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Energy of central and middle graph of a regular graph *

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Abstract An eigenvalue of a graph G is the eigenvalue of its adjacency matrix. The energy $E(G)$ of G is the sum of the absolute values of its eigenvalues. Two graphs having same energy and same number of vertices are called *equienergetic* graphs. If $\mu_1, \ldots, \mu_n$ are the eigenvalues of the adjacency matrix of G and $\theta_1, \ldots, \theta_n$ are the eigenvalues of the adjacency matrix of all-one matrix (J_n), then the energy of the Central graph of a simple, connected, r-regular graph $E(C(G))$ and the energy of a Middle graph of a simple, connected, 2-regular graph $E(M(G))$ is derived in terms of the eigenvalues of the original graph G by us in this paper.

Key words Central graph, Middle graph, adjacency matrix, incident matrix, eigenvalues, energy of graph.

2010 Mathematics Subject Classification 05C50.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. For standard terminology and notations related to graph theory, we follow Balakrishnan and Ranganathan [4], while, for algebra we follow Lang [10].
The adjacency matrix $A(G)$ of a graph G with vertices $v_1, \ldots, v_n$ is an $n \times n$ matrix $[a_{ij}]$ such that, $a_{ij}= 1$ if v_i is adjacent to v_j, and 0 otherwise. The incidence matrix $R(G)$, is an $n \times m$ matrix $[r_{ij}]$ for which $r_{ij} = 1$ if v_i is incident with the edge e_j and 0 otherwise. The set of eigenvalues of the graph with their multiplicities is known as spectrum of the graph and it is denoted by Spec G.
In 1978 Gutman [7] defined the energy of a graph G as the sum of absolute values of the eigenvalues of graph G and denoted it by $E(G)$. Hence,

$$E\,(G) = \sum_{i=1}^{n} |\lambda_i(G)|$$

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Here it is also mentioned that the energy of a totally disconnected graph ${K_n}^c$ is zero while the energy of a complete graph K_n with maximum possible number of edges is $2(n-1)$.
In 2004, Bapat and Pati [5] proved that if the energy of a graph is rational then it must be an even integer, while Pirzada and Gutman [12] established that the energy of a graph is never the square root of an odd integer.
In 2011 Brouwer and Haemers [1] stated that if $\overline{G}$ is the complement graph of G then the adjacency matrix $A\left(\overline{G}\right) = J - I - A(G)$ where J is all-1 matrix of order n and its spectrum is $n^1, 0^{n-1}$ (exponents are its multiplicities).
In theoretical chemistry, using Hückel theory, the π-electron energy of a conjugated carbon molecule was computed, which coincides with the energy as defined here. A brief account of graph energy is given in [3] as well as in the books [6,11]. The results on graph energy assume special significance. The present work is intended to relate the graph energy to larger graphs obtained from the given graph by means of some graph operations. Vaidya and Popat [14] showed that the energy of splitting graph and shadow graph can be obtained from energy of the given graph G. In this paper we have obtained the energy of central graph and middle graph of a regular graph from the eigenvalues of the given graph G.

Lemma 1.1. *[15] Let P, Q, R, S denote four $n \times n$ matrices and suppose that P and R commute. Then the determinant $\det(M)$ of the $2n \times 2n$ matrix $M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ is equal to the determinant of the matrix $PS - RQ$.*

Lemma 1.2. *[13] Let G be an $r-$regular graph with an adjacency matrix $A(G)$ and an incident matrix $R(G)$. Then $RR^T = rI + A$.*

Definition 1.3. [2] Central Graph: Let G be a simple, connected, r-regular and undirected graph and let its vertex set and edge set be denoted by $V(G)$ and $E(G)$ respectively. The Central graph of G, denoted by $C(G)$ is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.

Definition 1.4. [2] Middle Graph: The Middle graph $M(G)$ is a graph which is obtained by subdividing each edge of G exactly once and join all the newly added middle vertices of the adjacent edges of G.

2 Energy of central graph

In this section we obtain the energy of Central graph of a simple, connected, r-regular graph G.

Theorem 2.1. *If G is a simple, connected, r-regular and undirected graph with eigenvalues $\mu_1, \ldots, \mu_n$ and $E(G) = \sum_{i=1}^{n} |\mu_i|$, then*

$$E(C(G)) = \sum_{i=1}^{n} \left| \frac{(\theta_i - 1 - \mu_i) \pm \sqrt{(1 - \theta_i + \mu_i)^2 + 4(\mu_i + r)}}{2} \right| + 0^{m-n},$$

where, $\theta_1, \ldots, \theta_n$ are the eigenvalues of the adjacency matrix of J_n.

Proof. Let $v_1, \ldots, v_n$ and $e_1, \ldots, e_m$ be the vertices and edges of G. Its adjacency matrix $A(G)$ and incident matrix $R(G)$ are given by,

$$A(G) = \begin{array}{c c} & \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_n \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array} & \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \end{array}$$

$$R(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & & \cdots & e_m \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2m} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nm} \end{bmatrix} \end{array}$$

Now consider $C(G)$. Let $c_1, \ldots, c_m$ be the subdivided vertices of G. By joining the non-adjacent vertices $v_1, \ldots, v_n$ we obtain the Central graph of G. Then the adjacency matrix $A(C(G))$ can be written as a block matrix as shown in the Fig 1.

$$A(C(G)) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_m \end{array} \begin{array}{c} \begin{array}{cccccccccccc} v_1 & v_2 & v_3 & \cdots & v_n & c_1 & c_2 & c_3 & \cdots & c_m \end{array} \\ \left[\begin{array}{ccccc|ccccc} 0 & 1-a_{12} & 1-a_{13} & \cdots & 1-a_{1n} & b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ 1-a_{21} & 0 & 1-a_{23} & \cdots & 1-a_{2n} & b_{21} & b_{22} & b_{23} & \cdots & b_{2m} \\ 1-a_{31} & 1-a_{32} & 0 & \cdots & 1-a_{3n} & b_{31} & b_{32} & b_{33} & \cdots & b_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-a_{n1} & 1-a_{n2} & 1-a_{n3} & \cdots & 0 & b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nm} \\ \hline b_{11} & b_{21} & b_{31} & \cdots & b_{n1} & 0 & 0 & 0 & \cdots & 0 \\ b_{12} & b_{22} & b_{32} & \cdots & b_{n2} & 0 & 0 & 0 & \cdots & 0 \\ b_{13} & b_{23} & b_{33} & \cdots & b_{n3} & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ b_{1m} & b_{2m} & b_{3m} & \cdots & b_{nm} & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}$$

Fig. 1: Block Matrix of the adjacency matrix $A(C(G))$.

The edge set of G and $\overline{G}$ together form the edge set of the complete graph K_n. We observe that the adjacency matrix of the complete graph is $J_n - I_n$ (where J_n is the all-one matrix). Hence

$$A(G) + A\left(\overline{G}\right) = J_n - I_n$$

$$\Longrightarrow \quad A\left(\overline{G}\right) = \begin{bmatrix} 0 & 1-a_{12} & 1-a_{13} & \cdots & 1-a_{1n} \\ 1-a_{21} & 0 & 1-a_{23} & \cdots & 1-a_{2n} \\ 1-a_{31} & 1-a_{32} & 1-a_{33} & \cdots & 1-a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-a_{n1} & 1-a_{n2} & 1-a_{n3} & \cdots & 0 \end{bmatrix}$$

Hence the above block matrix shown in the Fig. 1 can be written as,

$$A(C(G)) = \begin{bmatrix} A(\overline{G}) & R(G) \\ R^T(G) & 0 \end{bmatrix}$$

Then, $\det\left[A(C(G)) - \lambda I_n\right] = \det\left[A(G) - \mu I_n\right]$ where $\mu = \frac{(-\lambda^2 + r - \lambda + \lambda J_n)}{(\lambda - 1)}$ (using the Lemmas 1.1 and 1.2). Hence corresponding to each eigenvalue μ_i of G,

$$\mu_i = \frac{(-\lambda_i^2 + r - \lambda_i + \lambda_i \theta_i)}{(\lambda_i - 1)}$$

where $\theta_1, \ldots, \theta_n$ are the eigenvalues of J_n. Then

$$\lambda_i^2 + (1 - \theta_i + \mu_i)\lambda_i - (\mu_i + r) = 0, \qquad i = 1, \ldots, n$$

which gives,

$$\lambda_i = \frac{(\theta_i - 1 - \mu_i) \pm \sqrt{(1 - \theta_i + \mu_i)^2 + 4(\mu_i + r)}}{2},\ 0^{m-n}$$

thus, $E(C(G)) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i is computed from the above relation.

□

Corollary 2.2. *If C_n is a cycle with eigenvalues $\mu_1, \ldots, \mu_n$ and $E(C_n) = \sum_{i=1}^{n} |\mu_i|$, then*

$$E(C(C_n)) = \sum_{i=1}^{n} \left| \frac{(\theta_i - 1 - \mu_i) \pm \sqrt{(1 - \theta_i + \mu_i)^2 + 4(\mu_i + 2)}}{2} \right|$$

where, $\theta_1, \ldots, \theta_n$ are the eigenvalues of the adjacency matrix of J_n.

Illustration 2.3. Consider the cycle C_4 and the Central graph of (C_4) as shown in the Fig. 2. The energy $E(C_4) = 4$ as $\mathrm{Spec}(C_4) = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

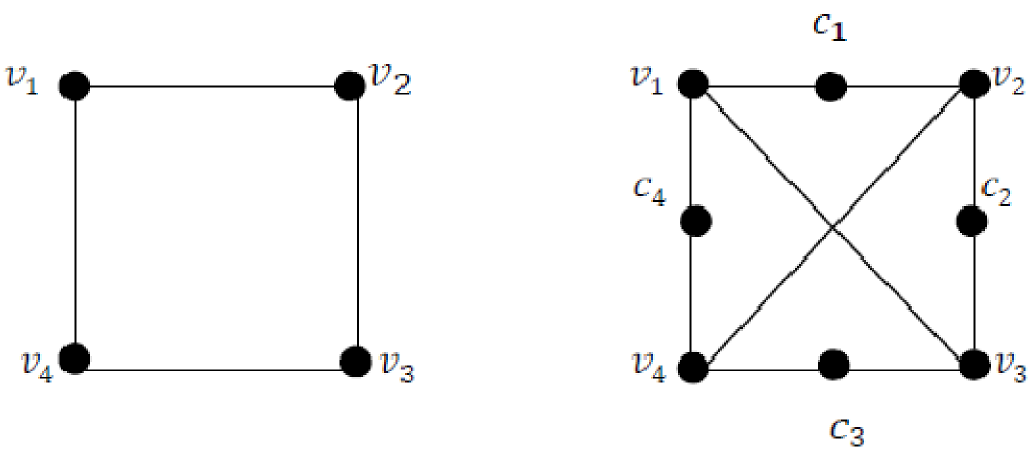

Fig. 2: The Central graph of C_4.

For, $J_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and the $\mathrm{Spec}(J_4) = \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix}$, when $\mu = 2$ and $\theta = 4$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2}$$
$$= 2.5616\ ,\ -1.5616.$$

When $\mu = -2$ and $\theta = 0$

$$\lambda_{3,4} = \frac{1 \pm \sqrt{1+0}}{2}$$
$$= 1,\ 0$$

and when $\mu = 0$ and $\theta = 0$

$$\lambda_{5,6,7,8} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$$
$$= 1,\ -2$$

$$\therefore E(C(C_4)) = \sum_{i=1}^{n} |\lambda_i| = 2.5616 + 1.5616 + 1 + 0 + 1 + 2$$
$$= 11.1232$$

and

$$\text{Spec}\; C\,(C_4) = \begin{pmatrix} 2.5616 & -1.5616 & 1 & 0 & -2 \\ 1 & 1 & 3 & 1 & 2 \end{pmatrix}.$$

3 Energy of middle Graph

In this section we obtain the energy of finite, simple, connected, $2-$regular and undirected graph G.

Theorem 3.1. *If G is a simple, connected, $2-$regular and undirected graph with eigenvalues $\mu_1, \ldots, \mu_n$ and $E\,(G) = \sum_{i=1}^{n} |\mu_i|$, then*

$$E(M(G)) = \sum_{i=1}^{n} \left| \frac{\mu_i \pm \sqrt{{\mu_i}^2 + 4(\mu_i + 2)}}{2} \right|.$$

Proof. Let $v_1, \ldots, v_n$ and $e_1, \ldots, e_n$ be the vertices and edges of G. Its adjacency matrix $A(G)$ and incident matrix $R(G)$ are given by,

$$A\,(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & & \cdots & v_n \end{array} \\ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \end{array}$$

$$R\,(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & & \cdots & e_n \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix} \end{array}$$

Now consider $M\,(G)$. Let $c_1, c_2, c_3, c_4, \ldots\ldots c_n$ be the subdivided vertices of G. By joining all the newly added middle vertices of adjacent edges of G we obtain the middle graph of G, i.e. $M(G)$. Then the adjacency matrix $A(M(G))$ can be written as a block matrix as shown in the Fig. 3.

$$A(M(G)) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{array} \begin{array}{c} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & \cdots & v_n & c_1 & c_2 & c_3 & \cdots & c_n \end{array} \\ \left[\begin{array}{ccccc:ccccc} 0 & 0 & 0 & \cdots & 0 & b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & 0 & 0 & \cdots & 0 & b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ 0 & 0 & 0 & \cdots & 0 & b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \\ \hdashline b_{11} & b_{21} & b_{31} & \cdots & b_{n1} & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ b_{12} & b_{22} & b_{32} & \cdots & b_{n2} & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ b_{13} & b_{23} & b_{33} & \cdots & b_{n3} & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & b_{3n} & \cdots & b_{nn} & a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{array}\right] \end{array}$$

Fig. 3: Block Matrix of the adjacency matrix $A(M(G))$.

Hence the above block matrix of the Fig. 3 can be written as,

$$A\left(M(G)\right) = \begin{bmatrix} 0 & R(G) \\ R^T(G) & A\left(G\right) \end{bmatrix}$$

Then, $\det\left[A\left(M\left(G\right)\right) - \lambda I_n\right] = \det\left[A(G) - \mu I_n\right]$ where $\mu = \frac{(\lambda^2-2)}{(\lambda+1)}$ (using the Lemmas 1.1 and 1.2). Hence corresponding to each eigenvalue μ_i of G,

$$\mu_i = \frac{(\lambda_i^2 - 2)}{(\lambda_i + 1)},$$

which gives,

$$\lambda_i^2 - \mu_i\lambda_i - (\mu_i + 2) = 0, \quad i = 1, \ldots, n,$$

hence,

$$\lambda_i = \frac{\mu_i \pm \sqrt{\mu_i^2 + 4(\mu_i + 2)}}{2},$$

and

$$E\left(M(G)\right) = \sum_{i=1}^{n} |\lambda_i|,$$

where, λ_i is computed using the above relation.

□

Illustration 3.2. Consider the cycle C_4 and the Middle graph $M(C_4)$ of the Fig. 4 . The energy $E\left(C_4\right) = 4$ as $\mathrm{Spec}\left(C_4\right) = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

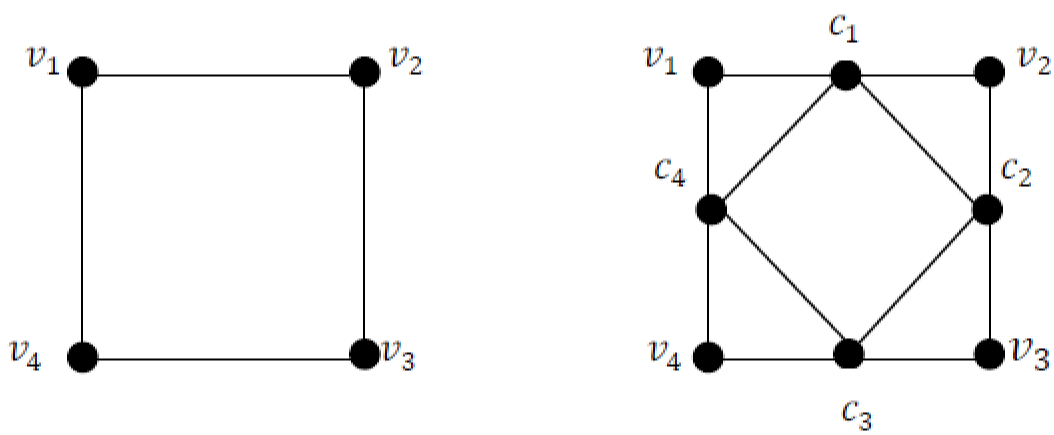

Fig. 4: The Middle graph of C_4.

When $\mu = 2$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4 + 4(2+2)}}{2} = 1 \pm \sqrt{5}$$
$$= 3.23606\ ,\ -1.23606.$$

When $\mu = -2$

$$\lambda_{3,4} = \frac{-2 \pm \sqrt{4 + 4(-2+2)}}{2} = -1 \pm 1$$
$$= 0\ ,\ -2.$$

When $\mu = 0$

$$\lambda_{5,6,7,8} = \frac{0 \pm \sqrt{0 \pm 4(0+2)}}{2} = \pm\sqrt{2}$$

$$= 1.41421\ ,\ -1.41421,$$

$$\therefore\ E\left(M(G)\right) = \sum_{i=1}^{n} |\lambda_i| = 3.2361 + 1.2361 + 0 + 2 + 1.414 + 1.414$$

$$= 9.65685,$$

and Spec $M\left(C_4\right) = \begin{pmatrix} 3.23606 & -1.23606 & 0 & -2 & 1.41421 & 1.41421 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$

4 Conclusion

The energy of a graph is one of the emerging concepts in graph theory which serves as a frontier between chemistry and mathematics. In this paper we have obtained the energy of the Central graph and the Middle graph of a regular graph from the energy of the given graph G.

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