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AN EPIDEMIC MODEL WITH IMMIGRATION AND NON-MONOTONIC INCIDENCE RATE

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Abstract
The present mathematical model deals with the study of SIR's epidemic model with immigration and non-monotonic function type. We start from formulation of model and analyze it. It also carries out a qualitative analysis of a SIR model with immigration and non-monotone incidence rate and studies the existence and stability of the disease free and endemic equilibrium.

Keywords: Mathematical Model, Equilibrium Point, Global Stability Analysis, Saturated incidence, Psychological effect.

1. INTRODUCTION

Mathematical model have become important tools to study and analyze the spread and control of infectious diseases. Almost all mathematical models of diseases start from the same basic premise; that the population can be subdivided into a set of distinct classes, dependent upon their experience with respect to the diseases. In this discipline V.H. Badshah and Amit Kumar [3] gave a primary result of mathematical modeling. Most of the proposed mathematical models which describe the transmission of infectious disease have been derived from the classical susceptible infective recover SIR model, which was suggested originally by Kermack and Mc. Kendrick[9] and also gave the result on simple mass action [9]. In that model the susceptible individuals and then the infected individuals may recover and transfer to removal individuals at a specific rate. Numerous mathematical models were developed to study and analyze the spread of infectious diseases in order to prevent or minimize their transmission through quarantine and other measures. The incidence in an epidemiological model is the rate at which the susceptible become infectious. Cappaso and Serio [4] introduced a saturated incidence rate into epidemic model. Mena – Lorca and Hethcote [13] also analyzed a SIRS model with the same saturation incidence. Ruan and Wang [19] studied an epidemic model with a specific nonlinear incident rate. Liu et al. [11,12], Derrick and Ven den Driessche[5] Hethcote and Ven Den Driessche [7] proposed an epidemic model with non-monotonic incidence rate. After that Xiao and Raun [2001] discussed non-monotonic incidence rate. Several different incidence rates have been proposed by many researchers, see for instance, Anderson and May [1], Elteva and Matias [6], Hethcote and Driesech [7], Ruan and Wang [19], Liu et al.[11,12] Derrick and Ven den Driessche [5] , Alexander and Moghadas [2] and Xiao and Raun [22]. Recently Porwal, et al. [15,16,17,18]] presented their work in this field .

2. THE MATHEMATICAL MODEL

2.1 Basic Model

Dongmei Xiao, and Shigui Raun [22] have proposed an epidemic model with non- monotonic incidence which describe psychological effect of certain serious diseases on the community when the number of infective is getting larger. The governing differential equations are given by

$$\frac{dS}{dt}=b-dS-\frac{kIS}{1+\alpha I^2}+\gamma R, \tag{1}$$

$$\frac{dI}{dt}=\frac{kIS}{1+\alpha I^2}-(d+\mu)I, \tag{2}$$

$$\frac{dR}{dt}=\mu I-(d+\gamma)R, \tag{3}$$

Where, $S(t)$, $I(t)$, $R(t)$ represent the number of susceptible, infective and recovered individual respectively. Here b is the recruitment rate of population, d is the natural death rate of the population, k is the proportionality constant, μ is the natural recovery rate of the infective individual, γ is the rate at which recovered individuals loss immunity and return to susceptible class, α is the parameter measures of the psychological or inhibitory effect..

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2.2 Model for Immigration

The model with immigration is given by:

$$\frac{dS}{dt}=b-dS-\frac{kIS}{1+\alpha I^2}+\gamma R+\beta, \tag{4}$$

$$\frac{dI}{dt}=\frac{kIS}{1+\alpha I^2}-(d+\mu)I, \tag{5}$$

$$\frac{dR}{dt}=\mu I-(d+\gamma)R. \tag{6}$$

The section (2.2) of the parameters have similar meanings as for as the model (2.1).

Part I: SIR Model with $0\le I\le I_0$.

In this case the system (4) to (6) reduce to

$$\frac{dS}{dt}=b-dS-\frac{kIS}{1+\alpha I^2}+\gamma R+\beta, \tag{7}$$

$$\frac{dI}{dt}=\frac{kIS}{1+\alpha I^2}-(d+\mu)I, \tag{8}$$

$$\frac{dR}{dt}=\mu I-(d+\gamma)R. \tag{9}$$

The system of equation (7) to (9) always has the DFE $E_0\left(\beta+b\big/_d,0,0\right)$ for any set of parameter values.

To find the endemic equilibrium of (4) to (6), let

$$b-dS-\frac{kIS}{1+\alpha I^2}+\gamma R+\beta=0, \tag{10}$$

$$\frac{kIS}{1+\alpha I^2}-(d+\mu)=0, \tag{11}$$

$$\mu I-(d+\gamma)R=0, \tag{12}$$

Now by equation (6),

$$\Rightarrow \quad R=\frac{\mu I}{(d+\gamma)},$$

and equation (5)

$$\Rightarrow \quad S=\frac{(d+\mu)(1+\alpha I^2)}{k}\mu I-(d+\gamma)R.$$

Thus equation (4),

$$\Rightarrow \quad b-d\left[\frac{(d+\mu)(1+\alpha I^2)}{k}\right]-\frac{kI}{1+\alpha I^2}\left[\frac{(d+\mu)(1+\alpha I^2)}{k}\right]+\gamma\cdot\frac{\mu I}{d+\gamma}+\beta=0,$$

$$\Rightarrow \quad bk-d(d+\mu)-d\alpha I^2(d+\mu)-kI(d+\mu)+\frac{\gamma\mu Ik}{d+\gamma}+\beta k=0,$$

$$\Rightarrow \quad \alpha d(d+\mu)I^2+k\left(d+\mu-\frac{\gamma\mu}{d+\gamma}\right)I+d(d+\gamma)-(b+\beta)k=0. \tag{13}$$

Basic reproduction number

$$R_0=\frac{(b+\beta)k}{d(d+\gamma)},$$

Unique positive equilibrium $E^*=(S^*,I^*,R^*)$ called the endemic equilibrium.

$$\Rightarrow \quad R^*=\frac{\mu}{d+\gamma}I^*,$$

Now equation (11),

$$\Rightarrow \quad \frac{kS}{1+\alpha I^2}-d+\mu=0,$$

$$\Rightarrow \quad \frac{kS}{1+\alpha I^2}=d+\mu.$$

Therefore equation (10)

$$\Rightarrow \quad b-dS^*-\frac{kS^*I^*}{1+\alpha L^2}+\gamma R^*+\beta=0,$$

$$\Rightarrow \quad S^*=\frac{1}{d}\left[(b+\beta)-\left(d+\mu-\frac{\gamma\mu}{d+\gamma}\right)I^*\right].$$

Now by equation (7) we get

quadratic equation

$$I^*=-k\left(d+\mu-\frac{\gamma\mu}{d+\gamma}\right)+\sqrt{\Delta}, \tag{14}$$

where

$$\Delta = k^2 - k\left(d + \mu - \frac{\gamma\mu}{d+\gamma}\right)^2 - 4\alpha d^2 (d+\mu)^2 [1 - R_0],$$

$$R_0 = \frac{(b+\beta)k}{d(d+\mu)},$$

3. MATHEMATICAL ANALYSIS

Lemma 3.1: The plane $S + I + R = (b+\beta)/d$ is an invariant manifold of system (2.1), which is attracting in the first octant.

Adding up the three equation in (2.1)

$$\Rightarrow \quad \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = (b+\beta) - d(S+I+R),$$

$$\Rightarrow \quad \frac{dN}{dt} = (b+\beta) - dN, \tag{15}$$

$\{\because S + I + R = N\},$

It is clear that $N(t) = (b+\beta)/d$ is a solution of equation (15) and for any $N(t_0) \geq 0$, the general solution of equation (15) is.

$$\Rightarrow \quad \frac{dN}{dt} = (b+\beta) - dN,$$

$$\Rightarrow \quad \frac{dN}{dt} + dN = (b+\beta),$$

$$\Rightarrow \quad I.F. = e^{\int \rho dx},$$

$$\Rightarrow \quad = e^{\int d.dt} = e^{dt},$$

$$\Rightarrow \quad N(t)e^{dt} = \int \left(e^{dt} \times (b+\beta)\right) dt + C,$$

Now

$$\Rightarrow \quad C = Ne^{dt_0} - \frac{(b+\beta)}{d} e^{dt_0},$$

$$\Rightarrow \quad N(t) = \frac{1}{d}\left[(b+\beta) - \left((b+\beta) - dN(t_0)\right)e^{-d(t-t_0)}\right],$$

$$\{\because \text{ for } N(t_0) \geq 0\}.$$

Thus

$$\lim_{t\to\infty} N(t) = \frac{(b+\beta)}{d}.$$

which implies the conclusion it is clear that the limit section (2.1) is on the plane $S + I + R = (b+\beta)/d$ thus we focus on the reduced system.

$$\frac{dI}{dt} = \frac{kI}{1+\alpha I^2}\left(\frac{(b+\beta)}{d} - I - R\right) - (d+\mu)I = P(I,R),$$

$$\left\{\because S + I + R = \frac{(b+\beta)}{d}\right\},$$

$$\frac{dR}{dt} = \mu I - (d+\gamma)R = Q(I,R), \tag{16}$$

$$DP(I,R) = \left(\frac{b}{d} - I - R\right) - (d+\mu)\frac{1+\alpha I^2}{kI}, \tag{17}$$

$$DQ(I,R) = \mu\left(\frac{1+\alpha I^2}{k}\right) - (d+\gamma)\frac{1+\alpha I^2}{kI}, \tag{18}$$

Theorem 3.2: System (16) does not have nontrivial periodic orbits.

Proof. Consider system (16) for $I > 0$ and $R > 0$. Take Dulac function

$$D(I,R) = \frac{1+\alpha I^2}{kI},\ \{differentiate\ equation\ (17), (18)\ w.r.t\ I\ and\ R, and\ adding\},$$

We have

$$\frac{\partial(DP)}{\partial I} + \frac{\partial(DQ)}{\partial R} = -1 - \frac{2\alpha(d+\mu)I}{k} - \frac{1+\alpha I^2}{kI}(d+\gamma)R < 0,$$

The conclusion follows

In order to study the properties of the $DFE\ E_0$, and the equilibrium E^*, we rescale (16) by

Substituting, $x = \frac{k}{(d+\gamma)}I,$ $\qquad y = \frac{k}{(d+\gamma)}R,$

$$\tau = (d+\gamma)t,$$

$$d\tau = (d+\gamma)dt,$$

$$\frac{d\tau}{dt} = (d+\gamma),$$

$$x = \frac{k}{(d+\gamma)}I,$$

$$\Rightarrow \qquad \frac{dx}{d\tau} = \frac{k}{d+\gamma}\cdot\frac{dI}{d\tau},$$

$$\Rightarrow \qquad \frac{dx}{d\tau} = \frac{k}{d+\gamma}\cdot\frac{\frac{dI}{dt}}{\frac{d\tau}{dt}},$$

$$\Rightarrow \qquad = \frac{k}{d+\gamma}\left\{\frac{\frac{kI}{1+\alpha I^2}\left(\frac{(b+\beta)}{d} - I - R\right) - (d+\mu)I}{d+\gamma}\right\},$$

$$\Rightarrow \qquad = \frac{kI}{\frac{1+\alpha(d+\gamma)^2}{k^2}\cdot\frac{k^2}{(d+\gamma)^2}\cdot I^2(d+\gamma)}\left(\frac{(b+\beta)k}{d(d+\gamma)} - \frac{kI}{d+\gamma} - \frac{kR}{d+\gamma}\right) - \frac{(d+\mu)}{d+\gamma}\cdot\frac{kI}{d+\gamma}$$

Where $p = \frac{\alpha(d+\gamma)^2}{k^2},$ $\qquad A = \frac{(b+\beta)k}{(d+\gamma)},$

$$m = \frac{(d+\mu)}{(d+\gamma)}, \qquad q = \frac{\mu}{d+\gamma},$$

We obtain

$$\frac{dx}{d\tau}=\frac{x}{1+px^2}(A-x-y)-mx, \tag{19}$$

$$\Rightarrow \quad y=\frac{k}{d+\gamma}R$$

$$\Rightarrow \quad \frac{dy}{d\tau}=\frac{k}{d+\gamma}\frac{dR}{dt},$$

$$\Rightarrow \quad \frac{dy}{d\tau}=\frac{k}{d+\gamma}\cdot\frac{\frac{dR}{dt}}{\frac{d\tau}{dt}},$$

$$\Rightarrow \quad \frac{dy}{d\tau}=\frac{\mu}{d+\gamma}\cdot\frac{kI}{d+\gamma}-\frac{kR}{d+\gamma},$$

$$\Rightarrow \quad \frac{dy}{d\tau}=qx-y, \tag{20}$$

$$\Rightarrow \quad \left.\begin{aligned}\frac{dx}{d\tau}&=\frac{x}{1+px^2}(A-x-y)-mx\\ \frac{dy}{d\tau}&=qx-y,\end{aligned}\right\}, \tag{21}$$

Note that the trivial equilibrium $(0,0)$ of system (21) is the disease free equilibrium E_0 of model (2.1) and the unique positive equilibrium (x^*, y^*) of system (21) is the endemic equilibrium E^* of model (2.1) iff $m-A<0$, where

$$x^*=\frac{-(1-q)+\sqrt{(1+q)^2-4mp(m-A)}}{2mp},$$

$$y^*=qx^*,$$

We first determine the stability and topological type of $(0,0)$, the jacobian matrix of system (21) at $(0,0)$, is differentiating (21), model w.r.to. x and y we get

$$M_0=\begin{bmatrix}A-m, & 0\\ q & -1\end{bmatrix},$$

If $A-m=0$, then there exists a small neighborhood number of $(0,0)$ such that the dynamics of system (21) are equivalent to that of

$$\frac{dx}{d\tau}=-x^2-2xy+0\left((x,y)^3\right),$$

$$\frac{dy}{d\tau}=qx-y.$$

Hence we obtain the result.

Theorem 3.3. The $DFE\,(0,0)$ of system (18) is

(i) a stable hyperbolic node if $m-A>0$

(ii) a saddle node if $m-A=0$;

(iii) a hyperbolic saddle if $m-A<0$

when $m-A<0$, we discuss the stability and topological type of the endemic equilibrium (x^{*},y^{*}) the Jacobian matrix of (21) at (x^{*},y^{*}) is differentiate (21) model w.r.t. x and y.

$$M_0=\begin{bmatrix}\frac{x\left(Px^{*2}+2pqx^{*2}-2Apx^{*}-1\right)}{(1+px^{*})^2} & \frac{-x^{*}}{1+px^{*2}}\\ q & -1\end{bmatrix},$$

We have that

$$\Rightarrow \det(M_1)=\frac{-x^{*}\left(px^{*2}+2pqx^{*2}-2Apx^{*}-1\right)}{\left(1+px^{*2}\right)^2}+\frac{qx^{*}}{1+px^{*2}},$$

The sign of $\det(M_1)$ is determined by

$S_1=1+q+2Apx^{*}-(1+q)px^{*2}$,

Now that

$$mpx^{*2}+(1+q)x^{*}+m-A=0, \quad (22)$$

$mS_1+(1+q)\times 22,$

$$mS_1=m+mq+2Ampx^{*}-(1+q)mpx^{*2}, \quad (23)$$

$(1+q)\times equation\ (22)$

$$=(1+q)mpx^{*2}+(1+q)^2x^{*}+(1+q)(m-A), \quad (24)$$

Now adding equation (23) and (24)

$$mS_1=\left[-mp(1+q)+(1+q)mp\right]x^{*2}+\left[2Amp+(1+q)^2\right]x^{*}+m+mq+(1+q)(m-A),$$

$$=\left[2Amp+(1+q)^2\right]x^{*}+(1+q)(2m-A),$$

Substituting

$$x^{*}=\frac{-(1+q)+\Delta_1}{2mp},$$

Where

$$\Delta_1=\sqrt{(1+q)^2-4mp\,(m-A)}$$

Now

$$\Rightarrow mS_1=\frac{-(1+q)+\Delta_1}{2mp}\left\{2Amp+(1+q)^2\right\}+(1+q)(2m-A),$$

$$\Rightarrow mS_1=\frac{-(1+q)}{2mp}\left\{\frac{\Delta_1^2}{(1+q)^2-4m^2p+4mpA}\right\}+\frac{\Delta_1}{2mp}\left\{2Amp+(1+q)^2\right\},$$

$$\because \Delta_1^2=(1+q)^2-4mp(m-A)$$

$$\Rightarrow\because \Delta_1^2=\frac{1}{2mp}\left[-(1+q)\Delta_1^2-\Delta_1\left\{2Amp+(1+q)^2\right\}\right]$$

$$\Rightarrow \quad =\frac{4m^2p^2A^2}{(1+q)^2}+4m^2p>0,$$

It follows that $S_1>0$, hence, $\det(M_1)>0$ and (x^{*},y^{*}) is a node or a focus or a centre. Furthermore, we have the following result on the stability of (x^{*},y^{*}).

Theorem 3.4: Suppose $m-A<0$, then there is a unique endemic equilibrium (x^{*},y^{*}) of model (21) which is a stable node.

We know that the stability of (x^{*},y^{*}) is determined by $tr(M_1)$ we have

$$\Rightarrow tr(M_1) = \frac{-p^2x^{*4} + (1+2q)px^{*3} - 2(1+A)px^{*2} - x^* - 1}{(1+px^{*2})^2},$$

The sign of $tr(M_1)$, is determined by

$$\Rightarrow \quad S_2 = -p^2x^{*4} + (1+2q)px^{*3} - 2(1+A)px^{*2} - x^* - 1,$$

We claim that $S_2 \neq 0$. to see this, note that

$$\Rightarrow \quad mpx^{*2} + (1+q)x^*, + m - A = 0,$$

Then we have

$$\Rightarrow \quad x^{*2} = \frac{-(1+q)x^* - (m-A)}{mp},$$

$$\Rightarrow \quad m^3pS_2 = -m^3p^3x^{*4} + (1+2q)m^3p^2x^{*3} - 2(1+A)m^3p^2x^{*2} - m^3px^* - m^3p,$$

$$\Rightarrow \quad m^3p^3 = \left[\frac{-(1+q)x^* - (m-A)}{mp}\right]^2 + (1+2q)m^3p^2\left[\frac{-(1+q)x^* - (m-A)}{mp}\right]x^*$$

$$-2(1+A)m^3p^2\left[\frac{-(1+q)x^* - (m-A)}{mp}\right] - m^3px^* - m^3p$$

$$\Rightarrow \quad \frac{m^3p^3}{m^2p^2} = \left[(1+q)^2x^{*2} + (m-A)^2 + 2(1+q)x^*(m-A)\right]$$

$$+(1+2q)\frac{m^3p^2}{mp}\left[-(1+q)x^* - (m-A)\right]x^* - \frac{2(1+A)m^3p^2}{mp}$$

$$\left[-(1+q)x^* - (m-A)\right] - m^3px^* - m^3p$$

$$\Rightarrow \quad = -mp\left[(1+q)^2\left\{\frac{-(1+q)x^* - (m-A)}{mp}\right\} + (m-A)^2 + 2(1+q)x^*(m-A)\right]$$

$$+(1+2q)m^2p\left[-(1+q)x^{*2} - (m-A)x^*\right] - 2(1+A)m^2p\left[-(1+q)x^* - (m-A)\right]$$

$$-m^3px^* - m^3p$$

$$\Rightarrow \quad = -(1+q)^2\{-(1+q)x^* - (m-A)\} - (m-A)^2mp - 2(1+q)x^*(m-A)mp$$

$$-(1+2q)m^2p(1+q)x^{*2} - (m-A)x^*(1+2q)m^2p + 2(1+A)m^2p(1+q)x^*$$

$$+2(1+A)m^2p(m-A) - m^3px^* - m^3p,$$

$$\Rightarrow \quad = \begin{bmatrix} (1+q)^3 - 2(1+q)(m^2 - mpA) + (1+2q)m(1+q)^2 - (m-A)(m^2p + 2m^2pq) \\ +2(1+A)(m^2p + m^2pq) - m^3p \end{bmatrix} x^*$$

$$+(1+q)^2m - (1+q)^2A - (m^2 + A^2 - 2mA)mp + (1+2q)m(1+q)m - (1+2q)mA$$

$$(1+q) + 2m^2p(m-A) + 2Am^2p(m-A) - m^3p,$$

$$\Rightarrow \quad = \left\{(2mp + 3m^2p + 2mpq + 4m^2pq)A + (1+q)^3 + m(1+q)^2(1+2q) - (1+q)2m^3p\right\}x^*,$$

$$\left\{-(1+q)^2A - m(1+q)(1+2q)A + 2m^3pA + (1+q)^2m + m^2(1+q)(1+2q) - mp(1+2m)A^2\right\}$$

$$m^3pS_2 = (B_1A + B_2)x^* + (B_3A + B_4),$$

$$\Rightarrow \quad B_1 = mp(2 + 3m + 2q + 4mq),$$

$$B_2 = (1+q)\left[(1+q)^2 + m(1+q)(1+2q) - 2m^3p\right],$$

$$B_3 = -(1+q)^2 - m(1+q)(1+2q) + 2m^3p,$$

$$B_4 = m\left[(1+q)^2 + m(1+q)(1+2q) - p(1+2m)A^2\right],$$

When $m - A < 0$, we can see that $B_1A + B_2 > 0$,

$$\Rightarrow \quad Let\ \xi = mpx^{*2} + m - A,$$

Similarly we have

$$\Rightarrow \qquad (B_1 A + B_2)^2 \xi = m^3 p\rho s_2 + S_3,$$

Where ρ is a polynomial of $x*$ and

$$S_3 = m^3 p\left(1 + A^2 P + 2q + q^2\right)\begin{bmatrix}\left(A + 2Am - 2m^2\right)^2 p \\ +\left(1 + A - m + q\right)\left(1 + m + q + 2mq\right)\end{bmatrix},$$

Assume that $S_2 = o,$ since $\xi = 0,$ it follows that $S_3 = 0.$ However, when $m - A < 0,$ we have $S_3 > 0.$ therefore, $S_2 \neq 0,$ for any positive value of the parameters $p, q,$ and $A,$ that is, $t_r(M_1) \neq 0.$ thus $m - A < 0,$ implies that $(x * y*)$ does not change stability.

Take (i) $m = 4,\ A = 8,\ p = 1,\ q = 1$, then $x* =,0.780776 \qquad y* = 0.780776,$ $tr(M_1) = -4.51493 < 0.$

(ii) $m = 2,\ A = 4,\ p = 1,\ q = 1$, then $x* =,0.414214 \quad y* = 0.414214,$ $tr(M_1) = -2.55279 < 0.$ By the continuity of $tr(M_1)$ on the parameters,

We know that $tr(M_1) < 0$ for $m - A < 0.$

4. CONCLUSION

In this paper we have carried out a result on SIR model with immigration and studied the existence and stability of disease- free and endemic equilibria with the basic reproduction number $R_0 = \dfrac{(b+\beta)k}{d(d+\gamma)}$.

Our main result indicates that when $R_0 < 1,$ the diseases-free equilibrium is stable and when $R_0 > 1,$ the endemic equilibrium exist and locally asymptotically stable.

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