "New applications of reduced differential transform method", *Alexandria Engineering Journal*, 53, pp 243–247.
25. Soltanalizadeh B., 2011. "Differential transformation method for solving one-space-dimensional telegraph equation", *Computational and Applied Mathematics*, 30, 639–653.
26. Theodoros G., Papaioannou, ChristodoulosStefanadis., 2005. "Vascular Wall Shear Stress: Basic Principles and Methods", *Hellenic J Cardiol*, 46, pp 9-15.
27. https://www3.nd.edu/~nsl/Lectures/mphysics/MedicalPhysics/PartI.PhysicstheBody/Chapter3.Pressure System of the Body Physics of the cardiovascular system/ Physics of the cardiovascular system.pdf.
28. Mohammad A., Hussain , Subir Kar, Ram R. Puniyani., 1999. "Relationship between power law coefficients and major blood constituents affecting the whole blood viscosity", *Journal of Biosciences*,24 (3), pp 329-337.