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Odd graceful labeling in cycle with extended bistar *

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Abstract Graph labeling creates a new direction towards research areas in graph theory and has various applications in coding approach, communication networks and many more. In 1991, Gnanajothi (Topics in Graph Theory, Ph.D. Thesis, Madurai Kamraj University, Tamilnadu, India) introduced a labeling method called odd graceful labeling. A graph G with q edges is odd graceful if there is an injection, $f : V(G) \to \{0, 1, 2, \ldots, (2q-1)\}$ such that when each xy edge is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, (2q-1)\}$. In this paper, we prove the odd graceful labeling in a cycle with extended bistar.

Key words Odd Graceful labeling, Extended bistar, Cycle.

2010 Mathematics Subject Classification 05C78.

1 Introduction

The course on graph labeling embarks with the initiation to β- valuation by Rosa [7] in 1967. Golomb [4] called this β-valuation as graceful labeling in 1972. A graph G is said to admit a *graceful labeling* if there exists an injection from its q edges to its vertex set V given by $f : V(G) \to \{0, 1, 2, \ldots, q\}$ with the property that the resulting edge labels are also unique, where an edge incident with vertices u and v is assigned the label $|f(u) - f(v)|$. In 1991, Gnanajothi [3] introduced a labeling method called the odd graceful labeling. A graph G with q edges is odd graceful if there is an injection, $f : V(G) \to \{0, 1, 2, \ldots, 2(q-1)\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, (2q-1)\}$. Gnanajothi [3] confirmed that the class of odd graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs. In [3], it is shown that the following graphs are odd graceful: the path P_n, the cycle C_n if and only if n is even, the comb $P_n \odot K_1$ (graphs obtained by linking a single pendant edge to each vertex of P_n), books,

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crowns $C_n \odot K_1$ (graphs obtained by connecting a single pendant edge to each vertex of C_n) if and only if n is even, the disjoint union of copies of C_4, the one-point union of copies of C_4, caterpillars, rooted trees of height 2, the graphs obtained from $P_n (n \geq 3)$ by adding exactly two leaves at each vertex of degree 2 of P_n. Ibrahim Moussa [5] showed that the graph $C_m \cup P_n$ is odd graceful if m is even. Eldergill [1] proved that the one-point union of any number of copies of C_6 is odd graceful. Sekar [8] showed that the splitting graph of P_n, the splitting graph of C_n when n is even, lobsters, banana trees and regular bamboo trees are odd graceful. For an overall embracing survey on graph theory we refer to the dynamic survey by Gallian [2].
Graph labeling creates huge applications in graph theory and has rigorous requisitions in coding theory, transmission networks, optimal circuits layouts and graph disintegration problems. The main objective of this paper is to prove that the extended bistar attachment with cycle is odd graceful.

Definition 1.1. [6] A Extended bistar $\langle K_{1,k} : n \rangle$, as shown in Fig.1 below is the graph obtained by attaching a path of length n with the center vertices of two copies of the star $K_{1,k}$ whose vertices are denoted by $r_1, r_2, \ldots, r_n, u_1, u_2, \ldots, u_k, s_1, s_2, \ldots, s_k$ and the edges by $r_1 u_i, \ldots, r_n s_i \, (1 \leqslant i \leqslant k)$ and $r_1 r_j \, (1 \leqslant j \leqslant k)$.

Fig. 1: The Extended bistar.

2 Main result

We now prove the main result of this paper in the form of the Theorem 2.1 as follows:

Theorem 2.1. *The graph G obtained by attaching each vertex of a cycle C_m with the isomorphic extended bistar of $\langle K_{1,k} : n \rangle$ is odd graceful, where m is even, n is odd and when $m \equiv 0 \, (\mathrm{mod} 4)$.*

Proof. Let G be the graph obtained by attaching m isomorphic copies of Extended bistar of $\langle K_{1,k} : n \rangle$ graph at every vertex of cycle C_m where $m \equiv 0 \, (\mathrm{mod}\ 4)$.
We narrate the graph G as follows: the vertices in the cycle in G are expressed as $C_1, C_2, C_3, \ldots, C_m$ in the clockwise direction. Extended bistar E is described as follows: consider m mirror images of isomorphic extended bistar of $\langle K_{1,k} : n \rangle$. Denote the m copies of E as $E^1, E^2, \ldots, E^m$. The first internal vertices of the first copy of the extended bistar E^1 at the vertex C_1 of cycle are denoted by $u_1^1, u_2^1, u_3^1, \ldots, u_{n-1/2}^1$ and the second internal vertices of the first copy of the Extended bistar E^1 at the vertex C_1 of cycle are denoted as $v_1^1, v_2^1, v_3^1, \ldots, v_{n-1/2}^1$. The pendant vertices attached with u_1^1 are denoted as $x_1^1, x_2^1, x_3^1, \ldots, x_r^1$ and the pendant vertices attached with v_1^1 are denoted as $y_1^1, y_2^1, y_3^1, \ldots, y_r^1$. The first internal vertices of the second copy of the Extended bistar E^2 at vertex C_2 of cycle are denoted as $u_1^2, u_2^2, u_3^2, \ldots, u_{n-1/2}^2$ and the second internal vertices of the second copy of the Extended bistar E^2 at vertex C_2 of cycle are denoted as $v_1^2, v_2^2, v_3^2, \ldots, v_{n-1/2}^2$. The pendant vertices attached with u_1^2 are denoted as $x_1^2, x_2^2, x_3^2, \ldots, x_r^2$ and the pendant vertices attached with v_1^2 are denoted as $y_1^2, y_2^2, y_3^2, \ldots, y_r^2$. Correspondingly, the first internal vertices of the m^{th} copy of the extended bistar E^m at vertex C_m of cycle are denoted as $u_1^m, u_2^m, u_3^m, \ldots, u_{n-1/2}^m$ and the second internal vertices of the m^{th} copy of the extended bistar E^m at vertex C_m of cycle are denoted as $v_1^m, v_2^m, v_3^m, \ldots, v_{n-1/2}^m$. The pendant

vertices attached with u_1^m are denoted as x_1^m, x_2^m, x_3^m,..., x_r^m and the pendant vertices attached with v_1^m are denoted as y_1^m, y_2^m, y_3^m,..., y_r^m .
Now we attach the m copies of the Extended bistar E^m, namely as $E^1, E^2, E^3, \ldots, E^m$ at the m vertices of the cycle C_m in such a way that the first internal vertices u_1^1, u_2^1, u_3^1,..., $u_{n-1/2}^1$ and the second internal vertices v_1^1, v_2^1, v_3^1,..., $v_{n-1/2}^1$ of the first copy of the Extended bistar E^1 are attached with the vertex C_1 which is attached with the cycle, the first internal vertices u_1^2, u_2^2, u_3^2,..., $u_{n-1/2}^2$ and the second internal vertices v_1^2, v_2^2, v_3^2,..., $v_{n-1/2}^2$ of the second copy of the Extended bistar E^2 are attached with the vertex C_2 which is attached with the cycle. In general, the first internal vertices u_1^m, u_2^m, u_3^m,..., $u_{n-1/2}^m$ and the second internal vertices v_1^m, v_2^m, v_3^m,..., $v_{n-1/2}^m$ of the m^{th} copy of the Extended bistar E^m are attached with the vertices of the cycle C_m for which $m \equiv$ (mod4). Thus we obtain the new graph G exhibited in Fig.2.

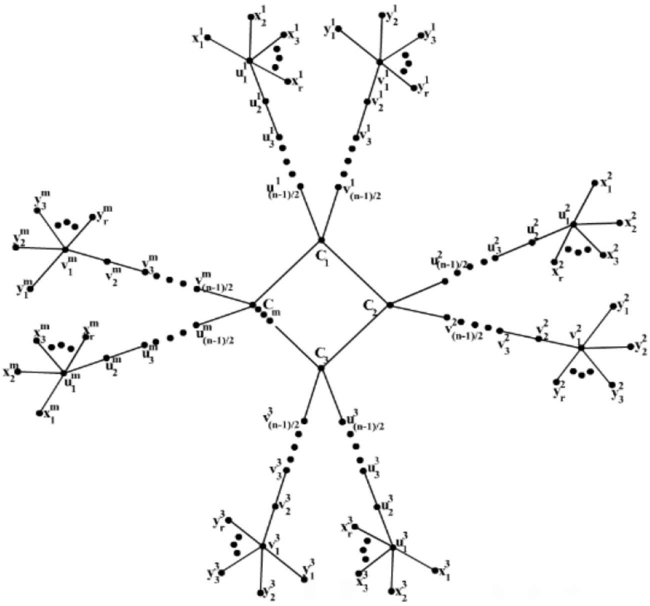

Fig. 2: The graph G obtained by joining each vertex of C_m with the isomorphic copies of extended bistar of $\langle K_{1,k} : n\rangle$ graph.

The vertices of G are classified based on the parameters m as follows:
The number of pendant edges in the extended bistar are denoted by "r" while, the number of internal vertices in one copy of coconut tree are denoted as "n".

Case 1: When n is odd

The vertex labels for the first internal vertices of the extended bistar $\langle K_{1,k} : n\rangle$ where $n = 5, 9, 13, 17, \ldots$ are given below:

$$f\left(u_{2i-1}^{2k-1}\right) = 2i - 2 + 2(k-1)(n+2r), \qquad 1 \le i \le \frac{n-1}{4} \text{ and } 1 \le k \le \frac{m}{2}$$

$$f\left(u_{2i}^{2k-1}\right) = \begin{cases} 2q - 2r + 2i - 2(k-1)(n+2r) - 3, & 1 \le i \le \left(\frac{n-1}{4}\right),\ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q - 2r + 2i - 2(k-1)(n+2r) - 1, & 1 \le i \le \left(\frac{n-1}{2}\right),\ \left(\frac{m}{4}\right) + 1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

$$f\left(u_{2i-1}^{2k}\right) = \begin{cases} 2q - 1 - 2(r+n) - (2i-2) + 2(k-1)(n+r), & 1 \le i \le \left(\frac{n-1}{4}\right),\ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q + 1 - 2(r+n) - (2i-2) + 2(k-1)(n+r), & 1 \le i \le \left(\frac{n-1}{4}\right), \left(\frac{m}{4}\right) + 1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

$$f\left(u_{2i}^{2k}\right) = (2n+2) + (2i-2) + 2(k-1)(n+2r), \qquad 1 \le i \le \left(\frac{n-1}{4}\right),\ 1 \le k \le \left(\frac{m}{2}\right).$$

The vertex labels for the second internal vertices of extended bistar $\langle K_{1,k} : n\rangle$ are given below,

$$f\left(v_{2i-1}^{2k-1}\right) = n-1+(2i-2)+2(k-1)(n+2r), \qquad 1 \le i \le \frac{n-1}{4}, \ 1 \le k \le \frac{m}{2}$$

$$f\left(v_{2i}^{2k-1}\right) = \begin{cases} 2q-1-2(n+2r-3)+(2i-2)+2(k-1)(n+2r), & 1 \le i \le \left(\frac{n-1}{4}\right), \ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q+1-2(n+2r-3)+(2i-2)+2(k-1)(n+2r), & 1 \le i \le \left(\frac{n-1}{2}\right), \left(\frac{m}{4}\right)+1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

$$f\left(v_{2i-1}^{2k}\right) = \begin{cases} 2q-1-2(2r+n-1)-(2i-2)-2(k-1)(n+2r), & 1 \le i \le \left(\frac{n-1}{4}\right), \ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q+1-2(2r+n-1)-(2i-2)-2(k-1)(n+2r), & 1 \le i \le \left(\frac{n-1}{4}\right), \left(\frac{m}{4}\right)+1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

$$f\left(v_{2i}^{2k}\right) = 2(n+r)-2+(2i-2)+2(k-1)(n+2r), \qquad 1 \le i \le \left(\frac{n-1}{4}\right), \ 1 \le k \le \left(\frac{m}{2}\right).$$

Now, vertex labels for the cycle C_m are given below,

$$f(C_{2j-1}) = \frac{(n-1)}{2}+2(k-1)(n+2r), \quad 1 \le i \le \left(\frac{n-1}{4}\right), \ 1 \le k \le \left(\frac{m}{2}\right)$$

$$f(C_{2j}) = \begin{cases} 2q-1-3\left(\frac{n-1}{2}\right)+4r-2(k-1)(n+2r), & 1 \le i \le \left(\frac{n-1}{4}\right), \ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q-1-3\left(\frac{n-1}{2}\right)+4r-2(k-1)(n+2r), & 1 \le i \le \left(\frac{n-1}{2}\right), \left(\frac{m}{4}\right)+1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

The vertex labels for the pendant vertex at the first internal vertices of the extended bistar $\langle K_{1,k} : n\rangle$ now follow:

$$f\left(x_i^{2k-1}\right) = \begin{cases} 2q-1-(2i-2)-2(k-1)(n+2r), & 1 \le i \le r, \ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q+1-(2i-2)-2(k-1)(n+2r), & 1 \le i \le r, \ \left(\frac{m}{4}\right)+1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

$$f\left(x_i^{2k}\right) = 2n+(2i-2)+2(k-1)(n+r), \qquad 1 \le i \le r, \ 1 \le k \le \frac{m}{2}.$$

Further the vertex labels for the pendant vertex at the second internal vertices of the extended bistar $\langle K_{1,k} : n\rangle$ are

$$f\left(y_i^{2k-1}\right) = \begin{cases} 2q-1-2(r-1)-(n+1)-(2i-2)-2(k-1)(n+2r), & 1 \le i \le r, \ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q+1-2(r-1)-(n+1)-(2i-2)-2(k-1)(n+2r), & 1 \le i \le r, \ \left(\frac{m}{4}\right)+1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

$$f\left(y_i^{2k}\right) = 2(n+r)+(2i-2)+2(k-1)(n+r), \qquad 1 \le i \le r, \ 1 \le k \le \frac{m}{2}.$$

Case 2: When n is odd

The vertex labels for the first internal vertices of the extended bistar $\langle K_{1,k} : n\rangle$ where $n = 7, 11, 15, 19, \ldots$ are as given below:

$$f\left(u_{2i}^{2k-1}\right) = 2i-2+2(k-1)(n+2r), \qquad 1 \le i \le \frac{n+1}{4}, \ 1 \le k \le \frac{m}{2}$$

$$f\left(u_{2i-1}^{2k-1}\right) = \begin{cases} 2q-2r+2i-2(k-1)(n+2r)-3, & 1 \le i \le \left(\frac{n-3}{4}\right), \ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q-2r+2i-2(k-1)(n+2r)-1, & 1 \le i \le \left(\frac{n-3}{2}\right), \left(\frac{m}{4}\right)+1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

$$f\left(u_{2i}^{2k}\right) = \begin{cases} 2q-1-2(r+n)-(2i-2)+2(k-1)(n+r), & 1 \le i \le \left(\frac{n-3}{4}\right), \ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q+1-2(r+n)-(2i-2)+2(k-1)(n+r), & 1 \le i \le \left(\frac{n-3}{4}\right), \left(\frac{m}{4}\right)+1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

$$f\left(u_{2i-1}^{2k}\right) = (2n+2)+(2i-2)+2(k-1)(n+2r), \qquad 1 \le i \le \left(\frac{n+1}{4}\right), \ 1 \le k \le \left(\frac{m}{2}\right).$$

The vertex labels for the second internal vertices of the extended bistar $\langle K_{1,k} : n\rangle$ are now given below:

$$f\left(v_{2i}^{2k-1}\right) = n-1+(2i-2)+2(k-1)(n+2r), \qquad 1 \le i \le \frac{n+1}{4}, \ 1 \le k \le \frac{m}{2}$$

$$f\left(v_{2i-1}^{2k-1}\right) = \begin{cases} 2q-1-2(n+2r-3)+(2i-2)+2(k-1)(n+2r), & 1 \le i \le \left(\frac{n-3}{4}\right), \ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q+1-2(n+2r-3)+(2i-2)+2(k-1)(n+2r), & 1 \le i \le \left(\frac{n-3}{2}\right), \left(\frac{m}{4}\right)+1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

$$f\left(v_{2i-1}^{2k}\right) = \begin{cases} 2q-1-2(2r+n-1)-(2i-2)-2(k-1)(n+2r), & 1 \le i \le \left(\frac{n-3}{4}\right), \ 1 \le k \le \left(\frac{m}{4}\right) \\ 2q+1-2(2r+n-1)-(2i-2)-2(k-1)(n+2r), & 1 \le i \le \left(\frac{n-3}{4}\right), \left(\frac{m}{4}\right)+1 \le k \le \left(\frac{m}{2}\right) \end{cases}$$

$$f\left(v_{2i}^{2k}\right) = 2(n+r)-2+(2i-2)+2(k-1)(n+2r), \qquad 1 \le i \le \left(\frac{n+1}{4}\right), \ 1 \le k \le \left(\frac{m}{2}\right).$$

Now, the vertex labels for the cycle C_m are given below:

$$f\left(C_{2j}\right)=\frac{(n-1)}{2}+2\left(k-1\right)\left(n+2r\right),\qquad 1\leq k\leq\left(\frac{m}{2}\right)$$

$$f\left(C_{2j-1}\right)=\begin{cases} 2q-1-3\left(\frac{n-1}{2}\right)+4r-2\left(k-1\right)\left(n+2\ r\right)\ , & 1\leq k\leq\left(\frac{m}{4}\right)\\ 2q-1-3\left(\frac{n-1}{2}\right)+4r-2\left(k-1\right)\left(n+2\ r\right), & \left(\frac{m}{4}\right)+1\leq k\leq\left(\frac{m}{2}\right).\end{cases}$$

The vertex labels for the first internal vertices of the extended bistar $\langle K_{1,k}:n\rangle$ are given below:

$$f\left(x_i^{2k-1}\right)=\begin{cases} 2q-1-(2i-2)-2\left(k-1\right)\left(n+2r\right), & 1\leq i\leq r,\ 1\leq k\leq\left(\frac{m}{4}\right)\\ 2q+1-(2i-2)-2\left(k-1\right)\left(n+2\ r\right), & 1\leq i\leq r,\ \left(\frac{m}{4}\right)+1\leq k\leq\left(\frac{m}{2}\right)\end{cases}$$

$$f\left(x_i^{2k}\right)=\ 2n+(2i-2)+2\left(k-1\right)\left(n+\ r\right),\qquad 1\leq i\leq r,\quad 1\leq k\leq\frac{m}{2}.$$

Lastly, we give the vertex labels for the second internal vertices of extended bistar $\langle K_{1,k}:n\rangle$ as under

$$f\left(y_i^{2k-1}\right)=\begin{cases} 2q-1-2\left(r-1\right)-\left(n+1\right)-\left(2i-2\right)-2\left(k-1\right)\left(n+2r\right), & 1\leq i\leq r,\ 1\leq k\leq\left(\frac{m}{4}\right)\\ 2q+1-2\left(r-1\right)-\left(n+1\right)-\left(2i-2\right)-2\left(k-1\right)\left(n+2r\right), & 1\leq i\leq r,\ \left(\frac{m}{4}\right)+1\leq k\leq\left(\frac{m}{2}\right)\end{cases}$$

$$f\left(y_i^{2k}\right)=\ 2(n+r)+(2i-2)+2\left(k-1\right)\left(n+\ r\right),\qquad 1\leq i\leq r,\quad 1\leq k\leq\frac{m}{2}.$$

From these equations, we see that the vertex labels are specified and also the edge labels can be computed and the edge labels are found to be odd and distinct. Therefore, the graph obtained by attaching each vertex C_m with the extended bistar graph is odd graceful, when m is even and n is odd.

□

Illustration 2.2. We illustrate the above theorem by means of the example depicted in Fig.3 when $m=4,k=4,n=9,q=68,2q-1=135$.

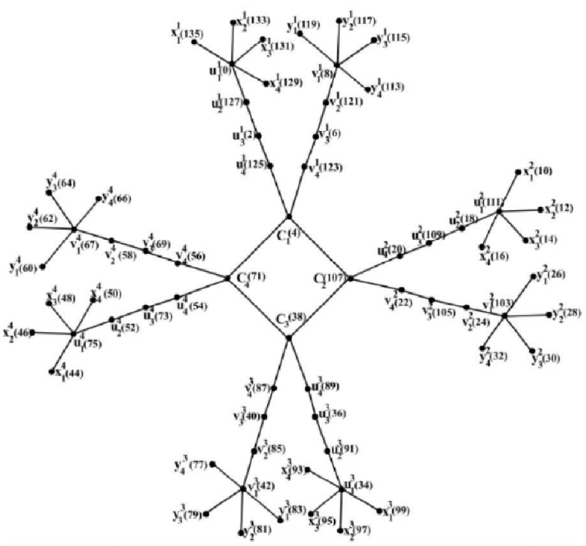

Fig. 3: Eight isomorphic copies of the extended bistar $\langle K_{1,4}:9\rangle$ graph attached with the cycle C_8.

3 Conclusion

It is shown that the m- isomorphic copies of the extended bistar $\langle K_{1,k}:n\rangle$ where n is odd, attached at each vertex of a cycle C_m when $m\equiv(\text{mod}4)$ admits odd graceful labeling.

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