Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics
Website : https : //www.bpasjournals.com/

*Bull. Pure Appl. Sci. Sect. E Math. Stat.*
**38E(Special Issue)**(2S), 60–64 (2019)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI: 10.5958/2320-3226.2019.00080.8


# Graceful labeling of some new graphs *

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**Abstract** A graceful labeling of a graph $G$ with $q$ edges is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ with the property that the resulting edge labels are distinct where the edge incident with the vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. In this paper, we prove that the series of isomorphic copies of Star graph connected between two Ladders are graceful.

**Key words** Graceful labeling, Path, Ladder Graph, Star Graph.

**2010 Mathematics Subject Classification** 05C60, 05C78.

## 1 Introduction

In 1967, Rosa [2] introduced the graceful labeling method as a tool to attack the Ringel-Kotzig-Rosa Conjecture or the Graceful Tree Conjecture that "All Trees are Graceful" and he also proved that caterpillars (a caterpillar is a tree with the property that the removal of its endpoints leaves a path) are graceful. Many graphs and families of graphs are shown to be graceful. Morgan [6] has shown that all lobsters with perfect matchings are graceful. Hrniar and Haviar [7] have shown that all trees of diameter five are graceful. Golomb [3] has shown that the complete bipartite graph $K_{m,n}$ is graceful. Wheels $W_n = C_n + K_l$ are graceful [8]. Helms are shown to be graceful in [9]. Sudha and Kanniga [5] proved that the arbitrary super subdivision of helms, centipedes and ladder graphs are graceful. Liu [11] proved that the gracefulness of the star graph with top sides. Let $G_1, \ldots, G_n$ be $(n \geq 2)$ copies of a graph $G$. Then the graph $G(n)$ obtained by adding an edge to $G_i$ and $G_{i+1}, i = 1, \ldots, n-1$ is called the path-union of $n$ copies of the graph $G$ [6]. Kaneria et al. [4] proved that the path union, cycle and star of complete bipartite graphs are graceful. Barrientos [10] introduced $kC_4$ -snakes graph as a generalization of the concept of triangular snake introduced by Rosa [2] and he proved that $kC_4$-snakes are graceful. For more results on graceful labeling we refer to the dynamic survey by Gallian [1].
In this paper, we prove that the series of isomorphic copies of Star graph connected between two Ladders are graceful.

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received March 16, 2019 / Revised October 15, 2019 / Accepted November 06, 2019. Online First Published on June 03, 2020 at https://www.bpasjournals.com/.
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**Refereed Proceedings of the National Conference on Mathematics and Computer Applications held at the Department of Mathematics, Women's Christian College, Chennai, India from January 29, 2019 to January 30, 2019.**

## 2 Definitions

**Definition 2.1. Ladder graph:** The Ladder graph $L_n$ is a planar undirected graph with $2n$ vertices and $3n-2$ edges. The ladder graph can be obtained as the Cartesian product of two path graphs, one of which has only one edge: $L_{n,1} = p_n \times p_2$.

**Definition 2.2. Star graph:** The Star graph $S_n$ of order $n$ is a tree on $n$ nodes with one node having vertex degree $n-1$ and the other $n-1$ having vertex degree 1. The star graph $S_n$ is therefore isomorphic to the complete bipartite graph $K_{1,n}$.

## 3 Main result

**Theorem 3.1.** *The series of isomorphic copies of Star graph connected between two Ladders are graceful.*

**Proof.** Let $E$ be a ladder graph whose vertices are denoted as follows: $a_1 - 1, a_2, \ldots, a_n$ denote the vertices on the path of the ladder on the left side from top to bottom and $b_1, b_2, \ldots, b_n$ denote the vertices on the path of ladder on the right side from top to bottom and $F$ be a star graph whose pendant vertices are denoted by $g_1, g_2, \ldots, g_k$ and the apex vertex is denoted by $h_1$ as shown in Fig. 1 below:

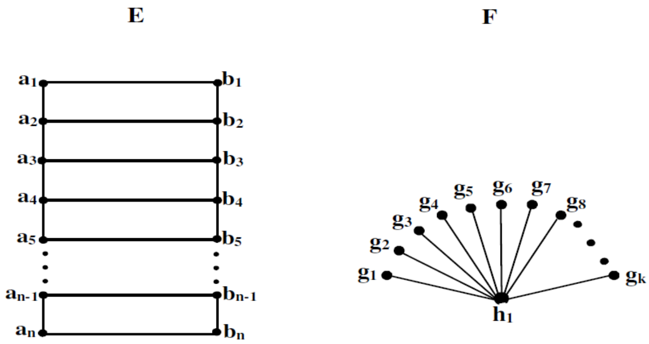


Fig. 1: The graphs $E$ and $F$.

Let $G$ be the series of isomorphic copies of star graph connected between two ladder graph as shown in Fig. 2. In the ladder graph, the vertices in the first copy are denoted as $u_1, u_2, \ldots, u_n$ on the path $P_n$ on the left side from the top to the bottom and $v_1, v_2, \ldots, v_n$ denote the vertices on the path $P_n$ of ladder on the right side from the top to the bottom and the vertices in the second copy attached at last are denoted as $c_1, c_2, \ldots, c_n$ on the path $P_n$ of ladder on the left side from the bottom to the top and $d_1, d_2, \ldots, d_n$ denote the vertices on the path $P_n$ of ladder on the right side from the bottom to the top. Let $m$ isomorphic copies of the star graph be attached on the path vertices between the two ladders and $x_1^1, x_2^1, \ldots, x_k^1$ indicate the pendant vertices of the star graph in the first copy, $x_1^2, x_2^2, \ldots, x_k^2$be the pendant vertices of the star graph in the second copy and $x_1^n, x_2^n, \ldots, x_k^n$be the pendant vertices of star graph in the $n^{\text{th}}$copy and $y_1, y_2, \ldots, y_m$ be the vertices of the star graph attached on the path.
Note that, in graph $G$, the vertices in the first copy on the path $P_n$ of ladder on the left side from the top to the bottom are denoted as $u_i \; (1 \le i \le n)$ and $v_i \; (1 \le i \le n)$ be the vertices on the path $P_n$ of ladder on the right side from the top to the bottom. The vertices in the second copy of the ladder graph attached at last are represented as $c_i \; (1 \le i \le n)$ on the path $P_n$ of ladder on the left side from the bottom to the top and $d_i \; (1 \le i \le n)$ be the vertices on the right side from the bottom to the top. The vertices of the star graph are defined as $x_l^j \; (1 \le l \le k, 1 \le j \le m)$ and the vertices of the star graph attached on path are denoted as $y_j(1 \le j \le m)$.The number of edges is denoted by $q$ and is defined by$q = 6n + m(k+1) - 3$ and the number of vertices are denoted by $p$ and is given by $p = 4n + m\,(k+1)$.
Now, we define the labels for the graph $G$ as follows:

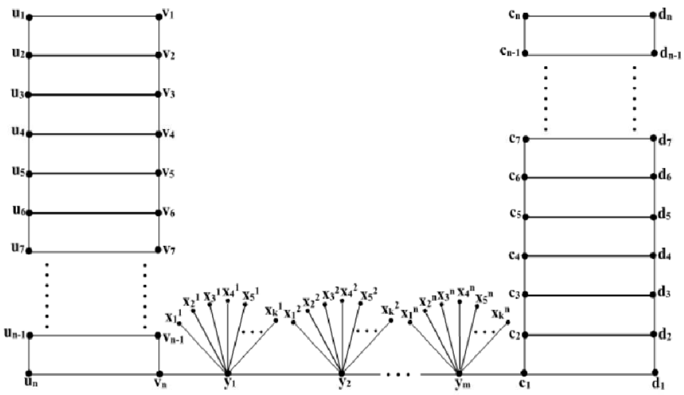

Fig. 2: The graph $G$ of generalized series with isomorphic copies of star graph connected between two ladder graphs.

Labeling for the first copy of Ladder graph:

$$f(u_1) = 0$$
$$f(u_{2i+1}) = 2 + 3\,(i-1) \quad \text{for} \quad 1 \leq i \leq n/2$$
$$f(u_{2i}) = q - 3\,(i-1) \quad \text{for} \quad 1 \leq i \leq n/2$$
$$f(v_{2i-1}) = q - 1 - 3\,(i-1) \quad \text{for} \quad 1 \leq i \leq n/2, \quad n \quad \text{is} \quad \text{even}$$
$$f(v_{2i-1}) = q - 1 - 3\,(i-1) \quad \text{for} \quad 1 \leq i < \lceil n/2 \rceil$$
$$f(v_{2i-1}) = q - 3\,(i-1) \quad \text{for} \quad i = \lceil n/2 \rceil\,, \quad n \quad \text{is} \quad \text{odd}$$
$$f(v_{2i}) = 3i \quad \text{for} \quad 1 \leq i < n/2$$
$$f(v_{2i}) = 3i - 1 \quad \text{for} \quad i = n/2$$

Labeling for second copy of Ladder graph:
When $m$ is odd:

$$f(c_1) = f(x_k^m) + 1$$
$$f(c_{2i+1}) = f(c_1) + 2 + 3(i-1) \quad \text{for} \quad 1 \leq i \leq n/2$$
$$f(d_1) = f(y_m) - 2$$
$$f(d_{2i+1}) = f(d_1) - 3i \quad \text{for} \quad 1 \leq i \leq n/2$$
$$f(c_2) = f(y_m) - 1$$
$$f(c_{2i}) = f(c_2) - 3i \quad \text{for} \quad 1 \leq i \leq n/2$$
$$f(d_2) = f(c_1) + 3$$
$$f(d_{2i}) = f(d_2) + 3i \quad \text{for} \quad 1 \leq i < n/2$$
$$f(d_{2i}) = f(d_2) + 3i - 1 \quad \text{for} \quad i = n/2$$

When $m$ is even:

$$f(c_1) = f(x_k^m) - 1$$
$$f(c_{2i+1}) = f(c_1) - 2 - 3(i-1) \quad \text{for} \quad 1 \leq i \leq n/2$$
$$f(d_1) = f(y_m) + 2$$
$$f(d_{2i+1}) = f(d_1) + 3i \quad \text{for} \quad 1 \leq i \leq n/2$$
$$f(c_2) = f(y_m) + 1$$
$$f(c_{2i}) = f(c_2) + 3i \quad \text{for} \quad 1 \leq i \leq n/2$$

$$f(d_2) = f(c_1) - 3$$
$$f(d_{2i}) = f(d_2) - 3i \text{ for } 1 \leq i < n/2$$
$$f(d_{2i}) = f(d_2) - 3i + 1 \text{ for } i = n/2$$

Labeling for star graph:

$$f(x_l^{2j-1}) = f(v_n) + l + (m+1)(j-1) \text{ for } 1 \leq l \leq k, 1 \leq j \leq \lceil m/2 \rceil$$
$$f(x_l^{2j}) = f(y_1) - l - (m+1)(j-1) \text{ for } 1 \leq l \leq k, 1 \leq j \leq m/2$$
$$f(y_{2j-1}) = f(v_{n-1}) - 1 - (m+1)(j-1) \text{ for } 1 \leq j \leq \lceil m/2 \rceil$$
$$f(y_{2j}) = f(x_k^1) + 1 + (m+1)(j-1) \text{ for } 1 \leq j \leq m/2$$

From the above definition, it is clear that all the vertex labels are distinct. The edge labels can be estimated for the above vertex labels and they are also found to be distinct from 1 to $q$. Therefore, the series of isomorphic copies of star graph connected between the two ladders is shown to be graceful. □

**Illustration 3.2.** An illustration for the above theorem is given below in Fig. 3 when $n = 8, m = 3, k = 6q = 66, p = 53$.

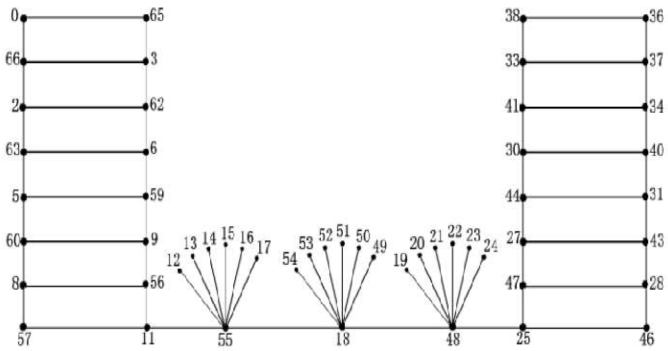


Fig. 3: Two ladder graphs with three isomorphic copies of star graph.

## 4 Conclusion

We have proved that the series of isomorphic copies of star graph connected between the two ladders are graceful. Further, we intend to prove this result for arbitrary copies of ladder and star graphs.

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