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Fibonacci divisor cordial labeling for helm and closed helm *

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Abstract In this paper we prove the Fibonacci divisor cordial labeling for helm and closed helm CH_n for $n = 4m$ and $n = 4m - 1$ for any positive integer m.

Key words Cordial labeling, divisor cordial labeling, Fibonacci divisor cordial labeling, helm and closed helm.

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1 Introduction

A graph labeling is an assignment of integers to the vertices or edges, or both subject to certain conditions. For the past five decades graph labeling techniques have gained much importance with wide applications [3] in other fields like coding theory, radar, astronomy, circuit design, missile guidance, communication network addressing, X-ray crystallography, data base management and major areas of computer science like data mining, image processing, cryptography, software testing and information security. In 1987, Cahit [1] introduced a labeling technique known as the *cordial labeling*. The *cordial labelling* of a graph G is an injection $f : V(G) \to \{0, 1\}$ such that each edge uv in G is assigned the label $|f(u) - f(v)|$ with the property that, $|v_f(0) - v_f(1)| \leq 1$and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ denotes the number of vertices with label i and $e_f(i)$ denotes the number of edges with label I for $i = 0, 1$. A graph is called *cordial* if it admits cordial labeling. In [2], Cahit proved that every tree is cordial, all fans are cordial, the wheel W_n when $n \not\equiv 3\,(mod4)$ is cordial, the complete graph K_n is cordial if and only if $n \leq 3$, the bipartite graph $K_{m,n}$ is cordial for all m and n, the friendship graph $C_3^{(t)}$ is cordial if and only if $t \not\equiv 2\,(mod4)$. Shee and Ho [4] proved that the path union of cycles, Petersen graphs, trees, wheels, unicycle graphs are cordial. Vaidya et al. [8] proved that the graph obtained by joining two copies of cycles by a path of arbitrary length is cordial.
In the year 2011, a variation of cordial labeling known as the *divisor cordial labeling* was introduced by Varatharajan et al. [9].The divisor cordial labeling of a graph G with vertex set V is a bijection from V

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to $\{1, 2, \ldots, V(G)\}$ such that, if each edge uv is assigned the label 1 when $f(u)\,|\,f(v)$ or, $f(v)\,|\,f(u)$ and 0 otherwise, then the number of edges labeled with *0* and the number of edges labeled with *1* differ by atmost 1. A graph that admits divisor cordial labeling is called a *divisor cordial graph.* Varatharajan et al. [9] have proved that the graphs such as paths, cycles, wheels, stars and some complete bipartite graphs are divisor cordial. Vaidya and Shah [6,7] have proved that the splitting graphs of stars, bistars and shadow graphs, the squares of bistars, helms, flower graphs , gears graph obtained by switching of a vertex in a cycle, switching of a rim vertex in a wheel and switching of an apex vertex in a helm are divisor cordial graphs.
Another variation of cordial labeling known as Fibonacci divisor cordial labeling, was introduced by Sridevi et al. [5] in 2013. A *Fibonacci divisor cordial labeling* of a graph G with vertex set $V(G)$ is a bijection $f : V(G) \to \{F_1, F_2, \ldots, F_p\}$, where F_i is the i^{th} Fibonacci number such that if each edge uv is assigned the label 1 when $f(u)\,|\,f(v)$ or, $f(v)\,|\,f(u)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph that admits Fibonacci divisor cordial labeling is called a *Fibonacci divisor cordial graph.* Sridevi et al. [6] have proved that graphs such as the path P_n, the cycle C_n, the subdivision of bistar $\langle B_{n,n} : w\rangle$, the n star $K_{1,n}$ (when $n = 11$), the wheel W_n (when $n = 7, 8, 9$ and 10) and $C_m \odot P_n$ are Fibonacci cordial graphs. In this paper we prove that the Helm H_n and the Closed helm CH_n for $n = 4m$ and $4m - 1$ where $1 = m = n$ admit Fibonacci divisor cordial labeling.

2 Preliminary definitions and main results

In this section we first define Fibonacci numbers, the helm and closed helm graphs and then we prove two theorems.

Definition 2.1. The *Fibonacci numbers* can be defined by the linear recurrence relation satisfying the following conditions:

$$F_n = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ F_{n-1} + F_{n-2}, & \text{if } n > 1. \end{cases}$$

This generates the infinite sequence of integers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$.

Definition 2.2. The *helm* H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each vertex of the wheel's rim . It contains three types of vertices: an apex of degree n, n vertices of degree *4* and n pendant vertices.

Definition 2.3. A *closed helm* CH_n is the graph obtained by taking a helm H_n and by adding edges between the pendant vertices.

Theorem 2.4. *A Helm H_n for every n admits the Fibonacci divisor cordial labeling.*

Proof. Let G be a graph obtained from a wheel W_n by attaching a pendant edge to each vertex of the wheel's rim. It contains three types of vertices: an apex of degree n, n vertices of degree 4 and n pendant vertices. Denote the apex vertex by v. The vertices of degree four are denoted by $v_1, v_2, \ldots, v_n$. Denote the pendant vertices of the helm $H_n$by $u_1, u_2, \ldots, u_n$.Denote the edges adjacent to the apex vertex by e_i*'s* where $1 \le i \le n$. Denote by $g_1, g_2, \ldots, g_n$ the edges of the wheel's rim and the edges of the pendant vertices incident on the rim of the wheel by $h_1, h_2, \ldots, h_n$.
Let $G = H_n$ be a graph with $2n+1$ vertices and $3n$ edges. Then $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. Define the vertex labelling $f : V(G) \to \{F_1, F_2, \ldots, F_{2n+1}\}$ as follows:

$$\begin{aligned} &f(v) = F_1, \\ &f(v_i) = F_4, \quad \text{for } i = 1, 2, \ldots, n, \\ &f(u_i) = F_2, \quad \text{for } i = 1, 2, \ldots, n. \end{aligned} \tag{2.1}$$

Case 1: When n is odd:
Define the vertex labeling for the wheel's rim v_{i+1} for $1 \le i \le n$ as follows:

$$f(v_{i+1}) = F_{3+2m}, \quad \text{for } 1 \le i \le n-1 \text{ and } 0 \le m \le n-2. \tag{2.2}$$

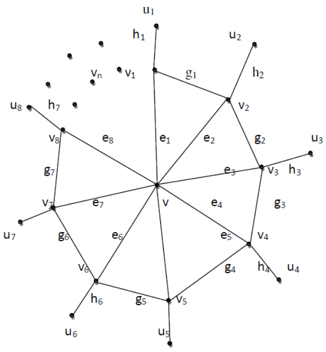

Fig. 1: The helm H_n.

Define the vertex labeling for the pendant vertex u_{i+1} (for $1 \le i \le n$) of the helm H_n as below:

$$\begin{array}{l} f(u_{i+1}) = F_{2(3+2m)} \text{ for } 1 \le i \le \frac{n-1}{2} \text{ and } 0 \le m \le \frac{n-3}{2}, \\ f(u_{i+1}) = F_{(8+4m)} \text{ for } \frac{n+1}{2} \le i \le n-2 \text{ and } 0 \le m \le \frac{n-5}{2}, \\ f(u_n) = F_{(2n+1)}. \end{array} \tag{2.3}$$

The edges are labeled 1 if $f(u)\,|f(v)$ or, $f(v)\,|f(u)$, else, the edges are labeled 0, i.e.,

$$e_f = \begin{cases} 1, \text{ if } f(u)\,|f(v) \text{ or, } f(v)\,|f(u), \\ 0, \text{ otherwise.} \end{cases} \tag{2.4}$$

The edges that are labeled 1 are the edges which are adjacent to the apex vertex e_i, where, $1 \le i \le n$ and the edges of the pendant vertices which are incident on the wheel's rim h_i, where, $1 \le i \le \frac{n+1}{2}$. The edges of the wheel's rim, g_i where, $1 \le i \le n$ and the edges of the pendant incident to wheel's rim h_i, where, $\frac{n+3}{2} \le i \le n$ are labelled 0. $F_m | F_n$ iff $m|n$ where, $m, n = 3$.
In general, for the edges labeled with 0 and 1 we have

$$\begin{array}{l} e_f(0) = \frac{3n+1}{2}, \\ e_f(1) = \frac{3n-1}{2}. \end{array} \tag{2.5}$$

Therefore, $|e_f(0) - e_f(1)| = \left|\frac{3n+1}{2} - \frac{3n-1}{2}\right| = 1$, hence, $|e_f(0) - e_f(1)| \le 1$. Thus $G = H_n$, when n is odd, admits Fibonacci divisor cordial labeling.

Case 2: When n is even:

The vertex labeling for the wheel's rim v_{i+1} for $1 \le i \le n$ is defined as follows:

$$f(v_{i+1}) = F_{3+2m}, \text{ for } 1 \le i \le n-1 \text{ and } 0 \le m \le n-2. \tag{2.6}$$

Define the vertex labeling for the pendant vertices of the helm H_n,u_{i+1} for $\frac{n+2}{2} \le i \le n$ as below:

$$\begin{array}{l} f(u_{i+1}) = F_{2(3+2m)} \text{ for } 1 \le i \le \frac{n-2}{2} \text{ and } 0 \le m \le \frac{n-4}{2}, \\ f(u_{i+1}) = F_{(8+4m)} \text{ for } \frac{n}{2} \le i \le n-2 \text{ and } 0 \le m \le \frac{n-4}{2}, \\ f(u_n) = F_{(2n+1)}. \end{array} \tag{2.7}$$

As mentioned in the Case 1 above the edges are labeled 1 if $f(u)\,|f(v)$ or,$f(v)\,|f(u)$, else, the edges are labeled 0 (see (2.4) above). The edges adjacent to apex vertex e_i where $1 \le i \le n$ and the edges of the pendant incident to wheel's rim $h_i$where, $1 \le i \le \frac{n}{2}$ are labeled 1. The edges of the wheel's rim, g_i where, $1 \le i \le n$ and the edges of the pendant incident to wheel's rim h_i, where, $\frac{n+2}{2} \le i \le n$ are labeled 0 and as mentioned in the Case 1 above that $F_m | F_n$ iff $m|n$ where, $m, n = 3$. It can be seen in this case that for the edges labelled 0 and 1, $e_f(0) = e_f(1) = \frac{3n}{2}$, which gives, $|e_f(0) - e_f(1)| = \left|\frac{3n}{2} - \frac{3n}{2}\right| = 0 \le 1$, showing thereby that the graph $G = H_n$, when n is even, admits

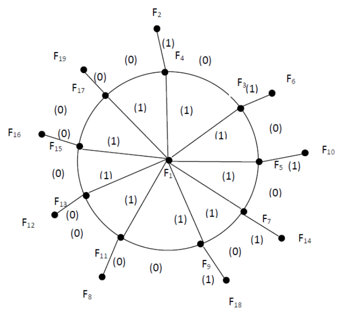

Fig. 2: The helm H_9.

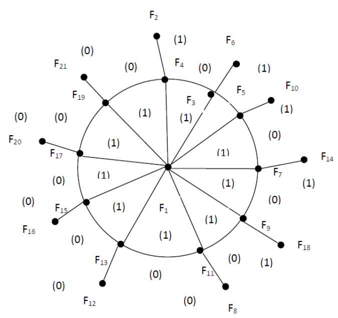

Fig. 3: The helm H_{10}.

Fibonacci divisor cordial labeling. Thus the helm graph H_n admits Fibonacci divisor cordial labelling for all n.

The illustration for the above theorem is given above in Figs. 2 and 3.

□

Theorem 2.5. *The closed helm CH_n, where $n = 4m$ and $4m - 1$ for any positive integer m, admits Fibonacci divisor cordial labeling.*

Proof. Let G be a graph obtained by taking a helm H_n and by adding edges between the pendant vertices. The graph G is thus closed helm CH_n. Denote the apex vertex of CH_n by u and the vertices of the inner rim of CH_n are denoted by u_i where, $1 \leq i \leq n$. Denote the vertices on the outer rim of CH_n by v_i where, $1 \leq i \leq n$. The edges incident on the apex vertex are denoted by $e_1, e_2, \ldots, e_n$ while, $g_1, g_2, \ldots, g_n$ denote the edges of the inner rim of CH_n. Denote the edges incident on the outer rim of CH_n by $h_1, h_2, \ldots, h_n$ and the edges of the outer rim of CH_n by $w_1, w_2, \ldots, w_n$. Let $G = CH_n$ be a graph with $2n+1$ vertices and $4n$ edges, then $|V(G)| = 2n+1$ and $|E(G)| = 4n$. Define the vertex labelling $f : V(G) \rightarrow \{F_1, F_2, \ldots, F_{2n+1}\}$ as

$$f(v) = F_1 \tag{2.8}$$

Define the vertex labeling for the inner rim of CH_n, i.e., u_i, where $1 \leq i \leq n$ in the following order, where the vertices are labeled in the clockwise direction:

$$\begin{aligned}
&F_2, F_{2^2}, \ldots, F_{2^{k1}} \\
&F_3, F_{3\cdot 2}, F_{3\cdot 2^2}, \ldots, F_{3\cdot 2^{k2}} \\
&F_5, F_{5\cdot 2}, F_{5\cdot 2^2}, \ldots, F_{5\cdot 2^{k3}} \\
&\ldots \\
&F_{(m-1)}, F_{(m-1)\cdot 2}, F_{(m-1)\cdot 2^2}, \ldots, F_{(m-1)\cdot 2^{kp}} \\
&F_m, \quad \text{where} \;\; m = 3 + 2r, 1 \leq r \leq n, p \geq 0.
\end{aligned} \tag{2.9}$$

Define vertex labeling for outer rim of CH_n, i.e., v_i, for $1 \leq i \leq n$, in the following manner, where the vertices are labeled in the anticlockwise direction:

$$\begin{aligned}
&F_{2m}, \quad \text{where,} \;\; m = 3 + 2r, \\
&F_{(m+1)}, F_{(m+1)\cdot 2}, F_{(m+1)\cdot 2^2}, \ldots, F_{(m+1)\cdot 2^{kp}} \\
&F_{(m+2)}, F_{(m+2)\cdot 2}, F_{(m+2)\cdot 2^2}, \ldots, F_{(m+2)\cdot 2^{kp}} \\
&\ldots \\
&F_n, F_{n\cdot 2}, F_{n\cdot 2^2}, \ldots, F_{n\cdot 2^{kp}}
\end{aligned} \tag{2.10}$$

As mentioned in the proof of the Theorem 2.4 that the edges are labeled 1 if $f(u)\,|f(v)$ or, $f(v)\,|f(u)$, else, the edges are labeled 0 (see (2.4) above). In the above labeling, we see that when the vertices are labeled with Fibonacci numbers, the edges are labelled 1 for consecutive adjacent vertices with even indices and for the edges incident with the apex vertex. Further, the vertices labeled with Fibonacci numbers with odd and even indices, if they divide they contribute 1, otherwise 0. The vertices labeled with odd indices for Fibonacci numbers whose edges are labelled contribute 0, since $F_m|F_n$ iff $m|n$ where, $m, n \geq 3$. The edges labeled with 0 and 1 are calculated as $e_f(0) = 2n = e_f(1)$, which gives, $|e_f(0) - e_f(1)| = |2n - 2n| = 0 \leq 1$. Thus, the graph $G = CH_n$ for $n = 4m$ and $4m - 1$, for any positive integer m admits Fibonacci divisor cordial labeling. This theorem is illustrated in Figs. 4 and 5.

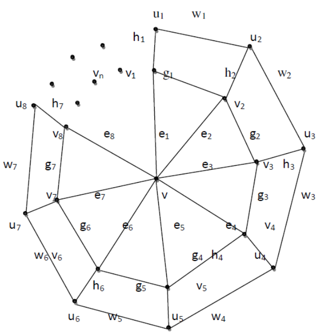

Fig. 4: The closed helm CH_n.

□

3 Conclusion

In this paper we have proved that the graphs – the Helm H_n for every n and the Closed helm $CH_n$where $n = 4m$ and $4m - 1$ for any positive integer m admits Fibonacci divisor cordial labelling. We would also like to find some other graphs that admit Fibonacci divisor cordial labeling.

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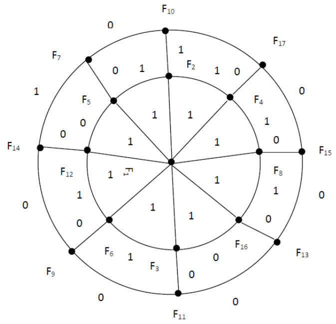

Fig. 5: The closed helm CH_8.

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