Bulletin of Pure and Applied Sciences
Section - E - Mathematics & Statistics
Website : https : //www.bpasjournals.com/

Bull. Pure Appl. Sci. Sect. E Math. Stat.
38E(Special Issue)(2S), 85–91 (2019)
e-ISSN:2320-3226, Print ISSN:0970-6577
DOI 10.5958/2320-3226.2019.00084.5

On gracefulness in the path union and cycle of $P_m\theta S_n$ *

J. Jeba Jesintha[1] and K. Subashini[2]

1. P.G. Department of Mathematics, Women's Christian College,
Affiliated to University of Madras, Chennai-600008, Tamil Nadu, India.
2. Research Scholar (Part-Time), P.G. Department of Mathematics, Women's Christian College,
Affiliated to University of Madras, Chennai-600008, Tamil Nadu, India.
1. E-mail: jjesintha_75@yahoo.com , 2. E-mail: k.subashinirajan@gmail.com

Abstract A *graceful labeling* of a graph G with q edges is an injection $f : V(G) \rightarrow \{0, 1, 2, ..., q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with the vertices u and v is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. In this paper, we prove that the path union of isomorphic copies of $P_m\theta S_n$ by fixing the middle vertex of the graph as the root is graceful and the graphs obtained by joining each vertex of cycle C_k with the middle vertex of the isomorphic copies of the $P_m\theta S_n$ are graceful where $m \equiv 1\,(\text{mod}2)$ and $k \equiv 0, 3\,(\text{mod}4)$.

Key words Graceful Labeling, Path Union, Cycle of graphs, Star graph.

2010 Mathematics Subject Classification 05C78, 05C99.

1 Introduction

The most famous and challenging graph labeling method is the graceful labeling of graphs introduced by Rosa [8] in 1967. A *graceful labeling* of a graph G with q edges is an injection $f : V(G) \rightarrow \{0, 1, \ldots, q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with the vertices u and v is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. A variety of graphs and families of graphs are known to be graceful for the past five decades. Caterpillars are proved to be graceful by Rosa [7]. Morgan [6] has shown that all lobsters with perfect matchings are graceful. Hrniar and Haviar [4] have shown that all trees of diameter five are graceful. Rosa [7] showed that the n-cycle C_n is graceful if and only if $n \equiv 0$ or $3\,(\text{mod}4)$. Wheels $W_n = C_n + K_l$ are graceful [3]. Helms are shown to be graceful [1]. Let $G_1, G_2, \ldots, G_n\,(n \geqslant 2)$ be the copies of a graph G. Then the graph $G\,(n)$ obtained by adding an edge to G_i and $G_{i+1}, i = 1, 2, \ldots, (n-1)$ is called the path-union of n copies of the graph G [5]. Kaneria et al. [5] have proved that the path union of finite copies of C_{4n} and C_{4n} with twin chords are graceful. For a cycle C_n, each vertex of C_n is replaced by connected graphs $G_1, G_2, G_3, \ldots, G_n$ and is known as

* Communicated, edited and typeset in Latex by *Lalit Mohan Upadhyaya* (Editor-in-Chief).
Received March 16, 2019 / Revised November 17, 2019 / Accepted December 14, 2019. Online First Published on June 03, 2020 at https://www.bpasjournals.com/.
Corresponding author K. Subashini, E-mail: k.subashinirajan@gmail.com
Refereed Proceedings of the National Conference on Mathematics and Computer Applications held at the Department of Mathematics, Women's Christian College, Chennai, India from January 29, 2019 to January 30, 2019.

cycle of graphs. We shall denote it by $C(G_1, G_2, G_3, \ldots, G_n)$. If we replace each vertex by a graph G, i.e., $G_1 = G, G_2 = G, G_3 = G, \ldots, G_n = G$ such cycle of a graph G is denoted by $C(n \cdot G)$. Solairaju et al. [8] proved that the graph $P_m\theta S_n$ is edge odd graceful. The graph $P_m\theta S_n$ is a tree obtained from the path P_m by adding a star graph S_n to each of the pendant vertices of P_m. For an exhaustive survey on this topic the reader is referred to the dynamic survey by Gallian [2]. In this paper, we prove that the path union of isomorphic copies of $P_m\theta S_n$ by fixing the middle vertex of the graph as the root is graceful and the graphs obtained by joining each vertex of cycle C_k with the middle vertex of the isomorphic copies of the $P_m\theta S_n$ are graceful, where $k \equiv 0, 3 \pmod 4$ and $m \equiv 1 \pmod 2$.

2 Preliminary definitions and main results

In this section we first recall the definitions for the cycle graph, the star graph and the path union of graphs. Later we prove that the path union of isomorphic copies of $P_m\theta S_n$ by fixing the middle vertex of the graph as the root is graceful and the graphs obtained by joining each vertex of cycle C_k with the middle vertex of the isomorphic copies of the $P_m\theta S_n$ are graceful where, $k \equiv 0, 3 \pmod 4$ and $m \equiv 1 \pmod 2$.

Definition 2.1. A sequence of vertices $[v_0, v_1, v_2, \ldots, v_n, v_0,]$ is a *cycle* of length $n+1$ if $v_{i+1}v_i \in E, i = 1, 2, \ldots, n$ and $v_n v_0 \in E$. A cycle of length n is denoted by C_n.

Definition 2.2. The *star graph* S_n of order n, sometimes simply known as an "n-star graph", is a tree on n vertices with one vertex of degree $n-1$ and the other $(n-1)$ vertices of vertex degree 1.

Definition 2.3. The graph $P_m\theta S_n$ is a tree obtained from the path P_m by adding a star graph S_n to each of the pendant vertices of P_m.

Definition 2.4. Let G be a graph and $G_1, G_2, \ldots, G_n (n \geqslant 2)$ be n copies of the graph G. Then the graph obtained by adding an edge from G_i to $G_{i+1}, i = 1, 2, \ldots, n-1$ is called the path union of G.

Theorem 2.5. *The path union of isomorphic copies of $P_m\theta S_n$ by fixing the middle vertex of the graph as the root is graceful where, $m \equiv 1 \pmod 2$.*

Proof. Let the graph $G = P_m\theta S_n$ be a tree obtained from the path P_m by adding a star graph S_n to each of the pendant vertices of P_m as shown in Fig.1. Let G' be the graph obtained from G by fixing the middle vertex of the path P_m where, $m \equiv 1 \pmod 2$ as a root of the tree which is shown in Fig.2.

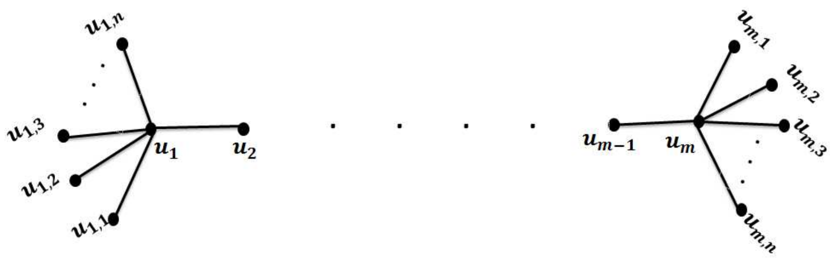

Fig. 1: The graph G.

Let $G_1', G_2', \ldots, G_t' (t \geqslant 2)$ be t copies of G' as shown in Fig.3. The first copy G_1' of G' is described as follows: denote the vertices in the path P_m as $u_1^1, u_2^1, \ldots, u_m^1$ and the vertices of the star S_n which are attached with the each end of the pendant vertices of P_m are denoted as $u_{1,1}^1, u_{1,2}^1, \ldots, u_{1,n}^1$ and $u_{m,1}^1, u_{m,2}^1, \ldots, u_{m,n}^1$ in the clockwise direction. The second copy G_2' of G' is described as follows: the vertices in the path P_m are denoted as $u_1^2, u_2^2, \ldots, u_m^2$ and the vertices of the star S_n which

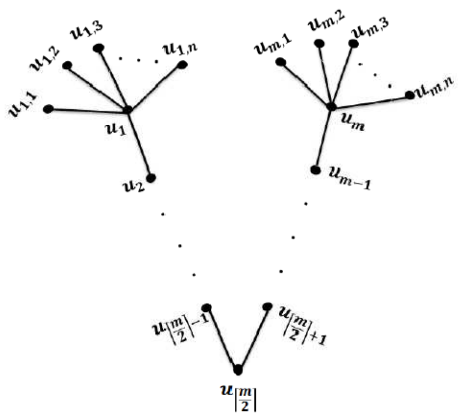

Fig. 2: The graph G'.

are attached with the each end of the pendant vertices of P_m are denoted as $u_{1,1}^2, u_{1,2}^2, \ldots, u_{1,n}^2$ and $u_{m,1}^2, u_{m,2}^2, \ldots, u_{m,n}^2$ in the clockwise direction. Finally the last copy $G_t^{'}$ of G' is described by denoting the vertices in the path P_m as $u_1^t, u_2^t, \ldots, u_m^t$ and the vertices of the star S_n which are attached with each end of the pendant vertices of P_m are denoted as $u_{1,1}^t, u_{1,2}^t, \ldots, u_{1,n}^t$ and $u_{m,1}^t, u_{m,2}^t, \ldots, u_{m,n}^t$ in the clockwise direction. Let G'' be the graph obtained by adding an edge e_i between the root vertex of the copies $G_i^{'}$ and $G_{i+1}^{'}, 1 \leq i \leq (t-1)$. The graph G'' so obtained is called the path union of $P_m \theta S_n$ which is shown in Fig.4. Note that in G'' the vertices in the path P_m are denoted as u_j^i for $1 \leq i \leq t$ and $1 \leq j \leq m$ and the vertices in the star S_n are denoted as $u_{j,s}^i$ for $1 \leq i \leq t, 1 \leq j \leq m$ and $1 \leq s \leq n$. If q, v denote the number of edges and vertices respectively in G'', then $q = (2n+m)\,t - 1$ and $v = (2n+m)\,t$.

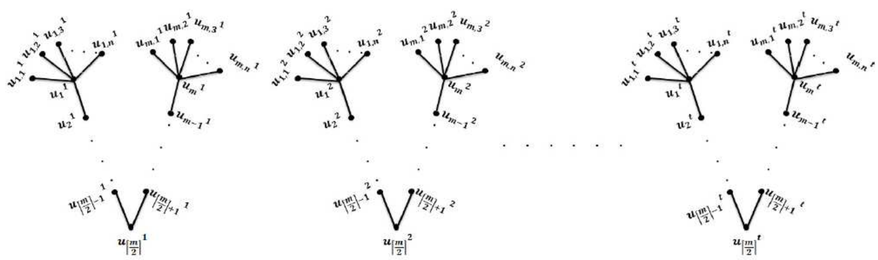

Fig. 3: The copies of the graph G'.

Labeling for the vertices u_j^i for $1 \leq i \leq t$ and $1 \leq j \leq m$

$$f\left(u_{2j-1}^{2i-1}\right) = q - (i-1)\,(2n+m) - (j-1) \;\text{ for } 1 \leqslant i \leqslant \left\lceil \frac{t}{2} \right\rceil, 1 \leqslant j \leqslant \left\lceil \frac{m}{2} \right\rceil$$

$$f\left(u_{2j}^{2i-1}\right) = n + (i-1)\,(2n+m) + (j-1) \;\text{ for } 1 \leqslant i \leqslant \left\lceil \frac{t}{2} \right\rceil, 1 \leqslant j \leqslant \left\lfloor \frac{m}{2} \right\rfloor$$

$$f\left(u_{2j-1}^{2i}\right) = 2n + \left\lfloor \frac{m}{2} \right\rfloor + (i-1)\,(2n+m) + (j-1) \;\text{ for } 1 \leqslant i \leqslant \left\lfloor \frac{t}{2} \right\rfloor, 1 \leqslant j \leqslant \left\lceil \frac{m}{2} \right\rceil$$

$$f\left(u_{2j}^{2i}\right) = q - \left(n + \left\lceil \frac{m}{2} \right\rceil\right) - (i-1)\,(2n+m) - (j-1) \;\text{ for } 1 \leqslant i \leqslant \left\lceil \frac{t}{2} \right\rceil, 1 \leqslant j \leqslant \left\lfloor \frac{m}{2} \right\rfloor$$

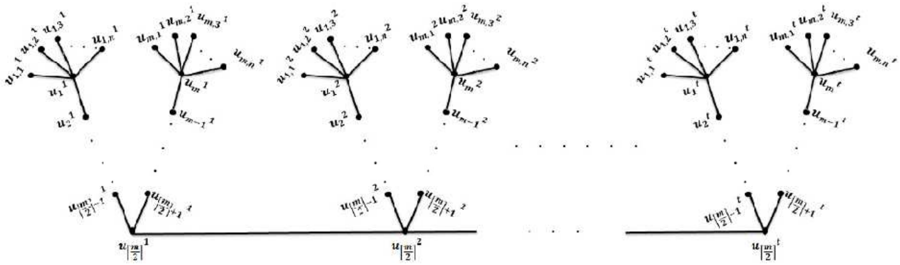

Fig. 4: The graph G'' which is the path union of $P_m\theta S_n$.

Labeling for the vertices $u_{j,s}^{i}$ for $1 \le i \le t$, $1 \le j \le m$ and $1 \le s \le n$

$$f\left(u_{j,s}^{2i-1}\right) = (i-1)(2n+m) + (s-1) \ \text{ for } 1 \leqslant i \leqslant \left\lceil \frac{t}{2} \right\rceil, 1 \leqslant s \leqslant n, j = 1$$

$$f\left(u_{j,s}^{2i-1}\right) = n + \left\lfloor \frac{m}{2} \right\rfloor + (i-1)(2n+m) + (s-1) \ \text{ for } 1 \leqslant i \leqslant \left\lceil \frac{t}{2} \right\rceil, 1 \leqslant s \leqslant n, j = m$$

$$f\left(u_{j,s}^{2i}\right) = q - \left(n + \left\lfloor \frac{m}{2} \right\rfloor\right) - (i-1)(2n+m) + (s-1) \ \text{ for } 1 \leqslant i \leqslant \left\lfloor \frac{t}{2} \right\rfloor, 1 \leqslant s \leqslant n, j = 1$$

$$f\left(u_{j,s}^{2i}\right) = q - (2n+m-1) - (i-1)(2n+m) + (s-1) \ \text{ for } 1 \leqslant i \leqslant \left\lfloor \frac{t}{2} \right\rfloor, 1 \leqslant s \leqslant n, j = m$$

From the above definition it is clear that all the vertex labels are distinct. The edge labels can be computed for the above vertex labels and they are also found to be distinct from 1 to q. Therefore, the path union of isomorphic copies of $P_m\theta S_n$ by fixing the middle vertex of the graph as the root is graceful where, $m \equiv 1\,(\text{mod}2)$ which is illustrated in Fig.5.

Illustration 2.6. Here $q = 84, n = 5, m = 7, t = 5$.

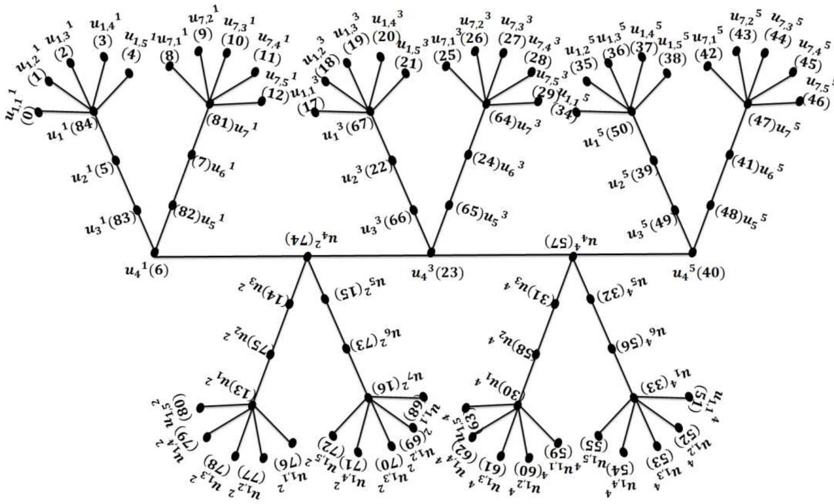

Fig. 5: The graceful labeling of path union of five copies of $P_7\theta S_5$.

□

Theorem 2.7. *The graphs obtained by joining each vertex of the cycle C_k with the middle vertex of the isomorphic copies of the $P_m\theta S_n$ are graceful where, $k \equiv 0,3\,(\mathrm{mod}4)$ and $m \equiv 1\,(\mathrm{mod}2)$.*

Proof. Let G be a cycle C_k with k vertices that are denoted as $v_1, v_2, \ldots, v_k$ in the anticlockwise direction and H be the connected graph $P_m\theta S_n$ whose path vertices are denoted as $u_1, u_2, \ldots, u_m$ nd the vertices of the star S_n which are attached with the each end of the pendant vertices of P_m are denoted as $u_{1,1}, u_{1,2}, \ldots, u_{1,n}$ and $u_{m,1}, u_{m,2}, \ldots, u_{m,n}$ as shown in Fig.6.

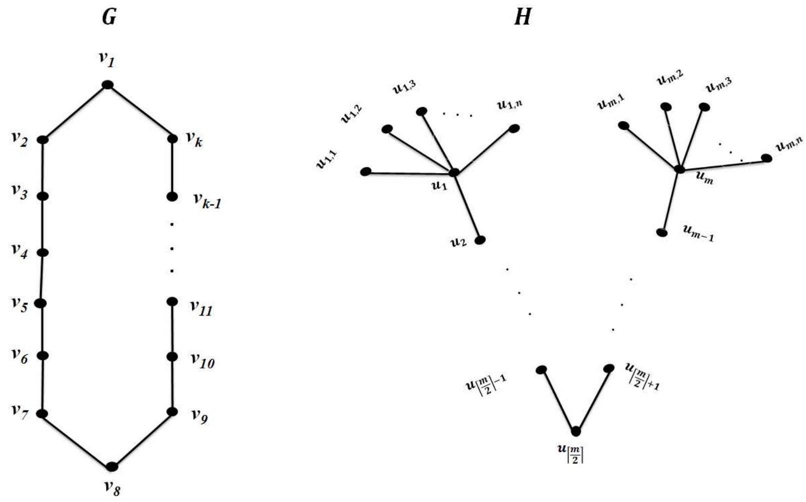

Fig. 6: The graphs G and H.

Let $C\,(k \circ H)$ be the cycle of graphs that is obtained by replacing each vertex of the cycle C_k by the graph H. In other words each vertex v_i for $1 \le i \le k$ is identified with the root vertex $u_{\lceil \frac{m}{2} \rceil}$ of the graph H. As the k copies of H are attached to the cycle C_k by the identification of v_i's of C_k with $u_{\lceil \frac{m}{2} \rceil}$ of each of the k copies of H. We shall denote these identified vertices as $v_1 = u^1_{\lceil \frac{m}{2} \rceil}$ for the first copy of H, $v_2 = u^2_{\lceil \frac{m}{2} \rceil}$ for the second copy of H, $v_k = u^k_{\lceil \frac{m}{2} \rceil}$ for the k^{th} copy of H. In general, $v_i = u^i_{\lceil \frac{m}{2} \rceil}$ for $1 \le i \le k$ for the i^{th} copy of H. Now we rename the path vertices in the first copy of H in $C\,(k \circ H)$ as $u^1_1, u^1_2, \ldots, u^1_m$ and the vertices of the star S_n which are attached with each end of the pendant vertices of P_m are denoted as $u^1_{1,1}, u^1_{1,2}, \ldots, u^1_{1,n}$ and $u^1_{m,1}, u^1_{m,2}, \ldots, u^1_{m,n}$ in the clockwise direction. The path vertices in the second copy of H are renamed as $u^2_1, u^2_2, \ldots, u^2_m$ and the vertices of the star S_n which are attached with each end of the pendant vertices of P_m are denoted as $u^2_{1,1}, u^2_{1,2}, \ldots, u^2_{1,n}$ and $u^2_{m,1}, u^2_{m,2}, \ldots, u^2_{m,n}$ in the clockwise direction. This continues and the path vertices in the last copy of H are denoted as $u^k_1, u^k_2, \ldots, u^k_m$ and the vertices of the star S_n which are attached with each end of the pendant vertices of P_m are denoted as $u^k_{1,1}, u^k_{1,2}, \ldots, u^k_{1,n}$ and $u^k_{m,1}, u^k_{m,2}, \ldots, u^k_{m,n}$ in the clockwise direction. Thus, u^i_j represent the path vertices in the i^{th} copy of H for $1 \le i \le k, 1 \le j \le m$ and the vertices in the star S_n are denoted as $u^i_{j,s}$ for $1 \le i \le k, 1 \le j \le m, 1 \le s \le n$. If q denotes the number of vertices and edges respectively in $C\,(k \circ H)$ then $p = q = (2n+m)\,k$. Also noting that the theorem is proved for $k \equiv 0,3\,(\mathrm{mod}4)$ the vertices of $C\,(k \circ H)$ are labeled as follows depending on the parameter k.

Labeling for the vertices u^i_j for $1 \le i \le k$ and $1 \le j \le m$

$$f\left(u^{2i-1}_{2j-1}\right) = q - (i-1)\,(2n+m) - (j-1) \;\text{ for } 1 \leqslant i \leqslant \left\lceil \frac{k}{4} \right\rceil, 1 \leqslant j \leqslant \left\lceil \frac{m}{2} \right\rceil$$

$$f\left(u^{2i-1}_{2j-1}\right) = q - 1 - (i-1)\,(2n+m) - (j-1) \;\text{ for } \left\lceil \frac{k}{4} \right\rceil < i \leqslant \left\lceil \frac{k}{2} \right\rceil, 1 \leqslant j \leqslant \left\lceil \frac{m}{2} \right\rceil$$

$$f\left(u_{2j}^{2i-1}\right)=n+(i-1)(2n+m)+(j-1)\ \text{for}\ 1\leqslant i\leqslant\left\lceil\frac{k}{2}\right\rceil, 1\leqslant j\leqslant\left\lfloor\frac{m}{2}\right\rfloor$$

$$f\left(u_{2j-1}^{2i}\right)=2n+\left\lfloor\frac{m}{2}\right\rfloor+(i-1)(2n+m)+(j-1)\ \text{for}\ 1\leqslant i\leqslant\left\lfloor\frac{k}{2}\right\rfloor, 1\leqslant j\leqslant\left\lceil\frac{m}{2}\right\rceil$$

$$f\left(u_{2j}^{2i}\right)=q-\left(n+\left\lceil\frac{m}{2}\right\rceil\right)-(i-1)(2n+m)-(j-1)\ \text{for}\ 1\leqslant i\leqslant\left\lceil\frac{k}{4}\right\rceil, 1\leqslant j\leqslant\left\lfloor\frac{m}{2}\right\rfloor$$

$$f\left(u_{2j}^{2i}\right)=q-1-\left(n+\left\lceil\frac{m}{2}\right\rceil\right)-(i-1)(2n+m)-(j-1)\ \text{for}\ \left\lceil\frac{k}{4}\right\rceil<i\leqslant\left\lfloor\frac{k}{2}\right\rfloor, 1\leqslant j\leqslant\left\lfloor\frac{m}{2}\right\rfloor$$

Labeling for the vertices $u_{j,s}^{i}$ for $1\leq i\leq k, 1\leq j\leq m$ and $1\leq s\leq n$

$$f\left(u_{j,s}^{2i-1}\right)=(i-1)(2n+m)+(s-1)\ \text{for}\ 1\leqslant i\leqslant\left\lceil\frac{k}{2}\right\rceil, 1\leqslant s\leqslant n, j=1$$

$$f\left(u_{j,s}^{2i-1}\right)=n+\left\lfloor\frac{m}{2}\right\rfloor+(i-1)(2n+m)+(s-1)\ \text{for}\ 1\leqslant i\leqslant\left\lceil\frac{k}{2}\right\rceil, 1\leqslant s\leqslant n, j=m$$

$$f\left(u_{j,s}^{2i}\right)=q-\left(n+\left\lfloor\frac{m}{2}\right\rfloor\right)-(i-1)(2n+m)+(s-1)\ \text{for}\ 1\leqslant i\leqslant\left\lceil\frac{k}{4}\right\rceil, 1\leqslant s\leqslant n, j=1$$

$$f\left(u_{j,s}^{2i}\right)=q-1-\left(n+\left\lfloor\frac{m}{2}\right\rfloor\right)-(i-1)(2n+m)+(s-1)\ \text{for}\ \left\lceil\frac{k}{4}\right\rceil<i\leqslant\left\lfloor\frac{k}{2}\right\rfloor, 1\leqslant s\leqslant n, j=1$$

$$f\left(u_{j,s}^{2i}\right)=q-(2n+m-1)-(i-1)(2n+m)+(s-1)\ \text{for}\ 1\leqslant i\leqslant\left\lceil\frac{k}{4}\right\rceil, 1\leqslant s\leqslant n, j=m$$

$$f\left(u_{j,s}^{2i}\right)=q-1-(2n+m-1)-(i-1)(2n+m)+(s-1)\ \text{for}\ \left\lceil\frac{k}{4}\right\rceil<i\leqslant\left\lfloor\frac{k}{2}\right\rfloor, 1\leqslant s\leqslant n, j=m$$

From the above definition it is clear that all the vertex labels are distinct. The edge labels can be computed for the above vertex labels and they are also found to be distinct from 1 to q. Therefore, the cycle $C(k\circ H)$ where H denotes the isomorphic copies of $P_m\theta S_n$ is graceful where $k\equiv 0,3\,(\mathrm{mod}4)$ is illustrated in Fig.7.

□

Illustration 2.8. Here $q=68, k=4, m=7, n=5$ (Fig.7).

3 Conclusion

In this paper we proved that the path union of isomorphic copies of $P_m\theta S_n$ by fixing the middle vertex of the graph as the root is graceful and the graphs obtained by joining each vertex of cycle C_k with the middle vertex of the isomorphic copies of $P_m\theta S_n$ are graceful where, $k\equiv 0,3\,(\mathrm{mod}4)$ and $m\equiv 1\,(\mathrm{mod}2)$. We intend to prove the same result for any number of vertices in our next step.

References

[1] Ayel, J. and Favaron, O. (1984). Helms are graceful, in Progress in Graph Theory (Waterloo, Ontario, 1982), Academic Press, Toronto, Ontario, 89–92.

[2] Gallian, J.A. (2015). A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, #DS6, 1–389.

[3] Hoede, C. and Kuiper, H. (1987). All wheels are graceful, *Util. Math.*, 14, 311–322.

[4] Hrnciar, P. and Haviar, A. (2001). All trees of diameter five are graceful, *Discrete Math.*, 233, 133–150.

[5] Kaneria, V.J., Makadia, H.M., Jariya, M.M. and Meghpara, M. (2014). Graceful labeling for complete bipartite graphs, *Appl. Math. Sci.*, 8 (103), 5099–5104.

[6] Morgan, D. (2002). All lobsters with perfect matchings are graceful, *Electron. Notes Discrete Math.*, 11, 503–508.

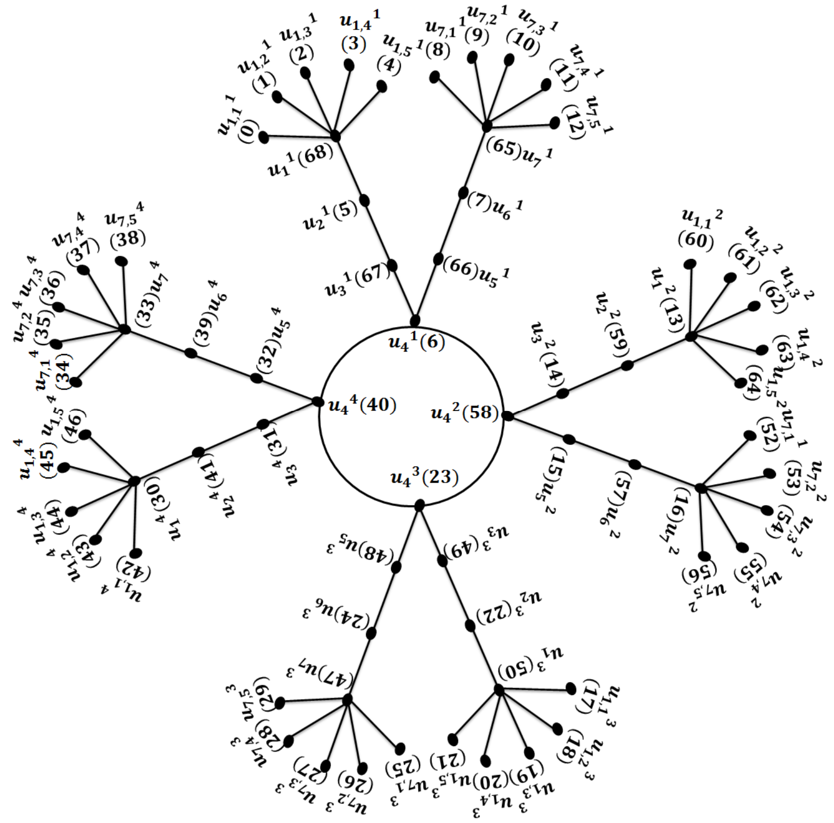

Fig. 7: The graceful labeling of $C\,(4 \circ H)$ where H denotes the isomorphic copies of $P_7 \theta S_5$.

[7] Rosa, A. (1967). On certain valuations of the vertices of a graph, Theory of Graphs (International Symposium, Rome, July (1966), Gordon and Breach, New York and Dunod Paris, 349– 355.

[8] Solairaju, A., Vimala, C. and Sasikala, A. (2010). Edge Odd gracefulness of $P_m \theta S_n$ where $m = 5, 6, 7, 8$, *International Journal of Computer Applications*, 9(11), 1–2.