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Graceful labeling on a new family of graphs *

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Abstract Let G be a graph with $V(G)$ as the vertex set and q edges. Let u, v be in $V(G)$ and define an injective function f from $V(G)$ to $\{0, 1, 2, \dots, q\}$ such that the resulting edges are also distinct and each edge is assigned the value $|f(u) - f(v)|$. This function f is called the graceful labeling of G . A graph which accepts a graceful labeling is called a graceful graph. In this paper, we prove that the graphs obtained by the join of two complete bipartite graphs and the join of two ladder graphs by a path of arbitrary length are graceful.

Key words Graceful labeling, path, complete bipartite graph, ladder graph.

2010 Mathematics Subject Classification Primary: 05C78, 05C90, 05C99.

1 Introduction

The most interesting and famous graph labeling method is the *graceful labeling* of graphs introduced by Rosa [8] in 1967. A variety of graphs and families of graphs are known as to be graceful for the past five decades. Caterpillars are proved to be graceful by Rosa [8]. Sethuraman and Jeba Jesintha [4, 5] proved that all banana trees and extended banana trees are graceful. Rosa [8] showed that the n cycle C_n is graceful if and only if $n \equiv 0$ or $3 \pmod{4}$. Bloom and Golomb [2] proved that the complete bipartite graph $K_{m,n}$ is graceful. Sudha and Kanniga [6] proved that the arbitrary super subdivision of helms, centipedes and ladder graphs are graceful. In 2005, Barrientos [3] introduced kC_4 -snakes graph as a generalization of the concept of triangular snake graphs introduced by Rosa [8] and he proves that kC_4 -snakes are graceful. Graceful labeling is actively being used in many research fields such

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as communication in sensor networks, designing fault tolerant systems, automatic channel allocation, coding theory problems, X-ray, optimal circuit layout and additive number theory.

In this paper, we prove that the graphs obtained by the join of two complete bipartite graphs and the join of two ladder graphs by a path of arbitrary length are graceful.

Definition 1.1. A simple graph $G = (V, E)$ is called bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$ and every edge of G is of the form $\{a, b\}$ with one of the vertices a and b in V_1 and the other in V_2 . If every vertex in V_1 is joined to every vertex in V_2 we can obtain a complete bipartite graph.

Example 1.2.

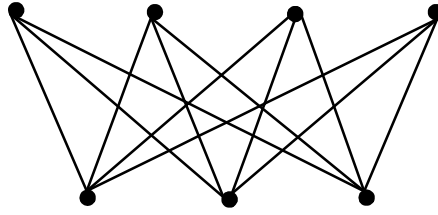


Fig. 1: Complete Bipartite Graph with $m = 4$ and $n = 3$.

Definition 1.3. An undirected planar graph with $2n$ vertices and $3n - 2$ edges is called a ladder graph L_n . It can be acquired as the Cartesian product of two path graphs.

Example 1.4.

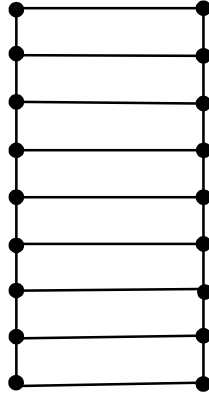
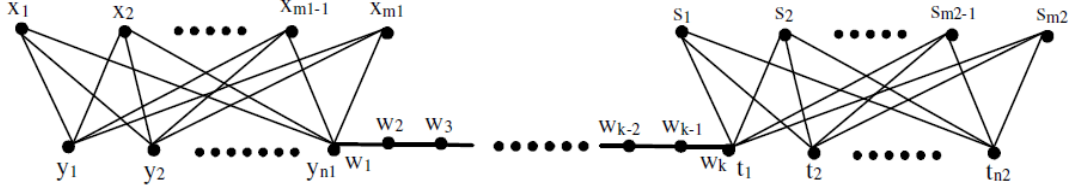


Fig. 2: The Ladder Graph.

2 Main Results

Theorem 2.1. *The graph obtained by joining two complete bipartite graphs by a path of arbitrary length is graceful.*

Proof. Let $G = (V, E)$ be the graph obtained by joining two complete bipartite graphs by a path P_k of length k . Let $|V(G)| = p$ and $|E(G)| = q$. Let $x_1, x_2, \dots, x_{m_1}$ and $y_1, y_2, \dots, y_{m_1}$ be the vertices of the first complete bipartite graph K_{m_1, n_1} and let $s_1, s_2, \dots, s_{m_2}$ and $t_1, t_2, \dots, t_{m_2}$ be the vertices of the second complete bipartite graph K_{m_2, n_2} and $w_1, w_2, \dots, w_k$ be the vertices of the path P_k with $y_{m_1} = w_1$ and $w_k = t_1$. Thus, for the graph G one has $p = m_1 + n_1 + m_2 + n_2 + k - 2$ and $q = m_1 n_1 + m_2 n_2 + k - 1$ as shown in Fig. 3.

Fig. 3: The Generalized Graph G .

The vertex labels for the graph G are as follows:

$$\begin{aligned}
 f(x_i) &= q - n_1(i - 1), \text{ for } 1 \leq i \leq m_1, \\
 f(y_j) &= j - 1, \text{ for } 1 \leq j \leq n_1, \\
 f(w_{2l}) &= f(x_{m_1}) - r, \text{ for } 1 \leq r \leq \left\lfloor \frac{k}{2} \right\rfloor, \\
 f(w_{2l+1}) &= f(y_{n_1}) + r, \text{ for } 1 \leq r \leq \left\lfloor \frac{k}{2} \right\rfloor.
 \end{aligned} \tag{2.1}$$

Case 1: If k is odd, then

$$\begin{aligned}
 f(s_g) &= f(w_{k-1}) - 1 - n_2(g - 1), \text{ for } 1 \leq g \leq m_2, \\
 f(t_h) &= f(w_{k-2}) + h, \text{ for } 1 \leq h \leq n_2.
 \end{aligned} \tag{2.2}$$

Case 2: If k is even, then

$$\begin{aligned}
 f(s_g) &= f(w_{k-1}) + 1 + n_2(g - 1), \text{ for } 1 \leq g \leq m_2, \\
 f(t_h) &= f(w_{k-2}) - h, \text{ for } 1 \leq h \leq n_2.
 \end{aligned} \tag{2.3}$$

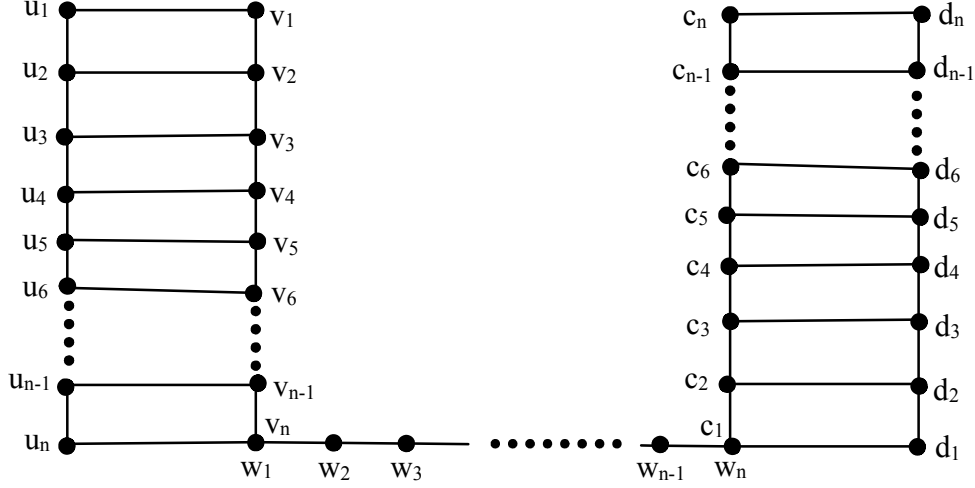
From equations (2.1) to (2.3) we see that the vertex labels $0, 1, 2, 3, \dots, q$ are distinct. Thus the graph G is graceful.

□

Theorem 2.2. *The graph obtained by joining two ladder graphs by a path of arbitrary length is graceful.*

Proof. Let $G = (V, E)$ be the graph obtained by joining two ladder graphs by a path P_k of length k . Let $|V(G)| = p$ and $|E(G)| = q$. Let $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ be the vertices on the path P_n of the first ladder graph from the top to the bottom on the left side and the right side respectively. Similarly the vertices on the left side and the right side of the path P_n of the second ladder graph from the bottom to the top are $c_1, c_2, \dots, c_n$ and $d_1, d_2, \dots, d_n$ respectively. Let $w_1, w_2, \dots, w_k$ be the vertices of the path P_k with $v_n = w_1$ and $w_k = c_1$. The graph G has $p = 4n + k - 2$ vertices and $q = 6n + k - 5$ edges as shown in the Fig. 4.

Define $f : V \rightarrow \{0, 1, \dots, q\}$, where $q = 6n + k - 5$ as follows:

Fig. 4: The Generalized Graph G .

The vertices of the first ladder graph are denoted as:

$$\begin{aligned}
 f(u_1) &= 0, \\
 f(u_{2i+1}) &= 2 + 3(i-1), \text{ for } 1 \leq i \leq \frac{n}{2}, \\
 f(u_{2i}) &= q - 3(i-1), \text{ for } 1 \leq i \leq \frac{n}{2}, \\
 f(v_{2i-1}) &= q - 1 - 3(i-1), \text{ when } n \text{ is even, for } 1 \leq i \leq \frac{n}{2}, \\
 f(v_{2i-1}) &= q - 1 - 3(i-1), \text{ for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\
 f(v_{2i-1}) &= q - 3(i-1) \text{ when } n \text{ is odd, for } i = \left\lceil \frac{n}{2} \right\rceil, \\
 f(v_{2i}) &= 3i, \text{ for } 1 \leq i \leq \frac{n}{2}, \\
 f(v_{2i}) &= 3i - 1, \text{ for } i = \frac{n}{2}.
 \end{aligned} \tag{2.4}$$

Labeling for the path P_n is given by:

Case 1: When n is even

$$\begin{aligned}
 f(w_{2j-1}) &= f(v_n) + j - 1, \text{ for } 1 \leq j \leq \left\lceil \frac{k}{2} \right\rceil, \\
 f(w_j) &= f(v_{n-1}) - j, \text{ for } 1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor.
 \end{aligned}$$

Case 2: When n is odd

$$\begin{aligned}
 f(w_{2j-1}) &= f(v_n) - j - 1, \text{ for } 1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor, \\
 f(w_j) &= f(v_{n-1}) + j, \text{ for } 1 \leq j \leq \left\lceil \frac{k}{2} \right\rceil.
 \end{aligned} \tag{2.5}$$

Vertex labeling of the second ladder graph attached at last is given by:

Case 1: When k is odd,

$$f(c_1) = \begin{cases} f(w_{k-2}) + 1, & \text{for } n \text{ is even,} \\ f(w_{k-2}) - 1, & \text{for } n \text{ is odd,} \end{cases}$$

$$\begin{aligned}
f(c_2) &= \begin{cases} f(w_{k-1}) + 1, & \text{for } n \text{ is odd,} \\ f(w_{k-1}) - 1, & \text{for } n \text{ is even,} \end{cases} \\
f(c_{2i+1}) &= f(c_1) + 2 + 3(i-1), \text{ for } 1 \leq i \leq \frac{n}{2}, \\
f(c_{2i}) &= f(c_2) - 3, \text{ for } 1 \leq i \leq \frac{n}{2}, \\
f(d_1) &= f(w_{k-2}) - 2, \\
f(d_2) &= f(c_1) + 3, \\
f(d_{2i+1}) &= f(d_1) - 3i, \text{ for } 1 \leq i \leq \frac{n}{2}, \\
f(d_{2i}) &= \begin{cases} f(d_2) + 3i, & \text{for } 1 \leq i \leq \frac{n}{2}, \\ f(d_2) + 3i - 1, & \text{for } i < \frac{n}{2}. \end{cases} \tag{2.6}
\end{aligned}$$

Case 2: When k is even,

$$\begin{aligned}
f(c_1) &= \begin{cases} f(w_{k-2}) - 1, & \text{for } n \text{ is even,} \\ f(w_{k-2}) + 1, & \text{for } n \text{ is odd,} \end{cases} \\
f(c_2) &= \begin{cases} f(w_{k-1}) + 3, & \text{for } n \text{ is odd,} \\ f(w_{k-1}) - 3, & \text{for } n \text{ is even,} \end{cases} \\
f(c_{2i+1}) &= f(c_1) - 2 - 3(i-1), \text{ for } 1 \leq i \leq \frac{n}{2}, \\
f(c_{2i}) &= \begin{cases} f(c_2) - 3(i-1), & \text{when } n \text{ is odd, for } 2 \leq i \leq \frac{n}{2}, \\ f(c_2) + 3(i-1), & \text{when } n \text{ is even, for } 2 \leq i \leq \frac{n}{2}, \end{cases} \\
f(d_1) &= \begin{cases} f(w_{k-1}) - 2, & \text{for } n \text{ is odd,} \\ f(w_{k-1}) + 2, & \text{for } n \text{ is even,} \end{cases} \\
f(d_2) &= \begin{cases} f(w_k) + 3, & \text{for } n \text{ is odd,} \\ f(w_k) - 3, & \text{for } n \text{ is even,} \end{cases} \\
f(d_{2i+1}) &= f(y_1) - 3i, \text{ for } 1 \leq i \leq \frac{n}{2}, \\
f(d_{2i}) &= \begin{cases} f(w_k) + 3i, & \text{when } n \text{ is odd, for } 1 \leq i \leq \frac{n}{2}, \\ f(w_k) - 3i, & \text{when } n \text{ is even, for } 1 \leq i \leq \frac{n}{2}. \end{cases} \tag{2.7}
\end{aligned}$$

From (2.4) to (2.7), it is clear that vertex labels of graph G are distinct. Thus the graph G is graceful. $\square$

3 Conclusion

In this paper, we have shown that the graphs obtained by the join of two complete bipartite graphs and the join of two ladder graphs by a path of arbitrary length are graceful. Further we intend to prove the same for the step grid graph and the double quadrilateral snake graph.

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