Original Article

Bulletin of Pure and Applied Sciences.
Vol. 36E (Math & Stat.), No.2, 2017.P.143-151
Print version ISSN 0970 6577
Online version ISSN 2320 3226
DOI: 10.5958/2320-3226.2017.00016.9

FORMING A MIXED QUADRATURE RULE USING AN ANTI-GAUSSIAN QUADRATURE RULE

Bibhu Prasad Singh*, Dr. Rajani Ballav Dash**

Author Affiliation:
* Institute of Mathematics and Application, Andharua, Bhubaneswar, Odisha, India
E-mail:bpsmath78@gmail.com
** Ex Reader in Mathematics, Ravenshaw University, Cuttack, Odisha, India.
E-mail: rajani_bdash@rediffmail.com

Corresponding Author:
Bibhu Prasad Singh, Institute of Mathematics and Application, Andharua, Bhubaneswar, Odisha 751003, India

E-mail: bpsmath78@gmail.com

Received on 19.06.2017, **Accepted on** 27.11.2017

Abstract

A mixed quadrature rule of higher precision for approximate evaluation of real definite integrals has been constructed using an anti-Gaussian rule. The analytical convergence of the rule has been studied. The relative efficiency of the mixed quadrature rule has been shown with the help of suitable test integrals. The error bounds have been determined asymptotically.

Keywords: Gauss Legendre three point rule; Anti-Gaussian four point rule; Lobatto four point rule, Mixed quadrature rule; Error analysis

2000 Mathematics Subject Classification: 65D30, 65D32

1. INTRODUCTION

The Concept of mixed quadrature was first coined by R.N Das and G.pradhan [6]. The method of mixing quadrature rules is based on forming a mixed quadrature rule of higher precision by taking linear/convex combination of two quadrature rules of lower precision. Though in literature we find precision enhancement through Richardson Extrapolation [4] and Kronord extension [4, 10, 11] taking respectively trapezoidal rule and Gaussian quadrature as base rules, these methods are quite cumbersome. On the other hand, the precision enhancement through mixed quadrature method is very simple and easy to handle. Many authors [6, 12-20] have developed mixed quadrature rules for numerical evaluation of real definite integrals. Authors [5, 7-10] have also developed mixed quadrature rules for approximate evaluation of the integrals of analytic functions
following F .Lether [3].

So far this is the first paper in which an anti-Gaussian four point quadrature rule has been used to construct a mixed quadrature rule.

Dirk P. Laurie [1,2] is first to coin the idea of anti-Gaussian quadrature formula . An anti-Gaussian quadrature formula is an (n+1) point formula of degree (2n-1) which integrates all polynominals of degree up to (2n+1) with an error equal in magnitude but opposite in sign to that of n-point Gaussian formula.

If $H^{(n+1)}(p) = \sum_{i=1}^{(n+1)} \lambda_i \, f(\xi_i)$ be (n+1) point anti-Gaussian formula and $G^{(n)}(p)$ be n point Gaussian formula then by hypothesis,
$I(p) - H^{(n+1)}(p) = -(I(p) - G^{(n)}(p)),\ p \in P_{2n+1}$ where p is a polynominal of degree $\leq 2n+1$..

In this paper we design a four point anti-Gaussian rule following LAURIE. We mix this anti-Gaussian four point rule with Lobatto four point rule.

The relative efficiency of the mixed rule has been shown by numerically evaluating some test integrals.

2. CONSTRUCTION OF ANTI-GAUSSIAN FOUR POINT RULE FROM GAUSS-LEGENDRE THREE POINT RULES

We choose the Gauss-Legendre three point rule ,

$$G_w^3(f) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \text{..(1)}$$

and develop a four point anti-Gaussian rule $H_w^4(f)$ from three point Gaussian rule $G_w^3(f)$.

Using the principle $I(p) - H_w^{n+1}(f) = -(I(p) - G_w^n(f))$ as adopted in Laurie [1], we get

$$RH_w^4(f) = 2\int_{-1}^{1} f(x)dx - (G_w^3(f)) \quad \text{......... (2)}$$

$$\Rightarrow \alpha_1 f(\xi_1) + \alpha_2 f(\xi_2) + \alpha_3 f(\xi_3) + \alpha_4 f(\xi_4) = 2\int_{-1}^{1} f(x)dx - (G_w^3(f)) \text{ , ...(3)}$$

Taking $RH_w^4(f) = \alpha_1 f(\xi_1) + \alpha_2 f(\xi_2) + \alpha_3 f(\xi_3) + \alpha_4 f(\xi_4)$...........(4)

In order to obtain the unknown weights and nodes, we assume that
(i)The rule is exact for all polynomial of degree ≤ 4.
(ii)The rule integrates all polynomials of degree up to six with an error equal in magnitude and opposite in sign to that of Gaussian rule. Thus we obtain a system of eight equations having eight unknowns using

$$\alpha_i, \xi_i \ , \ i = 1,2,3,4$$

For $f(x) = x^i, i = 0,1,2,3,4,5,6,7$.
Solving the systems of equation, we get

$$\alpha_1 = \frac{35(3+\sqrt{681})}{3\times\sqrt{681(39+\sqrt{681})}} = \alpha_4, \alpha_2 = \frac{35(\sqrt{681}-3)}{3\times\sqrt{681}(39+\sqrt{681})} = \alpha_3$$

$$\xi_1 = \sqrt{\frac{39+\sqrt{681}}{70}} = -\xi_4 \ , \xi_2 = \sqrt{\frac{39-\sqrt{681}}{70}} = -\xi_3$$

$$RH_w^4(f) = \frac{35(3+\sqrt{681})}{3\times\sqrt{681(39+\sqrt{681})}}\, f(\sqrt{\frac{39+\sqrt{681}}{70}}) + \frac{35(\sqrt{681}-3)}{3\times\sqrt{681}(39+\sqrt{681})}\, f(\sqrt{\frac{39-\sqrt{681}}{70}})$$

$$+\frac{35(\sqrt{681}-3)}{3\sqrt{681}(39-\sqrt{681})}\, f(-\sqrt{\frac{39-\sqrt{681}}{70}}) + \frac{35(3+\sqrt{681})}{3\sqrt{681}(39+\sqrt{681})}\, f(-\sqrt{\frac{39+\sqrt{681}}{70}})$$

But the anti-Gaussian four point rule computed as

$$RH_w^4(f) = \alpha_1\{f(-\xi_1) + f(\xi_1)\} + \alpha_2\{f(-\xi_2) + f(\xi_2)\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Hence, by taylors series expansion, we have

$$RH_w^4(f) = 2(\alpha_1+\alpha_2)f(0) + 2(\alpha_1\xi_1^2+\alpha_2\xi_2^2)\frac{f^{ii}(0)}{2!} + 2(\alpha_1\xi_1^4+\alpha_2\xi_2^4)\frac{f^{iv}(0)}{4!}$$

$$+2(\alpha_1\xi_1^6+\alpha_2\xi_2^6)\frac{f^{iv}(0)}{6!} + 2(\alpha_1\xi_1^8+\alpha_2\xi_2^8)\frac{f^{viii}(0)}{8!} + \ldots$$

By putting the values of α_1, α_2 and ξ_1, ξ_2 in the above equation, we have

$$RH_w^4(f) = 2f(0) + 2\frac{f^{ii}(0)}{3!} + 2\frac{f^{iv}(0)}{5!} + 2\frac{29\times f^{iv}(0)}{6!\times 175} + 2\frac{297105136\times f^{viii}(0)}{8!} + \ldots$$

We have,

$$I(f) = \int_{-1}^{1} f(x)dx = 2f(0) + 2\frac{f^{ii}(0)}{3!} + 2\frac{f^{iv}(0)}{5!} + 2\frac{f^{iv}(0)}{7!} + 2\frac{f^{viii}(0)}{9!} + \ldots$$

The error of the anti-Gaussian four point rule is computed as

$$EH_w^4(f) = \int_{-1}^{1} f(x)dx - RH_w^4(f) = -2\frac{28\times f^{vi}(0)}{7!\times 175} - 2\frac{2673946223\times f^{viii}(0)}{9!} + \ldots$$

$$EH_w^4(f) = \int_{-1}^{1} f(x)dx - RH_w^4(f) = -\frac{f^{vi}(0)}{75\times 210} - \frac{5347892446\times f^{viii}(0)}{9!} + \ldots\ldots\ldots\ldots(6)$$

3. CONSTRUCTION OF MIXED QUADRATURE RULE BY USING ANTI-GAUSSIAN FOUR POINT RULE WITH LOBATTO FOUR POINT RULE

We have the Lobatto four point rule,

$$Lob_w^4(f) = \frac{1}{6}[f(1) + f(-1) + 5\{f(-\frac{1}{\sqrt{5}}) + f(\frac{1}{\sqrt{5}})\}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Hence, by Taylor's series expansion, we have

$$Lob_w^4(f) = 2f(0) + 2\frac{f^{ii}(0)}{3!} + 2\frac{f^{iv}(0)}{5!} + \frac{26\times f^{iv}(0)}{6!\times 75} + \frac{42\times f^{viii}(0)}{8!\times 125} + \ldots\ldots\ldots\ldots\ldots(8)$$

The error associated with Lobatto four point rule is computed as

$$ELob_w^4(f) = I(f) - Lob_w^4(f) = -\frac{2}{315\times 75} f^{vi}(0) - \frac{128}{9!\times 125} f^{viii}(0) - \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

The error associated with the anti-Gaussian four point rule is

$$EH_w^4(f) = \int_{-1}^{1} f(x)dx - RH_w^4(f) = -\frac{f^{vi}(0)}{75\times 210} - \frac{5347892446\times f^{viii}(0)}{9!} + \ldots\ldots\ldots\ldots\ldots(10)$$

Eliminating $f^{vi}(0)$ from the equation (9) and (10), we have

$$(\frac{2}{315} - \frac{1}{210})I(f) = \frac{2}{315} RH_w^4(f) - \frac{1}{210} Lob_w^4\ (f)$$

$$+ (\frac{128}{9!\times 125\times 210} - \frac{2\times 534789244}{9!\times 315}) f^{viii}(0) - \ldots$$

$$I(f) = 4RH_w^4(f) - 3Lob_w^4\ (f) + 630(\frac{128}{9!\times 125\times 210} - \frac{2\times 534789244}{9!\times 315}) f^{viii}(0) - \ldots$$

$$I(f) = 4RH_w^4(f) - 3Lob_w^4\ (f) + 4EH_w^4(f) - 3ELob_w^4\ (f)$$

$$I(f) = RH_w^4 Lob_w^4\ (f) - EH_w^4 Lob_w^4\ (f)$$

$$RH_w^4 Lob_w^4\ (f) = 4RH_w^4(f) - 3Lob_w^4\ (f)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

$$EH_w^4 Lob_w^4\ (f) = 4EH_w^4(f) - 3ELob_w^4\ (f)$$

$$RH_w^4(f) = 4[\frac{(3+\sqrt{681})\times 35}{3\times\sqrt{681(39+\sqrt{681})}} f(\sqrt{\frac{39+\sqrt{681}}{70}}) + \frac{35(\sqrt{681}-3)}{3\times\sqrt{681}(39+\sqrt{681})} f(\sqrt{\frac{39-\sqrt{681}}{70}})$$

$$+ \frac{35(\sqrt{681}-3)}{3\sqrt{681}(39-\sqrt{681})} f(-\sqrt{\frac{39-\sqrt{681}}{70}}) + \frac{35(3+\sqrt{681})}{3\sqrt{681}(39+\sqrt{681})} f(-\sqrt{\frac{39+\sqrt{681}}{70}})] -$$

$$\frac{1}{2}[f(1) + f(-1) + 5\{f(\frac{1}{\sqrt{5}}) + f(-\frac{1}{\sqrt{5}})\}]$$

This is the desired mixed quadrature rule of precision seven. For the approximate evaluation of $I(f)$. The truncation error generated in this approximation is given by.

$$EH_w^4 Lob_w^4\ (f) = 4EH_w^4(f) - 3ELob_w^4\ (f)]\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

or $$EH_w^4 Lob_w^4\ (f) = 630(\frac{128}{9!\times 125\times 210} - \frac{2\times 534789244}{9!\times 315}) f^{viii}(0) + \ldots$$

$$|\ EH_w^4 Lob_w^4\ (f)\ | \le 630(\frac{128}{9!\times 125\times 210} - \frac{2\times 534789244}{9!\times 315})|\ f^{viii}(\eta)\ |, -1\langle\eta\langle 1,\ldots\ldots\ldots\ldots(13)$$

The rule $RH_w^4 Lob_w^4\ (f)$ is called a mixed type rule of precision seven as it is constructed from two different types of the rules of the same precision (i.e. precision 5).

4. ERROR ANALYSIS

An asymptotic error estimate and an error bound of the rule (13) are given by.

Theorem - 4.1
Let $f(x)$ be sufficiently differentiable function in the closed interval $[-1,1]$. Then the error $EH_w^4 Lob_w^4\ (f)$ associated with the rule $RH_w^4 Lob_w^4\ (f)$ is given by

$$\left| EH_w^4 Lob_w^4\ (f) \right| \le \frac{424}{9!}\left(\frac{16}{2625} - \frac{267394622}{63}\right)\left| f^{viii}(\eta) \right|, -1\langle \eta \langle 1$$

Proof:
Theorem follows from equation (11) and (13)
We have $RH_w^4 Lob_w^4\ (f) = 4RH_w^4(f) - 3Lob_w^4\ (f)$

And the truncation error generated in this approximation is given by $EH_w^4 Lob_w^4\ (f) = 4EH_w^4(f) - 3ELob_w^4\ (f)$

Hence we have,

$$\left| EH_w^4 Lob_w^4\ (f) \right| \le \frac{424}{9!}\left(\frac{16}{2625} - \frac{267394622}{63}\right)\left| f^{viii}(\eta) \right|, -1\langle \eta \langle 1$$

Theorem- 4.2
The bound of the truncation error

$EH_w^4 Lob_w^4\ (f) = I(f) - RH_w^4 Lob_w^4\ (f)$ is given by

$$\left| EH_w^4 Lob_w^4\ (f) \right| \le \frac{2M}{75 \times 105}\left| \eta_2 - \eta_1 \right|, \eta_1, \eta_2 \in [-1,1]$$

where $M = \max\limits_{-1 \le x \le 1} \left| f^{vii}(x) \right|$

Proof: We have $EH_w^4(f) = -\frac{1}{75 \times 210} f^{vi}(\eta_1)$ ....(14)

and

$$ELob_w^4\ (f) = -\frac{2}{75 \times 315} f^{vi}(\eta_1) \qquad (15)$$

$$EH_w^4 Lob_w^4\ (f) = 4EH_w^4(f) - 3ELob_w^4\ (f) \qquad (16)$$

Putting the values of equation (14) and (15) in eq (16) ,
we have

$$\left| EH_w^4 Lob_w^4\ (f) \right| \le \frac{2}{75 \times 105}\left| f^{vi}(\eta_2) - f^{vi}(\eta_1) \right|, \eta_1, \eta_2 \in [-1,1]$$

$$= \frac{2}{75 \times 105} \int_{\eta_1}^{\eta_2} f^{vii}(x)dx, \text{ where } \eta_1, \eta_2 \in [-1,1]$$

$$\le \frac{2M}{75 \times 105}\left| \eta_2 - \eta_1 \right|$$

where $M = \max\limits_{-1 \le x \le 1} f^{vii}(x)$

which gives a theoretical error bound as η_1, η_2 are unknown points in $[-1,1]$. From this theorem it is clear that the error in approximation will be less if points are η_1, η_2 closer to each other.

Corollary - 1

The error bound for the truncation error $EH_w^4 Lob_w^4\ (f)$ is given by

$$\left|EH_w^4 Lob_w^4\ (f)\right| \le \frac{4M}{75 \times 105}$$

Proof:

The proof follows from theorem (4.2) and $|\eta_1 - \eta_2| \le 2.$.

5. NUMERICAL VERIFICATION BY TABLE AND GRAPHS

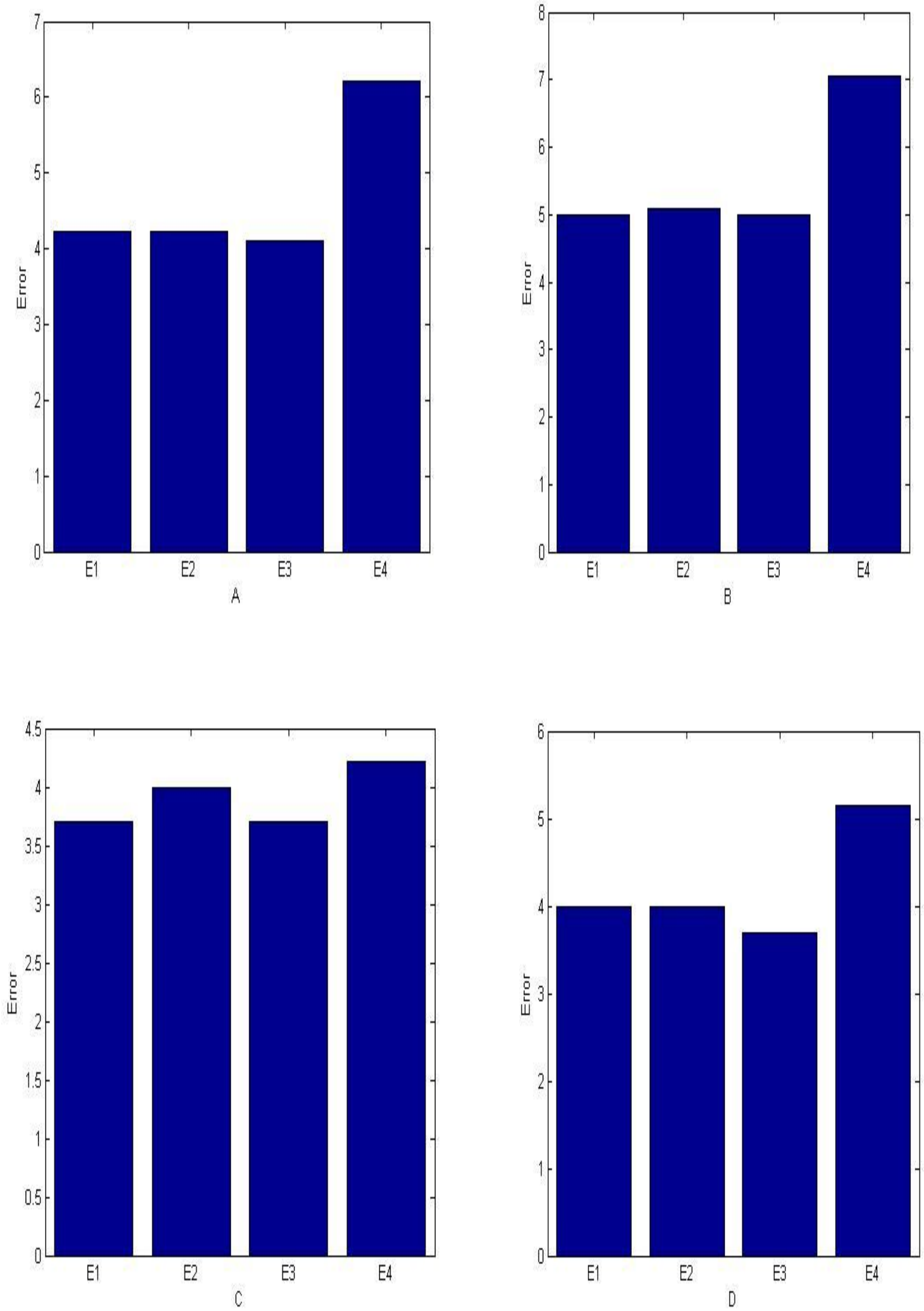
Table 1:

Sl No	Integrals	Exact Value	$G_w^3(f)/$ |E1|	$RH_w^4(f)/$ |E2|	$Lob_w^4\ (f)/$ |E3|	$RH_w^4 Lob_w^4\ (f)/$ | E4 |
1	$I_1 = \int_{-1}^{1} e^x dx$	2.350402387	2.350333692/ 0.000068	2.3504678/ 0.000065	2.3504899/ 0.000087	2.35040169/ 0.00000069
2	$I_2 = \int_0^1 e^{-x^2} dx$	0.746825	0.74681458/ 0.0000104	0.74683367/ 0.0000086	0.74683659/ 0.000011	0.746824901/ 0.000000099
3	$I_3 = \int_0^1 e^{x^2} dx$	1.4627	1.4624097/ 0.00029	1.46289391/ 0.00019	1.46297858/ 0.00027	1.46263989/ 0.0000601
4	$I_4 = \int_1^3 \left(\frac{\sin^2 x}{x}\right) dx$	0.7948251	0.79465267/ 0.000172	0.79499761/ 0.000172	0.79505264/ 0.000227	0,79483252/ 0.0000074
5	$I_5 = \int_0^1 \sqrt{x} dx$	0.666666	0.669174/ 0.002508	0.66429729/ 0.002369	0.6568258/ 0.00984	0.68671177/ 0.0200457

Where $E_1 = |\ I(f) - G_w^3(f)\ |$, $E_2 = |\ I(f) - RH_w^4(f)\ |$,

$E_3 = |\ I(f) - Lob_w^4\ (f)\ |$, $E_4 = |\ I(f) - RH_w^4 Lob_w^4\ (f)\ |$ are errors of various rules.

The graphical representation of these errors is given below in Figures: A, B, C, D.



Using the result of the table and the notations for the errors of different methods given above the table , four bar graphs for the errors of the mixed quadrature rule and its constituent rules have been constructed in figures A,B,C and D corresponding to $I_1 = \int_{-1}^{1} e^x dx$, $I_2 = \int_{0}^{1} e^{-x^2} dx$, $I_3 = \int_{0}^{1} e^{x^2} dx$ and $I_4 = \int_{1}^{3} (\frac{\sin^2 x}{x}) dx$ respectively.

In the above four graphs ,the error names of the mixed quadrature rule and its constituent rules have been embedded along X-axis and the respective values of the errors depicting heights of the bars are given along Y-axis. The unit in Y-axis is taken as below:

$$1 = -\log 10^{-1}, 2 = -\log 10^{-2}, 3 = -\log 10^{-3}, 4 = -\log 10^{-4}, 5 = -\log 10^{-5}, 6 = -\log 10^{-6}.$$

Thus from the graphs, we conclude that larger the height of the bar the smaller is the error. Here we derived most significant result that our mixed rule is more accurate than its constituent rules.

6. OBSERVATION

From the table as well as from the graphs it is observed that the absolute error corresponding to mixed quadrature rule $RH_w^4 Lob_w^4\ (f)$,is lesser than those corresponding to its constituent rules $G_w^3(f)$, $RH_w^4(f)$ and $Lob_w^4\ (f)$ are compared, when the test integrals are evaluated.

7. CONCLUSION

After observation one can smartly draw conclusion over the efficiency of the rule formed in this paper as follows:

The mixed rule $RH_w^4 Lob_w^4\ (f)$) is more efficient than its constituent rules $G_w^3(f)$, $RH_w^4(f)$ and $Lob_w^4\ (f)$.

REFERENCES

1. Dirk P. Laurie, Anti-Gaussian quadrature formulas, mathematics of computation, 65(1996)pp. 739-749.
2. Dirk P. Laurie, Computation of Gauss-type quadrature formulas, Journal of Computational and Applied mathematics of computation, 127(2001)pp. 201-217.
3. Lether F., (1976) On Birkhoff-Young quadrature of Analytic Function, J. Comput. Applied Math.2, 2(1976) , pp.81-84 .
4. Atkinson Kendall E.,An in introduction to numerical analysis , 2nd edition, John Wiley and Sons,Inc, (1989).
5. Acharya B.P and Das R.N., Compound Birkhoff –Young rule for numerical integration of analytic functions.Int.J.Math.Educ.Sci.Techno,l 14(1983),pp.91-101.
6. Das R.N.and Pradhan G., A mixed quadrature rule for approximate value of real Definite Integral, Int. J. Edu. Sci. Technol, 27 (1996), pp. 279 - 283.
7. Das R.N. and Pradhan G.,A mixed quadrature Rule for Numerical integration of analytic functions, Bull. Cal. Math. Soc., 89(1997) ,pp.37 - 42.
8. Dash R.B. and Jena S.R., A mixed quadrature of modified Birkhoff-Young using Richardson Extrapolation and Gauss-legendre-4 point transformed Rule, Int. J. Appl. Math. and Applic., 1(2)(2008),pp.111 - 117.
9. Dash R.B. and Mohanty S., A mixed quadrature rule for Numerical integration of analytic functions, Int. Jcomput.Applied math., 42(2009),pp.107-110.
10. Dash R.B. and Mohanty S., A mixed quadrature rule of Gauss-legendre-4 point transformed rule for Numerical integration of analytic functions,Int.J of math.Sci., Engg. Appls, 5(2011), pp.243-249.
11. Begumisa A. and Robinson I., Suboptional Kronrod extension formulas for numerical quadrature, Math, MR 92a (1991), pp.808-818.
12. Walter. Gautschi, Gauss-Kronrod quadrature - a Survey, In G.V. Milovanovic. Editor, Numerical Methods and Approxination Theory III, University of Nis, (1988). MR 89k: 41035, pp. 39-66.
13. Dash R.B. and Das D., A mixed quadrature rule by blending Clenshaw-Curtis Gauss-Legendre quadrature rule for Approximation of real definite integrals in adaptive environment, Proceedings of the International Multi Conference of Engineers and Computer Scientists, I(2011), pp.202-205.
14. Dash R.B. and Das D., A mixed quadrature rule by blending Clenshaw-Curtis Lobatto quadrature rules for Approximation of real definite integrals in adaptive environment,J. Comp,Math.Sci,3(2)(2012), pp.207-215

15. Pati A., Dash R.B. and Patra P., A mixed quadrature rule by blending Clenshaw-Curtis Gauss-Legendre quadrature rules for approximate evaluation of real definite integrals, Bull.Soc.for Math. Services and Standards SciPress Ltd., Swizerland, 2 (2012), pp. 10-15.
16. Mohanty K.P., Hota M.K. and Jena S.R., A Comparative study of mixed quadrature rule with the Compound quadrature rules, American International Journal of Reaserch in Science,Technology, Engineering Mathematics,7(1)(2014), pp.45-52.
17. Tripathy A.K., Dash R.B and Baral A., A mixed quadrature rule by blending Lobatto and Gauss-Legendre three point rule for approximate evaluation of real definite integrals,Int.J.Computing Science and Mathematics,6(1)(2015), pp.366-377.
18. Singh B.P., Dash R.B., and Pati A.,A mixed quadrature rule for numerical integration of analytic functions using open type formulae ,Int.J.of Math.Sci,Vol.10,No.3-4(2011),pp.189-193.
19. Singh B.P. and Dash R.B., Forming two mixed quadrature rule using anti-Gaussian Quadrature rule, Int.J.of Math Sciences and Engg.Appls.Vol.9, No.IV (2015), PP.27-38.
20. Singh B.P. and Dash R.B., Application of mixed quadrature rule using anti-Gaussian Quadrature rule in Adaptive quadrature, Journal of Orissa Mathematical Society, Vol.33.No.2(2014), pp.61-70.