Bull. Pure Appl. Sci. Sect. E Math. Stat.
42E(1), 1–4 (2023)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI 10.48165/bpas.2023.42E.1.1

Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics

Website : https : //www.bpasjournals.com/

A study on completely equivalent generalized normed spaces *

Jayashree Patil[1], Basel Hardan[2,†], Ahmed A. Hamoud[3], Kirtiwant P. Ghadle[4] and Alaa A. Abdallah[5]

1. Department of Mathematics, Vasantrao Naik Mahavidyalaya,
Cidco, Aurangabad, Maharshtra, India.
2,5. Department of Mathematics, Abyan University,
Abyan 80425, Yemen.
3. Department of Mathematics, Taiz University,
Taiz P.O. Box 6803, Yemen.
2, 4. Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad, Maharashtra, India.
1 . E-mail: jv.patil29@gmail.com , 2. E-mail: bassil2003@gmail.com
3. E-mail: ahmed.hamoud@taiz.edu.ye , 4. E-mail: ghadle.maths@bamu.ac.in
5. E-mail: maths.aab@bamu.ac.in

Abstract According to intuition, two spaces X and Y are equivalent if they may be bent, shrunk, or expanded into one another. Spaces that are homotopy-equivalent to a point are called contractible. A vector space can be equipped with more than one norm. In this paper, the necessary and sufficient conditions for n-norms to be completely equivalent on linear n-Banach spaces are obtained.

Key words Normed space, Banach space, Equivalent.

2020 Mathematics Subject Classification 46B40, 03D50, 54D15.

1 Introduction

Functional analysis as an independent mathematical discipline started at the turn of the 19^{th} century and was finally established in the 1920s and 1930s, it contains various kinds of mathematical concepts, from these concepts we will focus on n-normed and n-inner product spaces.
In [3, 4] Gähler introduced an attractive theory of n-norm and 2-norm on linear spaces. Konca and Idris [10] studied on two known 2-norms defined on the space of p-summable sequences of real numbers. Systematic development of linear n-normed spaces have been extensively made by Kim and Cho [9], Malceski [12], Misiak [13], and Gunawan [5]. Recently, the equivalence of n-norms in n-normed spaces is studied by Kristiantoo et al. [11]. This paper is devoted to obtaining some necessary and sufficient conditions for n-norms to be completely equivalent in a linear n-Banach space. For recent studies of the sections of functional analysis we refer to [1, 2], [6–8], [14–16].
For this work, we need the following definitions:

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received August 28, 2022 / Revised April 17, 2023 / Accepted May 07, 2023. Online First Published on June 18, 2023 at https://www.bpasjournals.com/.
†Corresponding author Basel Hardan, E-mail: bassil2003@gmail.com

Definition 1.1. [11] Let X be a real vector space of dim $\geq n$. An n-norm on X is a mapping $\|\cdot,\dots,\cdot\| : X^n \to \mathbb{R}$, which satisfies the following four conditions:

nN 1: $\|x_1,\dots,x_n\| = 0$, if and only if $x_1,\dots,\ x_n$ are linearly dependent,

nN 2: $\|x_1,\dots,x_n\| = \|x_{i_1},\dots,x_{i_n}\|$, for every permutation $(i_1,\dots i_n)$ of $(1,\dots,n)$,

nN 3: $\|\alpha x_1,\dots,x_n\| = |\alpha|\,\|x_1,\dots,x_n\|$ for $\alpha \in \mathbb{R}$,

nN 4: $\|x_1+\acute{x}_1,x_2,\dots,x_n\| \leq \|x_1,x_2,\dots,x_n\| + \|\acute{x}_1,x_2,\dots,x_n\|$, for all $x_1,\acute{x}_1,x_2,\dots,x_n \in X$. The pair $(X,\|\cdot\dots,\cdot\|)$ is called an n-normed space.

Example 1.2. [5] Let $X = \mathbb{R}^n$. Let us define the function $\|\cdot,\dots,\cdot\|$ on X by: $\|x_1,\dots,x_n\| = |\det(x_{ij})|$, for each $i,j = 1,\dots,n$. Then $(X,\|\cdot\dots,\cdot\|)$ is a linear n-normed space.

Definition 1.3. [11] A sequence (x_k) in an n-normed space $(X,\|\cdot,\dots,\cdot\|)$ is said to be converge to some $l \in X$ if for every $y_2,\dots,y_n \in X$,

$$\lim_{k\to\infty} \|x_k - l, y_2,\dots,y_n\| = 0. \tag{1.1}$$

Definition 1.4. [11] A sequence (x_k) in an n-normed space $(X,\|\cdot,\dots,\cdot\|)$ is said to be Cauchy if for every $y_2,\dots,y_n \in X$,

$$\lim_{k,m\to\infty} \|x_k - x_m, y_2,\dots,y_n\| = 0. \tag{1.2}$$

Definition 1.5. [5] If every Cauchy sequence in X converges to some $x \in X$, then X said to be complete where X is an n normed space, every complete n-normed space is said to be an n-Banach space.

Definition 1.6. [11] $\|\cdot,\dots,\cdot\|_1$ and $\|\cdot,\dots,\cdot\|_2$ are equivalent (in short, E1) if for every $x_2,\dots,x_n \in X$ there are constants such that:

$$A\|x_1,x_2,\dots,x_n\|_1 \leq \|x_1,x_2,\dots,x_n\|_2 \leq B\|x_1,x_2,\dots,x_n\|_1,$$

for every $x_1 \in X$, (especially for $x_1 \in X \setminus$ span $\{x_2,\dots,x_n\}$).

Theorem 1.7. *[11] $\|\cdot,\dots,\cdot\|_1$ and $\|\cdot,\dots,\cdot\|_2$ are sequentially equivalent (in short, SE1) if for every $x_2,\dots,x_n \in X$, we have*

$$\lim_{k\to\infty} \|x_k - x, x_2,\dots,x_n\|_1 = 0 \;\Leftrightarrow\; \lim_{k\to\infty} \|x_k - x, x_2,\dots,x_n\|_2 = 0\ ,$$

such that the vectors $x_2,\dots,x_n$ must depend on k.

2 Main results

In this section we prove necessary and sufficient conditions for n-norms to be completely equivalent on linear n-Banach spaces.

Definition 2.1. Two n-norms $\|\cdot,\dots,\cdot\|_1$ and $\|\cdot,\dots,\cdot\|_2$ on a linear n-Banach space X are said to be equivalent (in short E2), if there are positive constants $A < B$ such that:

$$A\|x_k - x_m, x_2,\dots,x_n\|_1 \leq \|x_k - x_m, x_2,\dots,x_n\|_2 \leq B\|x_k - x_m, x_2,\dots,x_n\|_1,$$

for every $x_2,\dots,x_n \in X$ and $k,m \in \mathbb{N}$.

Theorem 2.2. *$\|\cdot,\dots,\cdot\|_1$ and $\|\cdot,\dots,\cdot\|_2$ are completely equivalent (in short, CE2) if for every $x_2,\dots,x_n \in X$, we have*

$$\lim_{k,m\to\infty} \|x_k - x_m, x_2,\dots,x_n\|_1 = 0 \Leftrightarrow \lim_{k,m\to\infty} \|x_k - x_m, x_2,\dots,x_n\|_2 = 0.$$

where x_k and x_m are Cauchy sequences in X, and $k,m \in \mathbb{N}$, such that $k,m \notin span\{x_2,\dots,x_n\}$. We can verify that $\|\cdot,\dots,\cdot\|_1$ and $\|\cdot,\dots,\cdot\|_2$ are E2 if and only if they are CE2.

Proof. Suppose that $\|\cdot,\dots,\cdot\|_1$ and $\|\cdot,\dots,\cdot\|_2$ are E2 for every $x_2,\dots,x_n \in X$, there are constants $A>0, B>0$ and $A<B$ such that

$$A\|x_k - x_m, x_2,\dots,x_n\|_1 \le \|x_k - x_m, x_2,\dots,x_n\|_2 \le B\|x_k - x_m, x_2,\dots,x_n\|_1,$$

for every $x_2,\dots,x_n \in X$ and $k,m \in \mathbb{N}$, it follows that:

$$A \lim_{k,m\to\infty} \|x_k - x_m, x_2,\dots,x_n\|_1$$

$$\le \lim_{k,m\to\infty} \|x_k - x_m, x_2,\dots,x_n\|_2 \le B \lim_{k,m\to\infty} \|x_k - x_m, x_2,\dots,x_n\|_1,$$

that is

$$\lim_{k,m\to\infty} \|x_k - x_m, x_2,\dots,x_n\|_1 = 0$$

if and only if

$$\lim_{k,m\to\infty} \|x_k - x_m, x_2,\dots,x_n\|_2 = 0.$$

Hence, $\|\cdot,\dots,\cdot\|_1$ and $\|\cdot,\dots,\cdot\|_2$ are CE2.
For the converse part, suppose that the two n-norms are not CE2, then, we may assume the following two cases:

Case (i): We cannot find $A>0$, such that

$$A\|x_k - x_m, x_2,\dots,x_n\|_1 \le \|x_k - x_m, x_2,\dots,x_n\|_2,$$

for every $x_2,\dots,x_n \in X$, and $k,m \in \text{span }\{x_1\} \setminus \text{span}\{x_2,\dots,x_n\}$.

Case (ii): We cannot find $B>0$, such that

$$\|x_k - x_m, x_2,\dots,x_n\|_2 \le B\|x_k - x_m, x_2,\dots,x_n\|_1,$$

for every $x_2,\dots,x_n \in X$, and $k,m \in \text{span }\{x_1\} \setminus \text{span}\{x_2,\dots,x_n\}$.

In **Case (i)** Suppose that

$$\frac{1}{k+m}\|x_k - x_m, x_2,\dots,x_n\|_1 > \|x_k - x_m, x_2,\dots,x_n\|_2, \tag{2.1}$$

For every $x_2,\dots,x_n \in X$ define $y_k - y_m = \frac{1}{\sqrt{k+m}} \frac{x_k - x_m}{\|x_k - x_m, x_2,\dots,x_n\|_2}$, $k,m \in \mathbb{N}$. Then we have $\|y_k - y_m, x_2,\dots,x_n\|_2 = \frac{1}{\sqrt{k+m}} \to 0$, as $k,m\to\infty$. Using (2.1), as $k,m\to\infty$, we find that

$$\|y_k - y_m, x_2,\dots,x_n\|_1 = \frac{1}{\sqrt{k+m}} \frac{\|x_k - x_m, \ x_2,\dots,x_n\|_1}{\|x_k - x_m, x_2,\dots,x_n\|_2} > \frac{k+m}{\sqrt{k+m}} = \sqrt{k+m} \to \infty, \tag{2.2}$$

Then, $\{y_k\}$ and $\{y_m\}$ are Cauchy sequences with respect to $\|\cdot,\dots,\cdot\|_2$ but not with respect to $\|\cdot,\dots,\cdot\|_1$.
And $\|\cdot,\dots,\cdot\|_2$ is CE2, but $\|\cdot,\dots,\cdot,\|_1$ is not CE2.
Hence, $\|\cdot,\dots,\cdot\|_2$ is E2, but $\|\cdot,\dots,\cdot\|_1$ is not E2.
Similarly we can prove **Case (ii)**. □

Corollary 2.3. *The following relationships exist between the four equivalence relations:*

$$\begin{array}{ccc} SE1 & \Leftrightarrow & E1 \\ \Uparrow & & \Uparrow \\ CE2 & \Longleftrightarrow & E2. \end{array}$$

3 Concluding remarks

In this work equivalence relation of two n-norms $\|\cdot,\dots,\cdot\|_1$ and $\|\cdot,\dots,\cdot\|_2$ on a linear n-Banach space X is introduced. This study establishes the necessary and sufficient criteria for the complete equivalence of n-norms on linear n-Banach spaces.
This study can be developed further to discuss the equivalence of (n-norms) under specific conditions on generalized Banach spaces.

Acknowledgments The authors are highly grateful to the referees and the Editor-in-Chief for their valuable suggestions on the original draft of this paper which are incorporated in the final version of this paper.

References

[1] Ahire, , Patil, J., Hardan, B., Hamoud, A. A. and Bachhav, A. (2022). Recent advances on fixed point theorems, *Bull. Pure Appl. Sci. Sect. E Math. Stat.*, 41(1), 1–11.

[2] Bachhav, A., Patil, J., Hardan, B., Hamoud, A. and Emadifar, H. (2022). A new result on Branciari metric space using $(\alpha;\gamma)$-contractive mappings, *Topological Algebra and its Applications*, 10(1), 103–112.

[3] Gähler, S. (1969). Untersuchungen über verallgemeinerte m-metrishe räume I, *Math. Nachr.*, 40, 165–189.

[4] Gähler, S. (1964). Lineare 2-normierte räume, *Math. Nachr.*, 28, 1–43.

[5] Gunawan, H. (2002). On n-inner products, n-norms, and the Cauchy Schwarz inequality., *Sci. Math. Jpn.*, 55, 53–60.

[6] Hardan, B., Patil, J., Hamoud, A. A., Bachhav, A., Emadifar, H. and Guunerhan, H. (2022). Generalizing contractive mappings on b-rectangular metric space, *Advances in Mathematical Physics*, 2022, 10 pp.

[7] Hardan, B., Patil, J., Chaudhari, J. and Bachhav, A. (2020). Approximate fixed points for n-linear functional by (μ,σ)-nonexpansive mappings on n-Banach spaces, *Journal of Mathematical Analysis and Modeling*, 1(1), 20–32.

[8] Hardan, B., Patil, J., Chaudhari, A. and Bachhav, A. (2020). Caristi type fixed point theorems of contractive mapping with application, Proceedings of the One Day National Conference on Recent Advances in Sciences held on 13^{th} February, 2020 at the Department of Physics, Chemistry, Mathematics, Botany and Zoology, Shivaji Arts, Commerce and Science College, Kannad, District - Aurangabad, Maharshtra, India, 609–614.

[9] Kim, S. S. and Cho, Y. J. (1994). Strict convexity in linear n-norm spaces, *Demonstration Math.*, 29(4), 739–744.

[10] Konca, S. and Idris, M. (2015). Equivalence among three 2-norms on the space of p-summable sequences, *Journal of Inequalities and Special Functions*, 7(4), 218–224.

[11] Kristiantoo, T. R., Wibawa, R. A. and Gunawan, H. (2013). Equivalence relation of n-norms on a vector space, *Mat. Vesnik.*, 65(4), 488–493.

[12] Malceski, R. (2004). Strong n-convex n-normed spaces, Mat. Bull., 21, 81–102.

[13] Misiak, A. (1989). n-Inner product spaces, *Math. Nacher.*, 140, 299–319.

[14] Patil, J., Hardan, B., Hamoud, A. A., Bachhav, A., Emadifar, H., Ghanizadeh, A., Edalatpanah, S. A. and Azizi, H. (2022). On $(\eta,\gamma)\_(f,g)$-contractions in extended b-metric spaces, *Advances in Mathematical Physics*, Volume 2022, Article ID 9290539, 8 pp.

[15] Patil, J. and Hardan, B. (2019). On fixed point theorems in complete metric space, *Journal of Computer and Mathematical Sciences*, 10(7), 1419–1425.

[16] Patil, J., Hardan, B. and Bachhav, A. (2022). Suzuki type common fixed point result on h-metric space, *International Journal of Advanced Research in Science, Communication and Technology*, 2(3), 5 pp.