Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics
Website : https : //www.bpasjournals.com/

Bull. Pure Appl. Sci. Sect. E Math. Stat.
38E(Special Issue)(2S), 54–59 (2019)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI: 10.5958/2320-3226.2019.00079.1

Secure domination in degree splitting graphs of certain graphs *

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Abstract A secure dominating set of a graph $G = (V, E)$ is a dominating set $S \subseteq V$, if for each $u \in V - S$, there exists a v such that $v \in N(u) \bigcap S$ and $(S - \{v\}) \bigcup \{u\}$ is dominating. The minimum cardinality of a secure dominating set is the secure domination number, $\gamma_s(G)$. In this paper, we find $\gamma_s(G)$ for degree splitting graphs of few classes of graphs like paths, complete binary trees, complete graphs and complete bipartite graphs. Further, bounds for $\gamma_s(G)$ of degree splitting graphs of regular graphs and few classes of caterpillars are determined.

Key words Domination, Secure domination, Degree splitting graph.

2010 Mathematics Subject Classification 05C69.

1 Introduction

The graphs considered in this paper are finite, simple and undirected with vertex set $V(G)$ and edge set $E(G)$. We study the problem of using guards to defend the vertices of G against an attack. At each vertex of a graph, at most one guard is stationed. A guard at a vertex v can deal with the problem at any other vertex in its closed neighborhood. A *guard* can be considered as a unit of force (or server unit) capable of moving along an edge of a graph, whose purpose is to defend (or protect or secure) a vertex or set of vertices. In the process, a guard can protect the vertex at which it is located and can move to a neighboring vertex to defend an attack there. This paper deals with the "*secure*" version of the problem in which the configuration of guards induces a dominating set before and after an attack has been defended. The minimum number of guards required when at most one guard is allowed to move in order to defend an attack, so that the configuration of the guards induces a dominating set before and after an attack, is called the *secure domination number* and it is denoted by $\gamma_s(G)$. The notion of *secure domination* was introduced by E.J. Cockayne et al. [6]. In [6] four strategies for the protection of a graph, by placing guards at vertices were discussed. Several papers on secure domination have appeared, for example, [1, 2, 3, 4, 5, 6, 8, 9, 10]. We now give a formal definition of a secure dominating set.

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received March 16, 2019 / Revised November 08, 2019 / Accepted December 14, 2019. Online First Published on June 03, 2020 at https://www.bpasjournals.com/.
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Refereed Proceedings of the National Conference on Mathematics and Computer Applications held at the Department of Mathematics, Women's Christian College, Chennai, India from January 29, 2019 to January 30, 2019.

A set S is a *secure dominating set* (or SDS) if for each $u \in V - S$, there exists a v such that

$$v \in N(u) \cap S \quad \text{and} \quad (S - \{v\}) \cup \{u\} \quad \text{is dominating.} \tag{1.1}$$

We say that u is *S-defended* by v (or v *S-defends* u) if (1.1) is satisfied. The parameter $\gamma_s(G)$ is the minimum cardinality of an SDS of G and the corresponding set is called a $\gamma_s(G)$-set or γ_s-set of G.
In this paper, we find $\gamma_s(G)$ of degree splitting graphs of paths, complete binary trees, complete graphs and complete bipartite graphs. Further, bounds for $\gamma_s(G)$ of degree splitting graphs of regular graphs and few classes of caterpillars are determined.
Secure domination in degree splitting graphs finds its applications in computer communication networks, radio stations etc.

2 Definitions and Preliminaries

For graph theoretic notations and terminology in general, we follow [7].
For a vertex $v \in V$, the *open neighborhood of* v is the set $N(v) = \{u \in V : uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its open neighborhood is the set $N(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is the set $N[S] = N(S) \cup S$.
The *private neighborhood* $pn(v, S)$ of $v \in S$ is defined by $pn(v, S) = N(v) - N(S - \{v\})$. Equivalently, $pn(v, S) = \{u \in V : N(u) \cap S = \{v\}\}$. Each vertex in $pn(v, S)$ is called a *private neighbor of* v.
The *external private neighborhood* $epn(v, S)$ of v with respect to S consists of those private neighbors of v in $V - S$, while the *internal private neighborhood* $ipn(v, S)$ of v with respect to S consists of those private neighbors of v in S. Thus $epn(v, S) = pn(v, S) \cap (V - S)$ and $ipn(v, S) = pn(v, S) \cap S$, while $pn(v, S) = epn(v, S) \cup ipn(v, S)$.
A set $S \subseteq V$ is a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality among all dominating sets of G.
A *secure dominating set* (SDS) of G is a set $S \subseteq V$ if for each $u \in V - S$, there exists a v such that $v \in N(u) \cap S$ and $(S - \{v\}) \cup \{u\}$ is dominating. The minimum cardinality of a secure dominating set is the *secure domination number* and is denoted by $\gamma_s(G)$.
The *degree* of a vertex v in graph G is defined to be the number of edges incident to v and is denoted by deg v. The *maximum degree* of a graph G, denoted by $\triangle(G)$, is defined to be
$\triangle(G) = max\{\deg v : v \in V(G)\}$. A *leaf* is a vertex whose degree is one.
A *support* is a vertex which is adjacent to at least one leaf. A *weak support* is a vertex which is adjacent to exactly one leaf. A *strong support* is a vertex which is adjacent to at least two leaf vertices.
Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of vertices having at least two vertices and having the same degree and $T = V - \cup S_i$.
The *degree splitting graph* [10] of G is denoted by $DS(G)$ and it is obtained from G by adding vertices $w_1, w_2, \ldots, w_t$ and joining w_i to each vertex of S_i $(1 \le i \le t)$ see Fig. 1).

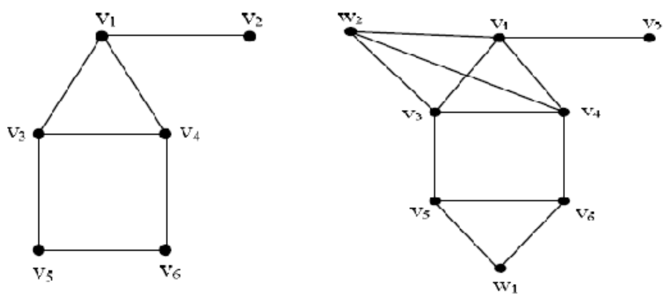

Fig. 1: A graph G and its degree splitting graph $DS(G)$.

A *caterpillar* is a tree whose removal of leaf vertices leaves a path called the *spine* of the caterpillar.
A *complete binary tree* is a rooted tree in which all leaves have the same depth and all internal vertices have degree three except the root, which is of degree two. If T is a complete binary tree with root vertex v, the set of all vertices with depth l are called vertices at level l.

A graph G is *complete* if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n.
A graph $G = (V,\ E)$ with bipartitions $V = (V_1,\ V_2)$ is said to be a *complete bipartite graph* if every vertex in V_1 is adjacent to every vertex of V_2. It is denoted by $K_{m,n}$.
For $v \in S \subseteq V$, $w \in V - S$ is an *$S-$ external private neighbor of* v (abbreviated *$S -$ epn of* v) if $N(w) \cap S = \{v\}$. Let $P(v,\ S)$ be the set of all $S-$ epn of v.

Proposition 2.1. *[6] Let S be a dominating set. Vertex $v \in S$ defends $u \in V - S$ if and only if $G[P(v,\ S)\ \cup \{u,\ v\}]$ is complete.*

Proposition 2.2. *[6] S is an SDS if and only if for each $u \in S$, there exists $v \in S$ such that $G[P(v,\ S)\ \cup \{u,\ v\}]$ is complete.*

Proposition 2.3. *[6] For a complete graph K_n, $\gamma(G) = \gamma_s(G) = 1$.*

3 Degree splitting graphs of certain trees

In this section secure domination number of degree splitting graphs of some classes of trees like paths and complete binary trees are determined.
We observe that for a path P_n, $\gamma_s(DS(P_n)) = 1$, if $n = 2$ and $\gamma_s(DS(P_n)) = 2$, if $n = 3$.

Theorem 3.1. *For a path P_n, $n \geq 4$, $\gamma_s(DS(P_n)) = \left\lfloor \frac{n+2}{3} \right\rfloor + 1$.*

Proof. Let $G = DS(P_n)$ and S be an SDS of G. Consider $V(P_n) = \{v_1,\ v_2 \ldots v_n\}$ and $V(G) = V(P_n) \cup \{w_1, w_2\}$, where $deg\ w_1 = n - 2$ and $deg\ w_2 = 2$. Suppose on contradiction there exists an SDS, S with $|S| < \left\lfloor \frac{n+2}{3} \right\rfloor + 1$. Let $|S| = \left\lfloor \frac{n+2}{3} \right\rfloor$. We have the following cases:
Case (i) $S \subseteq V(P_n)$
In this case, without loss of generality there exists a v_i, $1 \leq i \leq n$ such that the set $(S - \{v_i\}) \cup \{v_{i+1}\}$ is not a dominating set, which is a contradiction to the definition of S.
Therefore $|S| \geq \left\lfloor \frac{n+2}{3} \right\rfloor + 1$.
Case (ii) $S \subseteq V(P_n) \cup \{w_1\}$
The vertices $\left\lfloor \frac{n+2}{3} \right\rfloor - 1$ of S lie on the path P_n.
In this case, S is not a secure dominating set, which is a contradiction. Therefore $|S| \geq \left\lfloor \frac{n+2}{3} \right\rfloor + 1$.
Case (iii) $S \subseteq V(P_n) \cup \{w_2\}$
The same argument holds as in the Case (ii). Therefore we have, $|S| \geq \left\lfloor \frac{n+2}{3} \right\rfloor + 1$.
Case (iv) $S \subseteq V(P_n) \cup \{w_1, w_2\}$
In this case, $\left\lfloor \frac{n+2}{3} \right\rfloor - 2$ vertices belong to S.
Without loss of generality, there exists a v_i in $V - S$ such that $(S - \{w_1\}) \cup \{v_i\}$ is not a dominating set, which is a contradiction. Therefore, $|S| \geq \left\lfloor \frac{n+2}{3} \right\rfloor + 1$.
Hence in all the above cases we obtain, $|S| \geq \left\lfloor \frac{n+2}{3} \right\rfloor + 1$. Therefore $\gamma_s(DS(P_n)) = \left\lfloor \frac{n+2}{3} \right\rfloor + 1$ and a $\gamma_s(G)$-set is given by $\{w_1, w_2, v_2, v_5, v_8 \ldots v_{n-2}\}$ (see Fig. 2).

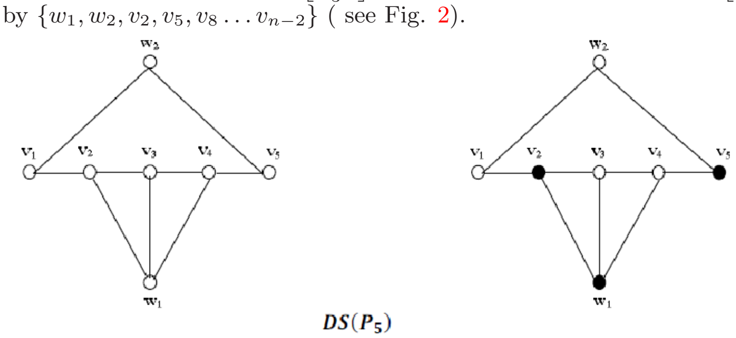

Fig. 2: Shaded vertices denote $\gamma_s(DS(P_5))$ set.

□

Theorem 3.2. *For a complete binary tree T of level l,* $\gamma_s(DS(T)) = \sum_{\substack{1 \leqslant i \leqslant l \\ k = 3i-2 \\ k \leqslant l}} 2^{l-k} + 2.$

Proof. Let $G = DS(T)$ and $V(G) = \{v_i : 1 \leq i \leq 2^{l-1} + 2^{l-2} + 2^{l-3}\} \cup \{u_i : 1 \leq i \leq 2^l\} \cup \{u\} \cup \{w_1, w_2\}$, where $\deg v_i = 3$ for each i, $1 \leq i \leq 2^{3l-6}$ and $\deg u_i = 1$ for each i, $1 \leq i \leq 2^l$ and $\deg u = 2$.
Let S be an SDS of G.
To prove that $|S| \geqslant \sum_{\substack{1 \leqslant i \leqslant l \\ k = 3i-2 \\ k \leqslant l}} 2^{l-k} + 2.$
Suppose on the contrary that $|S| < \sum_{\substack{1 \leqslant i \leqslant l \\ k = 3i-2 \\ k \leqslant l}} 2^{l-k} + 2$. Let $|S| = \sum_{\substack{1 \leqslant i \leqslant l \\ k = 3i-2 \\ k \leqslant l}} 2^{l-k} + 1.$
Since each u_i is adjacent to w_1, where $\deg w_1 = 2^l$, place a guard at w_1. Also place guards at each of the 2^{l-1} vertices at level $l-1$. Continuing in this way by placing guards at 2^{l-4} vertices of $l-4$ level and so on up to $l-k$ level, $k \leq l$ and $k = 3i-2$, $1 \leq i \leq l$, we see that S is a dominating set. But we can see that there exists a vertex v_i such that $(S - \{v_j\}) \cup \{v_i\}$, where $v_i \notin S$ and $i \neq j$, $v_j \in S$ is not a dominating set. Hence, this is a contradiction to the definition of S being an SDS of G satisfying the hypothesis of Proposition 2.1 and Proposition 2.2.
Therefore,

$$|S| \geqslant \sum_{\substack{1 \leqslant i \leqslant l \\ k = 3i-2 \\ k \leqslant l}} 2^{l-k} + 2.$$

Hence,

$$S = \left\{v_n^{l-1} : 1 \leqslant n \leqslant 2^{l-1}\right\} \cup \left\{v_n^{l-4} : 1 \leqslant n \leqslant 2^{l-4}\right\} \cup \ldots \cup \left\{v_n^{l-k} : 1 \leqslant n \leqslant 2^{l-k}\right\} \cup \{w_1, w_2\}$$

where v_n^{l-k}'s are $(l-k)$ level vertices and $k \leq l$, $k = 3i-2, 1 \leq i \leq l$, which yields, $|S| = \sum_{\substack{1 \leqslant i \leqslant l \\ k = 3i-2 \\ k \leqslant l}} 2^{l-k} + 2$, whence it follows that, $\gamma_s(G) = \gamma_s(DS(T)) = \sum_{\substack{1 \leqslant i \leqslant l \\ k = 3i-2 \\ k \leqslant l}} 2^{l-k} + 2.$

□

4 Degree splitting graphs of complete graphs

In this section, we find the secure domination number of degree splitting graphs of complete graphs like K_n and $K_{p,q}$ where, $p \leq q$.

Theorem 4.1. *For a complete graph K_n, $\gamma_s(DS(K_n)) = 1$.*

Proof. K_n is an $(n-1)$ regular graph. By Proposition 2.3, $\gamma_s(K_n) = 1$ and since $DS(K_n) \cong K_{n+1}$, so, $\gamma_s(DS(K_n)) = 1$, as K_{n+1} is a complete graph on $n+1$ vertices.

□

Theorem 4.2. *For a complete bipartite graph $K_{p,q}$, $p \leq q$,*

$$\gamma_s(DS(K_{p,q})) = \begin{cases} 3, & \text{if } p = q \geqslant 3, \\ 2, & \text{if } p = q = 2, \\ 2, & \text{if } p = 1, q > 1, \\ 3, & \text{if } p = 2, q > 2, \\ 4, & \text{if } p \geqslant 3. \end{cases}$$

Proof. Let $P = \{u_1, u_2 \dots u_p\}$ and $Q = \{v_1, v_2 \dots v_q\}$ be the defining partite sets of $K_{p,q}$.
Let $G = DS(K_{p,q})$ and S be an SDS of G. We consider the following cases:
Case (i) $p = q \geq 3$
Let $V(G) = P \cup Q \cup \{w\}$. To prove that $|S| \geq 3$.
Suppose on the contrary that $|S| < 3$. Assume that $|S| = 2$. The following sub cases arise:
Sub case (a) $S \subseteq P$
In this case, both the vertices of S belong to P. Clearly, S is not a dominating set. Hence a contradiction. Therefore $|S| \geq 3$.
Sub case (b) $S \subseteq Q$
Since both the vertices of S belong to Q, clearly, as in Case (i), S is not a dominating set, which is a contradiction. Hence $|S| \geq 3$.
Sub case (c) $S \subseteq P \cup Q$
Since $S \subseteq P \cup Q$, let $S = \{u_i, v_j\}$ for some i and j. Clearly, for $u_i \in S$, $(S - \{u_i\}) \cup \{v_k\}$, $v_k \in V - S$, $k \neq j$ is not a dominating set, which is a contradiction. Hence $|S| \geq 3$.
Sub case (d) $S \subseteq P \cup \{w\}$
In this case, $S = \{u_i, w\}$ for some i. Clearly, $(S - \{w\}) \cup \{u_j\}$, $i \neq j$ is not a dominating set, which is a contradiction. Hence $|S| \geq 3$.
Sub case (e) $S \subseteq Q \cup \{w\}$
In this case, $S = \{v_i, w\}$ for some i. Clearly, $(S - \{w\}) \cup \{v_j\}$, $j \neq i$ is not a dominating set, which is a contradiction. Hence $|S| \geq 3$.
All the above sub cases imply that $|S| \geq 3$.
Therefore $\gamma_s\ (G) = 3$ and a $\gamma_s\ (G)$-set is $\{u_i,\ v_j,\ w\}$ for any i and j.
Case (ii) $p = q = 2$
Let $G = DS(K_{2,2})$ and $V(G) = P \cup Q \cup \{w\}$. A similar argument as in Case (i) proves that $\gamma_s\ (G) = 2$ and a $\gamma_s\ (G)$-set is $\{w, u_1\}$.
Case (iii) $p = 1,\ \ q > 1$
Let $G = DS(K_{1,q})$ and $V(G) = \{u_1\} \cup \{v_1, v_2, \dots, v_q\} \cup \{w\}$. The deployment of a guard at a single vertex of G does not dominate the graph G. Therefore $|S| \geq 2$. Clearly, the set $\{w, u_1\}$ is the unique $\gamma_s\ (G)$-set of cardinality 2. Therefore $\gamma_s\ (G) = 2$.
Case (iv) $p = 2,\ \ q > 2$
Let $G = DS(K_{2,q})$ and $V(G) = P \cup Q \cup \{w_1, w_2\}$, where deg $w_1 = p$, deg $w_2 = q$. To prove that $|S| \geq 3$. Suppose on contrary that $|S| < 3$. By similar argument, as in Case (i), we obtain $\gamma_s\ (G) = 3$ and a $\gamma_s\ (G)$-set is $\{u_1,\ u_2,\ v_1\}$.
Case (v) $p \geq 3,\ \ p < q$
Let $G = DS(K_{p,q})$, $p \geq 3,\ \ p < q$ and $V(G) = P \cup Q \cup \{w_1, w_2\}$, where deg $w_1 = p$, deg $w_2 = q$.
To prove $|S| \geq 4$. Assume the contrary that $|S| < 4$. Analogous to the proof of Case (i), we obtain $|S| \geq 4$. Therefore $\gamma_s\ (G) = 4$ and a $\gamma_s\ (G)$-set is $\{u_1,\ u_2,\ v_1, w_2\}$.

□

5 Bounds

In this section we attempt to determine an upper bound for the degree splitting graphs of certain classes of caterpillars and regular graphs.

Theorem 5.1. *For a caterpillar T without strong support,* $\gamma_s(DS(T)) \leq \gamma_s(T) + 1$.

Proof. Let S be an $\gamma_s(T)$-set and let $L = \{l_1, l_2, \dots, l_k\}$ denote the set of leaf vertices of T. Consider $S' = \left\{s_1',\ s_2', \dots,\ s_{k-2}',\ s_{k-1}',\ s_k'\right\}$ to denote the set of support vertices of degree 3 in T except the vertices s_1' and s_k' which are of degree 2 in T. Let $V' = \left\{v_1',\ v_2', \dots, v_l', s_1', s_k'\right\}$ denote the set of vertices of degree 2 in T.
Let $G = DS(T)$ and $V(G) = \begin{cases} V(T) \cup \{w_1, w_2\} & \text{if } \Delta(T) = 2, \\ V(T) \cup \{w_1, w_2, w_3\} & \text{if } \Delta(T) = 3 \end{cases}$ where, $N(w_1) = L$, $N(w_2) = V'$ and $N(w_3) = S' - \left\{s_1', s_k'\right\}$.

Without loss of generality, any $v_i^{'}$, $1\leq i \leq l$ defends w_2, since the set $\left(S - \left\{v_i^{'}\right\}\right) \cup \{w_2\}$ is a secure dominating set of G, as $N(w_2) = V^{'}$. Consider the following cases:

Case (i) $\triangle(T) = 2$

In this case, $T \simeq P_n$ and we have the following sub cases:

Sub case (a)

There exists a leaf l_1 *or* $l_2 \in S$ such that, without loss of generality, the set $(S - \{l_1\}) \cup \{w_1\}$ is a dominating set of G. Therefore $\gamma_s(G) \leq \gamma_s(T)$.

Sub case (b)

There exists a $v_i^{'} \in V(G) - S$ or l_i, $i = 1$ or 2 which are not defended. Hence $\gamma_s(G) \leq \gamma_s(T) + 1$.

Case (ii) $\triangle(T) = 3$

In this case, the set $\left(S - \left\{s_i^{'}\right\}\right) \cup \{w_3\}$, for some i, $1 \leq i \leq k-2$ is a dominating set. If $w_1 \in S$, or otherwise S may or may not be a dominating set. It is a contradiction if S is not dominating set for G. Hence $\gamma_s(G) \leq \gamma_s(T) + 1$.

□

Remark 5.2. Bound is sharp for caterpillars like P_4, P_7 etc.

Theorem 5.3. *For a regular graph G, $\gamma_s(DS(G)) \leq \gamma_s(G)$ and the bound is sharp for complete graphs K_n.*

Proof. Let $V(DS(G)) = V(G) \cup \{u\}$ and let S be an $\gamma_s(G)$-set. Since G is regular, $N(u) = V(G)$. For any $v \in S$, $(S - \{v\}) \cup \{u\}$ is a dominating set of $DS(G)$. Hence S is an SDS of $DS(G)$. Therefore, $\gamma_s(DS(G)) \leq \gamma_s(G)$.

□

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