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# THE FORMAL-LOGICAL ANALYSIS OF THE FOUNDATION OF SET THEORY

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**Abstract**
The critical analysis of the foundation of set theory is proposed. The unity of formal logic and rational dialectics is the correct methodological basis of the analysis. The analysis leads to the following results: (1) the mathematical concept of set should be analyzed on the basis of the formal-logical clauses "Definition of concept", "Logical class", "Division of concept", "Basis of division", "Rules of division"; (2) the standard mathematical theory of sets is an erroneous theory because it does not contain definition of the concept "element (object) of set"; (3) the concept of empty set (class) is a meaningless, erroneous, and inadmissible one because the definition of the concept "empty set (class)" contradicts the definition of the logical class. (If the set (class) does not contain a single element (object), then there is no feature (sign) of the element (object). This implies that the concept of empty set (class) has no content and volume (scope). Therefore, this concept is inadmissible one); (4) the standard mathematical operations of union, intersection and difference of sets (classes) are meaningless, erroneous and inadmissible operations because they do not satisfy the following formal-logical condition: every separate element (object) of the set (class) must be in only one some set (class) and cannot be in two sets (classes). Thus, the results of formal-logical analysis prove that the standard mathematical theory of sets is an erroneous theory because it does not satisfy the criterion of truth.

**Keywords**: Set theory, Applications of set theory, Foundations of mathematics, General mathematics, General algebra, General topology, Methodology of mathematics; Logic, General logic, Temporal logic, Formal logic, Proof theory, Philosophy of mathematics, Mathematics education, Logic in the philosophy of science, Dialectics.

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## 1. INTRODUCTION

Recently, the progress of sciences, engineering, and technology has led to the rise of a new problem: the problem of rationalization of the fundamental sciences (for example, physics, mathematics). Rationalization of sciences is impossible without rationalization of thinking and critical analysis of the foundations of sciences within the framework of the correct methodological basis: the unity of formal logic and of rational dialectics.

As is well known, mathematics is widely and successfully used in the natural sciences. However, it does not mean that the problem of validity of pure mathematics is now completely solved, or that the foundations of mathematics are not in need of formal-logical and dialectical-materialistic analysis. The critical analysis within the framework of the correct methodological basis shows [1-27] that the foundations of theoretical physics and of mathematics (for example, classical geometry, the Pythagorean theorem, differential and integral calculus, vector calculus, trigonometry, theory of negative numbers) contain formal-logical errors.

In my opinion, the formal-logical errors in pure mathematics arise because mathematics abstracts the quantitative aspect from the qualitative aspect of real objects. Mathematics ignores the dialectical-materialistic principle of unity of the quantitative and qualitative aspects [28-32]. Therefore, mathematics does not satisfy the general-scientific criterion of truth: practice is criterion of truth. This gives reason to assert that pure "mathematics is a doctrine in which it is not known what we speak about and whether it is true that we speak" (Bertrand Russell).

In this connection, the problem of complete understanding of the essence of pure mathematics and, consequently, the problem of critical analysis of the foundations of pure mathematics within the framework of the correct methodological basis arise. In my opinion, standard mathematics cannot be considered as a science if there is no formal-logical and dialectical substantiation of it.

As is well known, set theory is a branch of pure mathematics (mathematical logic) that studies sets (classes) of elements of arbitrary nature [28-40]. (A set is an arbitrary collection of certain elements (objects) mentally united into a single whole).

The modern study of set theory was initiated by Georg Cantor and Richard Dedekind in the 1870s. After the discovery of paradoxes in naive set theory, such as the Russell's paradox, numerous axiom systems were proposed in the early twentieth century, of which the Zermelo–Fraenkel axioms are the best-known.

Set theory is commonly employed as a foundational system for mathematics, particularly in the form of Zermelo–Fraenkel set theory with the axiom of choice. Beyond its foundational role, set theory is a branch of mathematics in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals [28-40].

At present, set theory is the basis of many branches of mathematics: general topology, general algebra, functional analysis [28-40]. It had a significant impact on the modern understanding of the subject of mathematics. The methods of set theory are widely used in all areas of modern mathematics and mathematical logic. These methods are of fundamental importance for questions of substantiation of mathematics by logical facilities (means) [28-40]. However, the use of set theory for the logically irreproachable (perfect) construction of mathematical theories is complicated by the fact that it itself needs to be substantiated. Moreover, in the substantiation of set theory arises difficulties that have not been overcome even now [34].

There are no formal-logical and dialectical substantiation of the foundation of set theory in scientific literature. The purpose of the present work is to propose the critical analysis of the foundation of set theory within the framework of the correct methodological basis: the unity of formal logic and rational dialectics

## 2. THE METHODOLOGICAL BASIS

As is known, the correct methodological basis of sciences is the unity of formal logic and of rational dialectics. Use the correct methodological basis is a necessary condition for correct distinction between truth and falsehood. However, this fact is ignored by majority of scientists until now. Therefore, the main assertions of formal logic and of rational dialectics which are used in the present work should be stated.

### 2.1. The basic principles of formal logic

1. Formal logic is science of the laws of correct thinking as well as means of cognition of reality. Correct thinking represents uncontradictory, coherent, consistent, and sequential thinking. The conclusions resulting from correct thinking are true statements which reflect correctly the objective reality in the process of scientific cognition of the world. The basic formal-logical laws are the following four laws: the law of identity, the law of lack (absence) of contradiction, the law of excluded middle, the law of sufficient reason.

2. Thinking is the highest form of human cognitive activity which represents the process of reflection of objective reality in human consciousness. Human thinking is performed with the help of concepts and has different forms.

3. The form of thought reflecting and fixing the essential features (signs) of things, objects, and phenomena of reality is called concept. In other words, the concept is the thought which reflects things, objects from viewpoint of the general and essential features (signs). (Thing is an object that can be in relation to anything or have some property).

4. The essential features (signs) of the objects are chosen (are singled out) in objects and phenomena by thought. The essential features (signs) characterize the objects of given kind. Non-essential features (signs) do not characterize the objects of given kind. The characteristic which is used to determine similarity or difference of objects of thought is called essential feature (sign). In the most general view, features (signs) of objects can be reduced to properties (for example, large, small, white, black, good, bad, soft, hard, etc.), states (for example, state of rest, state of motion, energetic state, equilibrium state, etc.), actions (for example, it works, he reads, she performs her duties, etc.), and results of actions (for example, to have scored success, to have benefited, etc.), etc.

5. The first basic form of thought is a concept. Concepts are formed (created) with the help of logical methods such as analysis and synthesis, abstraction and generalization. Analysis is the mental decomposition (dissection) of the object of thought (thinking) in terms of the elements, the choice (separation) of the essential features (signs), and the consideration of the essential features (signs) separately. Analysis does not give knowledge of object or of phenomenon as a whole. Synthesis is the mental integration (association, combination, junction) of the elements of the object or of the phenomenon. Synthesis provides knowledge of object or of phenomenon as a whole (as a unity of parts, as a system). But this knowledge is not the reliable and complete one. Abstraction is the mental separation, the mental extraction of the certain, the essential features (signs) of object or of phenomenon and passing over all other features (signs) (i.e., abandonment of all other features (signs) without consideration). Generalization is the mental transition from features (signs) of individual, separate, single objects to features (signs) belonging to whole groups (classes) of these objects. Abstraction is the mental separation, the mental extraction of the certain, the essential features (signs) of the object or of the phenomenon and passing over all other features (signs) (i.e., abandonment of all other features (signs) without consideration). Generalization is the mental transition from features (signs) of individual, separate, single objects to features (signs) belonging to whole groups (classes) of these objects.

6. All the concepts can be divided into the following separate types: single concepts and general concepts. The concept which relates to only one certain object, separate phenomenon, separate event is called single (individual) concept. The concept which embraces (covers) a group (class) of similar things, objects is called general concept.

7. Each concept has two aspects: the volume (scope) of the concept and the content of the concept. The volume (scope) of the concept is all the objects (things, phenomena) which can be embraced (covers)

by given concept. The volume (scope) of the general concepts is expressed in the form of a logical class. The concept content is a set of all the essential features (signs) of objects (things, phenomena) embraced (covered) by the concept.

8. All the concepts can be divided into the following separate types: concrete concepts and abstract concepts. Concrete concept is the concept that relates to groups, classes of objects (things, phenomena) or to the separate objects (things, phenomena). Abstract concept is the concept of properties of objects (things, phenomena) if these properties are taken as the separate (independent) object of thought and are abstracted from objects.

9. There is a special kind of concepts that is called categories. Categories are the scientific concepts reflecting the most common properties of objects and phenomena, the most common, essential relations and connections in reality. For example, the concepts of "matter", "movement", "content", "form", "causality", "freedom", "necessity", "randomnicity", "essence", "phenomenon" are the categories.

10. There are the following relations between the concepts: identity relation; relation of subordination; relation of collateral subordination; relation of partial coincidence; relation of disagreement. (For example, the relation of disagreement exists between contradictory concepts and between opposite concepts). These formal-logical relations cannot be expressed in the mathematical (quantitative) form.

11. A concept is defined as follows. Each concept expresses essential features (signs) of homogeneous objects (things, phenomena). These features (signs) are the content of the concept. The definition of concept is the disclosure of the content of the concept, i.e. the indication of the essential features (signs) of objects (things, phenomena) expressed by the concept. Thus, definition of concept is indication of the essential features (signs) of those objects (things, phenomena) that are covered (embraced) by this concept; and these features (signs) should be indicated in their mutual connection. In other words, the definition of concept is the definition of those objects that are covered (embraced) by this concept. The definition of concept is a formal-logical operation which cannot be expressed in mathematical (quantitative) form.

12. A concept is defined in the way of indication of the proximal (nearest) genus and the species difference (specific difference). Logic determines the following method of definition, which does it possible to indicate the essential features (signs) of the definable objects. The definable concept is led (brought) under the other, more general, concept. Moreover, the definable concept is subordinated to a more general concept. The volume (scope) of the definable concept is part of the volume (scope) of the more general concept. Thereafter, the feature (sign) which expresses the difference of the definable concept from other concepts is indicated. The volumes (scopes) of the other concepts which are also subordinated to this general concept enter into the volume (scope) of this general concept.

Such method of definition is called definition by means of indication of the proximal (nearest) genus and the species difference (specific difference) (in Latin: "definitio per genus proximum et differentiam specificum"). This implies the following assertion. If one wants to define an object, one must, first of all, find the proximal (nearest) genus (in Latin: "genus proximum"), i.e. directly a wider class of objects into which the objects under consideration enter as a species. Then one must find a species difference (specific difference) (in Latin: "differentia specifica"), i.e. that feature (sign) which distinguishes (differentiates) the objects under consideration from objects of other species of the same class (genus). Thus, the definition by means of indication of the proximal (nearest) genus and species difference (specific difference) implies that all features (signs) of the definable object are not enumerated (listed), but only two features (signs) are indicated: the generic (the proximal (nearest) genus) and species features (signs).

The concept which one defines is called definable concept, and the concept by which the first concept is defined is called defining concept. In defining complex concepts, the species difference (specific difference) can include several features (signs) because one of some separate features (signs) can be insufficient feature (sign) in order to delimit (restrict, differentiate) given object from other objects of the same kind and to disclose its content. In these cases, defining the object (thing, phenomenon), it is necessary to indicate the genus and then the species difference (specific difference) consisting of several features (signs) which differentiates (distinguishes) given species from other species of the same genus. Logically, this set of features (i.e., the species difference) can be considered as one feature (sign), but the latter is a complex one consisting of several features (signs). (One cannot omit some of them without detriment to the completeness and concreteness of the definition!).

13. There are five rules that must be followed in order that the definitions to be logically correct. These rules are as follows.
a) The definition must be commensurate (ratable, rateable, proportionate, corresponding, appropriate, adequate) one. This means that the concept that is defined and the concept by which the first concept is defined must be the same in volume (scope). The definable concept and the defining concept must have the same volume (scope), and they can be permuted (rearranged, transposed, changed in places): the definable concept can be put in place of the defining concept, and the defining concept can be put in place of the definable concept. The ratability (commensurateness, proportionality) of the definition is also accuracy (adequacy) of the definition. The accurate definition is the definition which clearly confines, distinguishes the definable object from other similar objects;
b) The generic feature (sign) must indicate the proximal (nearest) higher concept without jumping over it. This means that, under definition of a concept, one must always find the proximal (nearest) genus. But one should not replace the proximal (nearest) genus by a more distant (remote) genus;
c) The feature (sign) which is attributable (inherent) to only the definable concept and is missing (absent) in other concepts related to the same genus must be a species feature (sign) (specific definition). This means that the definable concept as a species distinction (specific difference) must have such a feature (sign) which is absent (lack) in other collaterally subordinated concepts (i.e., in other concepts related to the same genus).
d) A definition should not be negative one. The negative definition would indicate what the given object is not, but not what it is. Thus, the negative definitions can be used only in the cases of the definition of purely negative concepts;
e) Every definition must be complete and clear one. The complete definition is the definition which indicates all essential features (signs) of the object. Consequently, the incomplete definition is the definition which although correctly indicates the features (signs) of the object but does not indicate all its essential features (signs). A clear definition is such definition which indicates only fully-known (completely known) features (signs) of the object. Consequently, an unclear definition is the definition which indicates such features (signs) of the definable object, which themselves are unknown features (signs) and they themselves need to be defined.

14. The most widespread (typical) errors in the definitions, that occur in practice are as follows.
a) The first error – the error of the incommensurability of the definition – is that the definition is either too narrow or too broad. Too narrow definition is a definition in which the volume of the defining concept is less than the volume of the definable concept. Too broad definition is a definition in which the volume of the defining concept is greater than the volume of the definable concept. The error is eliminated if the volumes of these concepts are equal;
b) The second error is a tautology in the definition (in Latin: “idem per idem”). The tautology in the definition is that the definable object is defined by itself;
c) The third error is the circle in the definition. The circle in the definition is that one concept in certain definition is defined by the second concept, and this second concept is defined by the first concept. The circle in the definition is reduced to tautology;
d) The fourth error is the definition of an unknown concept by another unknown concept. This error is called “ignotum per ignotius” (in Latin).

15. The significance of a definition is as follows. The definition of the concept discloses the content of the concept. In other words, the definition of the concept discloses (reveals) the essential features (signs) of the studied objects (things, phenomena) which are embraced (covered) by this concept. Therefore, definitions are the basis of the sciences. Definitions represent an essential aspect of the process of cognition of reality. Every object and every concept in scientific research should be accurately defined. Without an accurate definition, science will inevitably generate (create) ambiguities, a shift in concepts, etc. Thus, a definition is a short formula which expresses the most basic (fundamental, cardinal, principal) aspect in studied phenomenon. But this short formula does not fully characterize the phenomenon in the all diversity of its forms, connections and features (signs).

16. Every concept is characterized by volume (scope) and content. Definitions of the volume (scope) and content of the concept are as follows. The volume (scope) of the concept is all objects (things, phenomena) to which given concept can be applied. For example, the volume (scope) of the general concept “people” is all people who lived in the past, live or will live. The volume (scope) of the general

concept "tree" is all objects which are covered (embraced) by the concept "tree". The volume (scope) of an individual concept is only one object to which this concept is related.

The volume (scope) of general concepts is expressed in the form of class. A logical class is a collection (set) of objects which have common features (signs). Therefore, these objects are covered (embraced) by the general concept. (In other words, a class is all objects which form the volume (scope) of a concept). All objects forming a class have the same (identical) feature (sign). A class in a logical sense is all those objects that are expressed in a general concept. One class is higher (superior) class to relative other class if it includes the other class together with certain (other) classes. (For example, the class "trees" is the higher (superior) class to relative the class "birches" because the class "trees" includes the class "birches" together with other classes of trees ("spruces", "pines", etc.). A class which is higher (superior) class to relative the other class is called genus (in Latin: "genus"). A class which is lower to relative the genus is called species (in Latin: "species").

The concept expressing a class which is a genus is a generic concept. The concept expressing a class which is a species is a specific concept (superprdinate concept). The genus which is directly divided into species is called proximal (nearest) genus. For example, the class "coniferous trees" is the proximal (nearest) genus for the classes "spruces", "pines", etc. The class "trees" in general is the genus for the classes "spruces", "pines", and others. But the class "trees" is not the proximal (nearest) genus for the classes "spruces" and "pines". The class "trees" is the proximal (nearest) genus for the classes "deciduous trees" and "coniferous trees".

17. The relation between the volume (scope) and the content of the concept is as follows. The volume (scope) and the content of the concept are in the converse (inverse) relation: the content of the concept is decreased with the increase in the volume (scope) of the concept; and the volume (scope) of the concept is decreased with the increase in the content of the concept. The volume of an individual concept is one object to which this concept is related. The volume (scope) of an individual concept is a class containing only one object. A class that contains no object does not exist because there is no concept of non- existing object. Therefore, the concept "empty class" is a meaningless, erroneous and inadmissible concept.

18. The definition of division of a concept is as follows. In order to cognize objects (things, phenomena) embraced by the concept, one must disclose the volume (scope) of the concept: one must determine, ascertain, detect the circle of objects (objects, phenomena) that represents the volume (scope) of this concept. This goal is achieved by division of the concept. The division of a concept is a formal-logical (qualitative) operation which cannot be expressed in mathematical (quantitative) form.

The division of a concept is the distribution of objects (things, phenomena) into groups. These objects (things, phenomena) are embraced by the concept and constitute the volume (scope) of the concept. (The volume (scope) of the concept is all objects that are embraced given concept. I.e., the volume (scope) of the concept is all objects to which the concept is applied. It should be emphasized that the term "all objects" signifies "complete set of objects". The number of objects in the complete set is not an essential feature (property, characteristic) of the set). The volume (scope) of the concept is expressed as the class of the corresponding objects. This class can be divided into smaller classes. This is the division.

In operation of division of the concept, it is important to group the objects (things, phenomena) that are embraced by given concept but not to indicate and to enumerate all objects (things, phenomena) which are embraced by given concept. Therefore, division is as follows: one takes a concept and ascertains (determines, explores) its volume (scope), i.e. one determines which objects (things, phenomena) are embraced (covered) by this concept. Then one divide these objects (objects, phenomena), which are the volume (scope) of the given concept (i.e. class), into groups (into lower classes) in concordance with similar features (signs). A new concept is formed for each such class (group). Each such new concept can be divided as well.

For example, one considers the division of the concept "tree". The volume (scope) of this concept represents all the trees that exist in the world. These trees can be divided into coniferous trees and deciduous trees. The concepts "coniferous trees" and "deciduous trees" are subordinate concepts relative to the concept "trees". The concepts "coniferous trees" and "deciduous trees" are collaterally subordinated concepts relative to each other. The divisible concept "trees" is a generic concept for the concepts “coniferous trees” and “deciduous trees”. The concepts “coniferous trees” and “deciduous trees” are specific concepts (species concepts, superprdinate concepts) relative to the concept “trees”.

The concept that is divided is called divisible concept. The concepts into which the divisible concept is divided are called members of the division. This means that the “trees” are a generic (i.e., divisible) concept, and the “coniferous trees” and the “deciduous trees” are species concepts (specific concepts), i.e., members of division.

Since the volume (scope) of the concept is expressed as a class of objects (things, phenomena), the division is that the class of objects being a genus is divided into classes which are species. In other words, the division is that the genus is divided into species.

If the divisible concept is divided into two classes, then this division is called two-term one (dichotomy); if the divisible concept is divided into three classes, then this division is called three-term one (trichotomy); if the divisible concept is divided into a large number of classes, then this division is called polynomial one (polytomy).

19. The principle of the division is as follows. The division of the concept (i.e., the disclosure of the volume (scope) of the concept) cannot be performed without taking into account of the content of the concept because the volume (scope) and content of the concept are connected with each other. The feature (sign) of the concept, on the basis of which the volume (scope) of the divisible concept is divided into groups, is called basis of division (in Latin: principium divisionis). (For example, people can be divided into men and women. In this case, the basis of division is the sexual feature). Thus, in all cases of division, one must perform the following mental operations: (a) choose some essential feature (sign) which is proper (intrinsic, inherent) to the divisible concept; (b) divide all the objects embraced (covered) by this concept into groups in compliance with this feature (sign). The essential feature (sign) which is the basis of division is used as follows: the objects that represent the volume of the divisible concept are divided into groups either in compliance with changing this feature (sign) in each group of objects or in compliance with presence of this feature (sign) in one group and the lack of this feature (sign) in the other group of objects. (Note: any feature (sign) that is essential for some purpose can be chosen as the basis for division).

20. The rules of division, which must be observed in order that the division to be correct, are as follows.

a) There must be only one basis in each division. The basis of division is a feature (sign) that indicates (denotes) essential difference between the members of the division. This implies that any division is performed on the basis of one certain feature (sign), but not on the basis of different features (signs).

b) Members of the division must eliminate (exclude) each other. This implies that, under division of objects into groups in compliance with some feature (which is the basis of division), each individual (separate) object must be in only one group. (For example, if one divides the trees into coniferous and deciduous trees, then this division is correct because the members of the division eliminate (exclude) each other: the coniferous trees cannot be the deciduous trees at the same time, and the deciduous trees cannot be the coniferous trees at the same time; each tree can be either in a group of coniferous trees or in a group of deciduous trees, but each tree cannot be in both groups at the same time).

c) Members of division must be the proximal (nearest) species relative to the divisible concept: the members of division must be directly inferior concepts relative to the divisible concept; members of the division must be collaterally subordinated concepts relative to each other. This means that if one divides a class of objects into lower classes, then these lower classes (members of division) must be directly lower classes (members of division), i.e. the lower classes must collaterally adjoin the divisible class. Therefore, the divisible concept must be the proximal (nearest) genus (in Latin: genus proximum) relative to the members of division.

d) The sum of the volumes (scopes) of the members of the division must be equal to the volume (scope) of the divisible concept. For example, if the concept “trees” is divided into the concepts “deciduous trees” and “coniferous trees”, then the following relationship between the volumes (scopes) of these concepts must be fulfilled:

$$V_{(trees)} = V_{(foliage\ trees)} + V_{(coniferous\ trees)}$$

where $V$ is a volume (scope) of the concept.

21. The second, more complicated, form of thought is a proposition (logical judgment). The proposition (logical judgment) is the logical form of the expression of thought. The proposition (logical judgment) is the logical content of the grammatical sentence. The proposition (logical judgment) is a statement about the objects and phenomena of objective reality. The statement asserts the existence or lack (absence) of certain features (signs) in objects and phenomena. The proposition has the following two properties: (a) the proposition either asserts or denies (negates); (b) the proposition is either true or false. The proposition is always assertion or negation. The proposition is true if it reflects correctly the reality; and the proposition is false if it reflects incorrectly the reality. Every proposition represents a system of concepts. There are three elements in every proposition: subject, predicate, connective. The subject of the proposition is that what one states about. The predicate of the proposition is that what one states on the subject. The connective is an indication of the relation between subject and predicate. In any proposition, subject and predicate are concepts connected by connective. The connective in any proposition expressed by the word “is” or “is not”.

22. The third form of thought is an inference. The inference represents connection of propositions, which makes it possible to derive a new proposition from given one or more propositions. Those propositions from which one derives the new proposition are called premises, and the new proposition derived from the premises is called conclusion. Relation between the premises and the conclusion is relation between reason (basis) and consequence (logical corollary): the premises are the reason (basis) from which the conclusion follows as a consequence (logical corollary). Consequently, the inference is based on the law of sufficient reason.

Depending on number of premises, all the inferences are divided into two groups: immediate inferences and mediated inferences. The immediate inference is the inference in which the conclusion is consequence of one premise. The mediated inference is the conclusion in which a new proposition is derived from two or more propositions.

23. The mediated inferences can be of two types: deductive and inductive. The mediated deductive inference is called syllogism if a conclusion is derived from two premises. The inference is called inductive inference if the premises indicate features of separate objects or groups of separate objects, and the conclusion is extended to other objects of the same kind. Deduction and induction are in inseparable connection with each other and supplement each other. Mathematics uses mainly method of deduction.

24. Scientific induction is based on the determination of the causes. Therefore, the problem of causal connection of phenomena is important for scientific induction. The causal connection of phenomena is that one phenomenon is a cause for another phenomenon, and a change in the first phenomenon entails a change in the second phenomenon too. The phenomenon which necessarily entails another phenomenon is called cause, and the second phenomenon which is entailed by this cause is called effect of this cause. Thus, the connection of cause and of effect is a connection of two phenomena, two facts. In order to determine the cause of the phenomenon studied, one should use two basic logical methods of the inductive research: intercomparison of the circumstances in which given phenomenon occurs; comparison of these circumstances (in which given phenomenon occurs) with other circumstances (similar in other relations) in which given phenomenon do not occur.

25. The validity (trueness) of some proposition is determined with the help of proof. The proof is determination of the validity (trueness) of some proposition by the use of other true propositions from which the validity (trueness) of the given proposition follows. The proofs are based on the logical law of sufficient reason. The proof represents an indication of sufficient reason for whatever proposition. Whatever proof consists of three parts: thesis, arguments, demonstration (manifestation). The proposition is called thesis if one proves validity of this proposition. The propositions which are used for the proof of the thesis are called arguments (i.e., sufficient reason). Derivation of thesis from arguments is called demonstration (manifestation). In other words, demonstration (manifestation) is the propositions which show why the given thesis is substantiated (grounded) by the given arguments

**2.2. The basic principles of rational dialectics**

1. Rational dialectics (i.e., corrected dialectical materialism) is a science of programmed (predetermined) development: the science of the most common types of connections and laws of the development of the nature, of human society, and of thought. The universal connection exists not only

in the material world – in the nature and society – but also in thinking. Connection and interdependence of the forms of thought (for example, concepts) is (in the final analysis) reflection of the universal connection and of interdependence of the phenomena of the objective world in human consciousness. Since concepts are reflection of objects in human consciousness, the concepts are interconnected, and they can not be taken in isolation from each other. Concepts must correspond to the natural and social processes, must reflect their contents.

2. The basic laws of dialectics are as follows: the law of unity and struggle of opposites; law of transition of quantitative changes into qualitative changes; law of negation of negation. The law of transition of quantitative changes into qualitative changes is essential to analyze the foundations of mathematics. There are also the most common laws of dialectics, which do not belong to the basic ones. The paired (relative) categories of dialectics – necessity and chance, possibility and reality, form and content, essence and phenomenon, etc. – are the theoretical reflection of non-basic laws of dialectics. All the laws and categories of dialectics represent forms of thought, forms of cognition of the objective world, forms of reflection of the objective world in the human consciousness.

3. As is known, the cognitive psychical activity of man is performed in the following way (by the scheme):

$$\textit{(sensation, perception, representation)} \rightarrow$$
$$\textit{(concept)} \rightarrow \textit{(theory)} \rightarrow \textit{(practice)}.$$

Sensation is a result of influence (effect) of the outside world to the sense-organs of man; perception is an immediate (direct) sensuous reflection of the reality in the consciousness of man; representation is an image of an object or phenomenon (which is not perceived at given instant of time) in the consciousness of man. Thinking is carried out with the help of concepts. Concept is the form of thought reflecting and fixing the essential signs (features) of objects and phenomena of objective reality. Theory is a system of concepts.

4. The unity of sensuous and rational moments in the cognition is that sensuous cognition is the starting point, the first stage of the cognitive activity. A man, even at the level of logical thinking, continues to rely on (rest upon) sensuously perceivable material in the form of visual images, of various schemes, of symbols, on sensuous form of language.

5. Material activity of people represents practice. Practice is (first of all) a sensuous-objective activity aimed at satisfying human needs. Theoretical activity is derived from practice. Social practice is a starting and ending points of theory. The unity of theory and of practice is a starting point of epistemology. Practice is a driving force in development of cognition.

6. Social practice is criterion of truth. The criterion of truth can be found neither in the object of cognition nor in the consciousness of the subject. Practice is the experience of all humanity in its historical development. The absoluteness of practice as criterion of truth is that all knowledge proven by practice is an objective truth. But, at every given stage (step) of theoretical study, practice can not corroborate completely or refute all theoretical propositions – in this sense, practice is relative. Only the unity of formal logic and of practice can corroborate completely or refute all theoretical propositions at every given stage (step) of theoretical study.

7. The law of transition of quantitative changes into qualitative changes is essential to analyze the foundations of mathematics. The essence of this law is as follows: quantitative and qualitative changes represent the dialectical unity (interconnection) of the opposite and interdependent aspects.

Quality is inherent determinacy in the objects and phenomena. Quality is the organic unity of the properties, signs (features), and characteristics that makes it possible for to distinguish given object or phenomenon from the other ones. In other words, quality is the unity of structure and of elements. “There are not qualities, but only objects with qualities” (Friedrich Engels). Quality expresses specific character of an object or phenomenon in whole. Quality is not only holistic characteristic but also a relatively stable set of signs (features) which determines the specificity of given object. Quality is holistic characteristic of an object or phenomenon; and the property is one of the aspects (partial characteristics) of the object or phenomenon. Some properties express the qualitative determinacy of the object; other properties express the quantitative determinacy.

Quantity is inherent determinacy in the objects and phenomena, which expresses the number of inherent properties of objects and of phenomena, the sum of component parts of objects and of phenomena, the amount, the degree of intensity, the scale of development, etc. In other words, quantity is determinacy in objects and phenomena, expressed by a number. For example, noting in the object properties such as volume, weight, length, speed, etc., man ascertains simultaneously quantitative expression of these properties as well. The quantities of volume, weight, length, speed, etc. are the quantitative characteristic of these properties.

8. Quality and quantity are dialectically connected. They represent the unity of opposites. The qualitative determinacy does not exist without the quantitative determinacy, and vice versa. The unity of qualitative and quantitative determinacy is manifested in measure. The measure denotes existence of the interdependence of qualitative and quantitative aspects of the object or phenomenon. The measure expresses the limits (boundaries) within which objects and phenomena are themselves. Each state has its own measure. The violation of the measure leads to a change in the state. The transition from one state to another is a movement. Leading place belongs to quality in the unity of qualitative and quantitative determinacy. Quality determines the framework of quantitative changes. The qualitative changes can only result from the quantitative changes (i.e. quantitative movement).

9. The law of transition of quantitative changes into qualitative changes is essential to analyze systems. The important theoretical proposition of system analysis is as follows. The properties of the system determine the properties of the elements; and the properties of the elements characterize the properties of the system. The main problem is that the dependences of properties (qualitative and quantitative determinacy) of the system on number of the elements and on the qualitative and quantitative determinacy of the elements are not reliably known. (From this point of view, the Universe (System) cannot be cognized by mankind (the element of the System).

10. The law of transition of quantitative changes into qualitative changes is essential to analyze the foundations of mathematics. The question of the fundamental applicability of mathematical methods in all the areas of scientific cognition must be decided on the basis of the law of interdependence of qualitative and quantitative determinacy. The following fundamental statement results from this law: the operation of abstraction of quantitative determinacy from qualitative determinacy is inadmissible mental operation.

## 3. THE CRITICAL ANALYSIS OF STANDARD SET THEORY

As is known, there are the following relations between the concepts: the relation of identity, the relation of subordination, the relation of collateral subordination, the relation of partial coincidence, and the relation of disagreement (non-agreement). These formal-logical (qualitative) relations cannot be expressed in mathematical (quantitative) form.

The formal-logical analysis of the foundation of standard set theory is possible if the relation between the concept "set" and the logical concepts "group", "class" exists. The relation between these concepts is established by the following only correct statement: the concepts of set, group and class are identical concepts. By definition, a logical class is a set of objects that have common features (signs). Hereupon, these objects are embraced (covered) by a common concept. Consequently, the mathematical concept "set" should be analyzed on the basis of the clauses: "Definition of concept", "Division of concept", "Basis of division", "Rules of division" stated above.

1. If the class (set) $A$ contains elements (objects) $a$, then the elements $a$ (objects) are identical elements (objects). The definition of the concept "element (object) $a$" is the definition of the element (object) $a$. The definition of the concept "element (object) $a$" (i.e., the disclosure of the content of the concept) is an indication of the essential features of the element (object) $a$. Consequently, in formal-logical point of view, set theory is an erroneous one because it does not contain a definition of the concept of element (object) $a$.

2. If the class (set) does not contain any (a single) element (object), then the class (set) is called empty class (set) and is denoted by the symbol $\varnothing$ [29, 32, 33, 35-40]. In formal-logical point of view, the concept of empty class (set) $\varnothing$ is a meaningless, erroneous and inadmissible concept. Really, firstly,

this definition contradicts to the definition of logical class. If the set does not contain any (a single) element (object), then this set is not a logical class. Secondly, the definition of the concept of empty class (set) $\varnothing$ is a negative definition (i.e., an inadmissible definition). This definition indicates that $\varnothing$ is not a class (set) containing elements (objects). Thirdly, if $\varnothing$ is not a class (set) containing elements (objects), then there is no feature of the element (object). This implies that the concept of an empty class (set) $\varnothing$ has no content and volume. Thus, the concept of an empty class (set) $\varnothing$ represents a formal-logical error.

3. If the classes $A$ and $B$ containing the elements $a$ and $b$, respectively, are given, then the following mathematical expressions are valid:

$$a \in A,\ \ b \in B;\ \ a \notin B,\ \ b \notin A;\ \ a \neq b.$$

4. The classes $A$ and $B$ containing the elements $a$ and $b$, respectively, do not contain empty subclass $\varnothing$: $\varnothing \not\subset A$, $\varnothing \not\subset B$. Really, the standard assertion [29, 32, 33, 35-40] that $\varnothing$ is a subclass of a non-empty class leads to the following formal-logical contradiction:

*(existing element of the class)* $\equiv$
*(non-existing element of the class),*
*(element of the class)* $\equiv$ *(non-element of the class),*

$$a \equiv \overline{a},\ b \equiv \overline{b}$$

where $\overline{a}$ and $\overline{b}$ are non-elements, i.e., $\overline{a}$ and $\overline{b}$ are non-existing objects.

The contradiction is eliminated if the formal-logical law of lack of contradiction is satisfied:

*(existing element of the class)* $\neq$
*(non-existing element of the class),*
*(element of the class)* $\neq$ *(non-element of the class),*

$$a \neq \overline{a},\ b \neq \overline{b}.$$

The contradiction does not exist if the formal-logical law of identity is satisfied:

*(existing element of the class) =*
*(existing element of the class),*
*(element of the class) = (element of the class),*

$$a \equiv a,\ b \equiv b.$$

Thus, $\varnothing$ is not contained in $A$ and $B$: $\varnothing \not\subset A, \varnothing \not\subset B$.

5. The operation of union (join) of classes is an inverse operation relative to the operation of division of the class into subclasses. In other words, if the class $D$ (for example, "trees") is divided into the subclass $A$ (for example, "foliage trees") and the subclass $B$ (for example, "coniferous trees"), then the subclasses $A$ and $B$ can be united (unified) in the class $D$: $D = A \cup B$. In this case, the volumes of the concepts are connected by the following mathematical (quantitative) relationship: $V_D = V_A + V_B$. Consequently, the necessary and sufficient condition for the union of the classes $A$ and $B$ is that: (1) the concepts $A$ and $B$ (i.e., the concepts "element $a$" and "element $b$") must be collaterally subordinated concepts relative to each other; (2) the concept $D$ must be the proximal genus relative to $A$ and $B$; (3) $a \in A$, $b \in B$, $a \notin B$, $b \notin A$, $a \neq b$; (4) $V_D = V_A + V_B$. If the necessary and sufficient condition for the union of classes is fulfilled, then any element of the class $D$ belongs either to the class $A$ or to the class $B$.

The necessary and sufficient condition for the union of the classes $A$ and $B$ is violated, for example, in the following case (Figure 1).

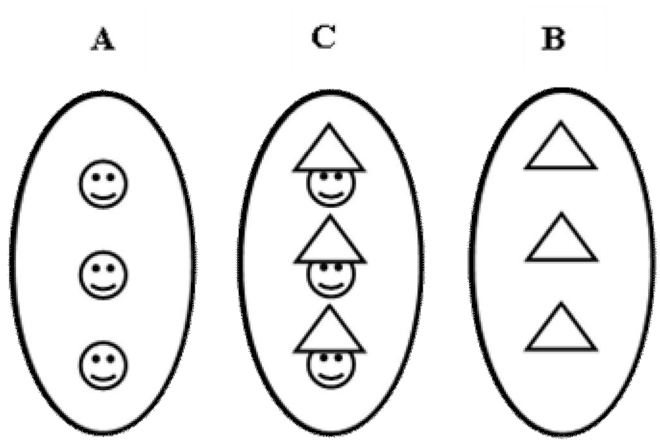


**Figure 1**: The division into three separate classes $A$, $B$, and $C$. The class $C$ represents neither union of the classes $A$ and $B$ nor intersection of the classes $A$ and $B$. The element $c$ of the class $C$ is a composite element: the element $c$ contains the elements $a$ and $b$ of both classes $A$ and $B$, respectively

Really, the class $A$ (containing the elements "human heads") and the class $B$ (containing the elements "triangular caps") on Figure 1 cannot be united because these classes do not have a common proximal genus. The proximal genus for the concept "human heads" is the concept "human bodies". The proximal genus for the concept "triangular caps" is the concept "caps". The concepts "human bodies" and "caps" are not collaterally subordinated concepts. Consequently, the union of these classes is a meaningless, erroneous and inadmissible operation.

Thus, the standard mathematical statement that sets (classes) $A$ and $B$ in an expression $D = A \cup B$ represent arbitrary sets (classes) is a formal-logical error. In other words, this implies that the standard mathematical definition of the union operation – "union of the sets $A$ and $B$ is the set of all objects that are a member of $A$, or $B$, or both" – is incorrect.

6. As is known [29, 32, 33, 35-40], the set that consists of elements belonging to both $A$ and $B$ is called intersection $A \cap B$ of sets $A$ and $B$ (Figure 2).

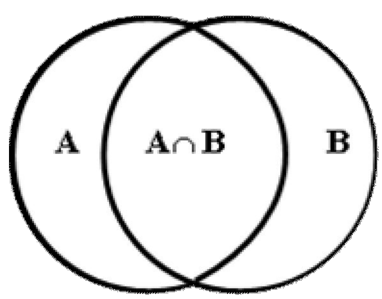


**Figure 2:** Venn diagram illustrating the intersection of two sets $A$ and $B$

But, in the point of view of formal logic, the mathematical operation of intersection of sets is a meaningless, erroneous and inadmissible operation. Really, according to the rule of division of concepts, the members of division must eliminate (exclude) each other, i.e., each separate (individual) element (object) must be in only one class and cannot be in two classes:

$$a \in A, \quad b \in B; \quad a \notin B, b \notin A; \; a \neq b.$$

(For example, only in this case, the formal-logical relationship

$$V_{(trees)} = V_{(foliage\ trees)} + V_{(coniferous\ trees)}$$

is valid (equitable, true)). Therefore, the operation $A \cap B$ is inadmissible one because the condition $a \notin B, b \notin A; \; a \neq b$ is not satisfied under the intersection of the sets.

If certain element (object) can be in two classes, then one must divide elements (objects) into three separate classes: for example, $A$, $B$, and $C$ (Figure 1), where $a$ is the element "human head", $b$ is the element "triangular cap", $c$ is the element "triangular cap on human head" (i.e., the element $c$ contains the elements "human head" and "triangular cap"). In this case, the basis of division is the feature (sign) that indicates on essential difference between the members of the division. Only such a division is correct one.

7. As is known [29, 32, 33, 35-40], the set of all elements of $A$ that are not contained in $B$ is called difference $A \setminus B$ of the sets $A$ and $B$. But, in point of view of formal logic, the mathematical operation of difference (subtraction) of classes is erroneous. Really, according to the rule of division of concepts, the members of division must eliminate each other, i.e. every separate (single, individual) element (object) must be in only one class:

$$a \in A, \quad b \in B; \quad a \notin B, b \notin A; \; a \neq b; \; A \setminus B = \varnothing.$$

But the mathematical operation of difference (subtraction) of sets does not satisfy the formal-logical rule of the division of concepts. Consequently, the operation $A \setminus B$ is meaningless, erroneous, and inadmissible one.

**4. DISCUSSION**

1. As is known, formal logic is the general science of the laws of correct thought. The laws of formal logic represent the theoretical generalization and reflection of practice in human consciousness. Consequently, formal logic exists in human consciousness and practice. Practice is criterion of validity (trueness, truth) of formal logic.

2. Dialectical materialism is the general science of the most common (general) kinds of connections and laws of development of nature, of human society, and of thought. The laws of dialectics represent the theoretical generalization and reflection of practice in human consciousness. Consequently, dialectics exists in human consciousness and practice. Practice is criterion of validity (trueness, truth) of dialectics.

3. The only correct methodological basis of sciences is the unity of formal logic and of rational dialectics. Mathematics is a science if and only if its foundations are formulated within the framework of correct methodological basis.

4. Pure mathematics is partial, special, non-general, non-common, and abstract science. Today, there is no complete understanding of the essence of pure mathematics by scientists. In my opinion, the essence of mathematics can be understood only within the framework of correct methodological basis. The critical analysis of mathematical concepts, theorems, and theories within the framework of correct methodological basis disclose the essence of mathematics.

5. As the critical analysis shows, the standard mathematical theories do not satisfy the criterion of truth. In order that the standard mathematical theories satisfy criterion of truth, the mathematical theories must satisfy formal logic and dialectics.

6. Set theory – branch of pure mathematics (mathematical logic) – does not satisfy criterion of truth because it contradicts to formal-logical laws and therefore represents a fiction, a useless intellectual game.

## CONCLUSION

Thus, the correct analysis of the foundation of set theory is possible only within the framework of correct methodological basis: the unity of formal logic and rational dialectics. The correct analysis leads to the following results:
1) the standard mathematical concept "set" should be analyzed on the basis of the formal-logical clauses: "Definition of concept", "Division of concept", "Basis of division", "Rules of division";
2) the standard mathematical theory of sets is an erroneous theory because it does not contain the definition of the concept "element (object) of set";
3) the concept of empty set (class) is a meaningless, erroneous and inadmissible concept because: (a) the definition of the concept "empty set (class)" contradicts to the definition of a logical class; (b) the definition of the concept "empty set (class)" is the negative definition (i.e., an inadmissible definition); (c) since the set (class) does not contain a single element (object), there is no feature (sign) of the element (object). This implies that the concept of empty set (class) has no content and volume and therefore is an inadmissible concept;
4) the standard mathematical operations of union, intersection and difference of sets (classes) are meaningless, erroneous and inadmissible operations because standard mathematical operations do not satisfy the following formal-logical condition: every separate (single) element (object) of the set (class) must be in only one set (class) and cannot be in two sets (classes).

Thus, the results of formal-logical analysis prove that the standard mathematical theory of sets represents an erroneous theory because it does not satisfy the criterion of truth.

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