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Prime labeling of certain graphs *

J. Jeba Jesintha[1,†], N.K. Vinodhini[2] and K.S.D. Subiksha[3]

1,3. P.G. Department of Mathematics, Women's Christian College,
University of Madras, Chennai, India.
2. Department of Mathematics, Anna Adarsh College for Women,
University of Madras, Chennai, India.
1. E-mail: jjesintha_75@yahoo.com , 2. E-mail: vinookaran@yahoo.co.in
3. E-mail: subikshastar1997@gmail.com

Abstract A graph $G = (V, E)$ with n vertices is said to admit prime labeling if its vertices can be labeled with distinct positive integers not exceeding n such that the label of each pair of adjacent vertices are relatively prime. In this paper we obtain the prime labeling for the middle graph of path, kite graph, one point union of shell graphs and the subdivided shell graph with star graph.

Key words Prime labeling, Middle graph, Kite graph, Shell graph.

2020 Mathematics Subject Classification 05C78.

1 Introduction

Graph theory is an area of discrete mathematics that deals with the study of graphs, which are structures that represent the pairwise relationships among discrete objects. Graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Rosa [9] introduced the labeling method called β – valuation which was renamed as graceful labeling by Golomb [6]. Later on different types of labelings were introduced. The concept of *Prime labeling* of a graph was introduced in a paper by Tout et al. [10]. Samuel and Kalaivani [1] proved that the brush graph admits prime labeling, while in [2] they proved the prime labeling for some vanessa related graphs. Jesintha and Hilda have shown in an earlier paper [4] that the subdivided shell flower graphs are odd graceful. Meena and Kavitha [7] proved that the butterfly related graphs admit prime labeling. Fu and Huang [3] proved that path P_n on n vertices is a prime graph. Meena and Vaithilingam [8] showed the prime labeling for some fan related graphs. For a detailed latest survey on graph labeling we refer to Gallian [5] . The real life application of prime labeling include hierarchical data representation in relational databases, scheduling the shift hours in any workstation. Apart from many applications prime labeling is also used to study the prime numbers. Our aim in this paper is to obtain the prime labeling for the middle graph of path, kite graph, one point union of shell graphs and the subdivided shell graph with star graph.

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†Corresponding author J. Jeba Jesintha, E-mail: jjesintha_75@yahoo.com

2 Definitions

In this section we give the definitions of the concept prime labeling, the middle graph, the kite, the shell graph and the subdivided shell graph .

Definition 2.1. Let $G = (V(G), E(G))$ be a graph with p vertices. A bijection $f : V(G) \longrightarrow \{1, 2, \ldots, p\}$ is called a **prime labeling** if for each edge $e = uv, \gcd\{f(u), f(v)\} = 1$. A graph which admits prime labeling is called a *prime graph*.

Definition 2.2. The *Middle graph* of a graph G, denoted by $M(G)$, and whose vertex set is $V(G) \cup E(G)$, is defined as follows:
Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following conditions holds:

2.2(i) x, y are in $E(G)$ and x, y is adjacent in G.

2.2(ii) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G. (see Fig. 1).

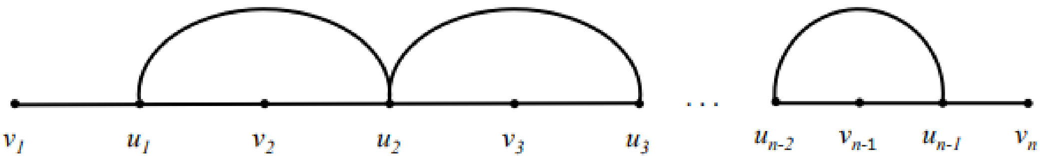

Fig. 1: The Middle graph of the Path P_n.

Definition 2.3. An (n, t)-*kite* is a cycle of length n with a t-edge path (the tail) attached to one of its vertices.

Definition 2.4. A *shell graph* is the join of a path P_m of m vertices and K_1. Th shell graphs are denoted by $C(m, m-3)$. A shell graph has $m-1$ petals.

Definition 2.5. A *Subdivided shell graph* [4] is defined to be a collection of edge disjoint sub divided shells that have their apex in common.

3 The main results

Theorem 3.1. *A middle graph of the path $M(P_n), n \geq 3$ admits prime labeling, where n is a positive integer.*

Proof. Let $v_1, \ldots, v_n$ be the vertices of the path P_n and $u_1, u_2, \ldots, u_{n-1}$ be the vertices introduced corresponding to the edges of the path P_n to obtain the middle graph of path $M(P_n)$. Thus the vertex set and the edge set of the middle graph $M(P_n)$ are $V = \{v_1, \ldots, v_n\} \bigcup \{u_1, \ldots, u_{n-1}\}$ and $E = \{v_i u_i / 1 \leq i \leq n-1\} \cup \{u_k v_{k+1} / 1 \leq k \leq n-1\} \cup \{u_j u_{j+1} / 1 \leq j \leq n-2\}$. Also $|V(M(P_n)| = 2n-1, |E(M(P_n)| = 3n-4$.
Define the prime labeling $f : V(M(P_n)) \longrightarrow \{1, \ldots, 2n-1\}$ as follows:

$$f(v_i) = 2i, \quad \text{for } 1 \leq i \leq n-1,$$

$$f(v_n) = 2n-1,$$

$$f(u_i) = 2i-1, \quad \text{for } 1 \leq i \leq n-1.$$

Note that for any edge,

$$v_i u_i \in E,\ \gcd\left(f\left(v_i\right), f\left(u_i\right)\right) = 1,\ \text{for } 1 \leq i \leq n-1,$$

$$u_k v_{k+1} \in E,\ \gcd\left(f\left(u_k\right), f\left(v_{k+1}\right)\right) = 1,\ \text{for } 1 \leq k \leq n-1,$$

$$u_j u_{j+1} \in E,\ \gcd\left(f\left(u_j\right), f\left(u_{j+1}\right)\right) = 1,\ \text{for } 1 \leq j \leq n-1.$$

Hence, the middle graph of the path admits prime labeling. Therefore, $M\left(P_n\right)$ is a prime graph. □

Theorem 3.2. *A (n,t)-kite $n \geq 3, t \geq 1$ admits prime labeling, where n is a positive integer.*

Proof. Let G be an (n,t)-kite such that $n \geq 3, t \geq 1$. Let $p_1, \ldots,, p_n$ be the vertices of the cycle. Let $q_1, \ldots, q_{t-1}$ be the vertices of the tail attached to one of the vertices in the cycle of the (n,t)- kite. Thus the vertex set and the edge set of the (n,t)-kite are respectively $V(G) = \{p_1, \ldots, p_n, q_1, \ldots, q_{t-1}\}$, $E(G) = \{p_i p_{i+1},\ 1 \leq i \leq n-1\} \cap \{p_n p_1\} \cup \{q_i q_{i+t},\ 1 \leq i \leq t-1\}$. Also $|V\left(G\right)| = |E\left(G\right)| = n+t-1$. Define a labeling from $f : V(G) \to\ \{1, 2, \ldots, n+t-1\}$ as follows:

$$f\left(p_i\right) = i-1, \quad \text{for } 2 \leq i \leq n,$$

$$f\left(p_1\right) = n,$$

$$f\left(q_j\right) = j+n, \quad \text{for} \ \ 1 \leq j \leq t-1.$$

Observe that all the vertex labels are distinct.
Then for any edge

$$p_i p_{i+1} \in E(G), \gcd\left(\left(f\left(p_i\right), f\left(p_{i+1}\right)\right)\right) = 1,\ \text{for } 1 \leq i \leq n-1,$$

$$q_i q_{i+1} \in E(G),\ \gcd\left(f\left(q_i\right), f\left(q_{i+1}\right)\right) = 1,\ \text{for } 1 \leq i \leq t,$$

$$p_n p_1 \in E(G),\ \gcd\left(f\left(p_n\right), f\left(p_1\right)\right) = 1.$$

Since, it is a prime labeling, therefore, the (n,t)-kite is a prime graph. □

Theorem 3.3. *The one point union of n copies of the shell graph admits prime labeling.*

Proof. Let $P_n^{(m)}$ be the graph obtained by the one point union of n copies of the shell graph $C\left(m, m-3\right)$. Let u_1 be the centre vertex of $P_n^{(m)}$ and let the vertices on the n copies be labelled as $u_2, \ldots, u_{mn+1}$ in the anticlockwise direction. The vertex set and the edge set of G are respectively

$$V = \left\{u_1, u_2, \ldots, u_{mn+1}\right\},$$

and

$$E = \left\{u_1 u_i / 1 \leq i \leq mn+1,\ \text{where } m, n \geq 2\right\} \bigcup \left\{u_{j+1} u_{j+2} / 1 \leq j \leq mn-1, \text{where}\ \ m, n \geq 2\right\}.$$

Also $\left|V(P_n^{(m)})\right| = nm+1$, $\left|E(P_n^{(m)})\right| = 2nm - n$.
Let us define the prime labeling $f : V\left(P_n^{(m)}\right) \longrightarrow \{1, 2, \ldots, nm+1\}$ of $P_n^{(m)}$ as follows:

$$f\left(u_i\right) = i,\ \text{for } 1 \leq i \leq mn+1.$$

Observe all the vertex labels are distinct.
Then for any edge,

$$u_1 u_i \in E,\ \gcd\ \left(f\left(u_1\right), f\left(u_i\right)\right) = 1,\ \text{for } 1 \leq i \leq mn+1,$$

$$u_{j+1} u_{j+2} \in E,\ \gcd\left(f\left(u_{j+1}\right), f\left(u_{j+2}\right)\right) = 1,\ \text{for } 2 \leq j \leq mn-1.$$

Thus, $P_n^{(m)}$ is a prime graph. □

Theorem 3.4. *The one point union of a subdivided shell graph with star graph is a prime graph.*

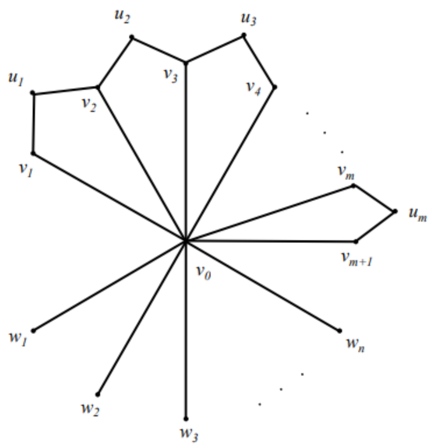

Fig. 2: The subdivided shell graph with star graph.

Proof. Let G be a one point union of a subdivided shell graph with star graph. Then G is one vertex union of a subdivided shell graph with m petals and a star $K_{1,n}$, where we note that $|V(G)| = 2m+n+2$ and $|E(G)| = 3m + n + 1$. Let m denote the number of petals in the subdivided shell graph and n denote the number of pendant vertices. The common vertex apex is denoted by v_0. Denote the pendant vertices of the graph G by $w_1, \ldots, w_{n-1}, w_n$ in the anticlockwise direction. The vertices in the first level (from the apex) of each petal are labeled as $v_1, \ldots, v_m, v_{m+1}$ in the anticlockwise direction. The vertices in the second level (from the apex) of each petal are labeled as $u_1, \ldots, u_{m-1}, u_m$ in the anticlockwise direction as shown Fig. 2.
Define the vertex labeling $f : V(G) \to \{0, 1, \ldots, 2m + 2n + 2\}$ as follows:

$$f(v_0) = 1,$$

$$f(u_i) = 2i, \text{ for } i = 1, 2, \ldots, m,$$
$$f(v_i) = 2i + 1, \text{ for } i = 1, 2, \ldots, m + 1,$$
$$f(w_i) = 2m + 2i, \text{ for } i = 1, 2,$$
$$f(w_i) = 2m + i + 1, \text{ for } i = 3, \ldots, n.$$

Observe that all the vertex labels are distinct.
Then for any edge,

$$v_0u_i \in E, \text{ gcd } (f(u_0), f(u_i)) = \text{gcd } (1, 2i) = 1, \text{for } 1 \le i \le m,$$

$$v_0v_i \in E, \text{ gcd } (f(u_0), f(v_i)) = \text{gcd } (1, 2i + 1) = 1, \text{ for } 1 \le i \le m + 1,$$
$$u_iv_i \in E, \text{ gcd } (f(u_i), f(v_i)) = \text{ gcd } (2i, 2i + 1) = 1, \text{ for } 1 \le i \le m,$$
$$u_iv_{i+1} \in E, \text{ gcd } (f(u_i), f(v_{i+1})) = \text{ gcd } (2i, 2i + 1) = 1, \text{ for } 1 \le i \le m.$$

Thus the one point union of the subdivided shell graph and star graph admits prime labeling. □

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